IDENTIFIABILITY CRITERIA FOR CAUSAL EFFECTS OF JOINT INTERVENTIONS

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The aim of this paper is to give an extended front door criterion and a mixed door criterion as new identifiability criteria for causal effects of a joint intervention when causal diagrams for empirical research have some unobserved variables. The criteria enable us to recognize from the structure of the graph whether the causal effect can be expressed by a subset of observed variables. If the answer is affirmative, the criteria provide a closed-form expression for the effect of the external joint intervention.

Key Words and Phrases: causal diagram, d-separation, inference rule, mixed door criterion, recursive factorization.

1. Introduction

The purpose of this paper is to give a probabilistic evaluation of the effects of a joint intervention from nonexperimental data when knowledge is encoded in the form of a causal diagram. We are given the topology of the diagram, but not the observation on all variables.

As an illustration, consider the following example of a production process, as depicted in Figure 1.

Figure 1. Causal Diagram (1)

The variables X_1 and X_2 stand for treatments that can be controlled by an operator. Z_1 and Z_2 represent intermediate variables that are influenced by X_1 and X_2 , respectively. And Y represents the final quality characteristic. The hidden variables U_1 , U_2 and U_3 represent covariates such as environment factors which have influence on X_1 , X_2 , and Y .

Our problem is as follows. Assume we have collected a large amount of data on the production process which is summarized in the form of joint distribution f of the observed five variables X_1 , X_2 , Z_1 , Z_2 , and Y . We wish to determine the effect of the joint intervention $(\tilde{x}_1, \tilde{x}_2)$ on the final quality characteristic Y , where $(\tilde{x}_1, \tilde{x}_2)$ means that the treatments are set to level $X_1 = x_1$ and

Received November, 1998. Revised March, 1999. Accepted April, 1999.

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$X_2 = x_2$ respectively by external interventions.

The methods developed in this paper will enable us to recognize whether the effect of a certain intervention can be evaluated from the joint distribution on the observed variables. If the answer is affirmative, the methods provide a closed-form expression for the effect of the external joint intervention. Also, our methods are applicable to epidemiologic research, where investigators must adjust for covariates that confound the effect of some control variable and yet are on the causal pathways of other control variables. This work and the graphical methodology on which it is founded, make the control of confounding more principled and more accessible to epidemiologists.

The first step in this discussion is to construct a causal diagram such as the one shown in Figure 1, which represents the analyst's understanding of the causal influences among quantities considered. The statistical implications of causal diagrams have been given by Kiiveri, Speed and Carlin [2] and Wermuth and Lauritzen [7], and its relation to interventions has been discussed in Pearl [4] [5]. Pearl [4] and Galles and Pearl [1] give graphical criteria for deriving probabilistic assessment of the effect of a single intervention in the presence of unobserved variables. Further, Pearl and Robins [6] extend the results of Pearl [4] to the case where $\{X_1, X_2, \dots, X_n\}$ stands for a joint intervention consisting of several single interventions which are implemented either concurrently or sequentially. The criterion is the extension of the "back door criterion" developed by Pearl [4].

We will establish new graphical criteria for recognizing identifiability of the causal effect of $\{X_1, X_2, \dots, X_n\}$ on Y . Our criterion is the extension of the "front door criterion" given by Pearl [4]. The criterion is useful in the case shown in Figure 1 to which the criterion given by Pearl and Robins [6] can not be applied. When the diagram satisfies our criterion, we can provide a closed-form expression for the distribution of outcome variable under the joint intervention. Furthermore, extending certain results given by Galles and Pearl [1] to the joint intervention, we develop the mixed door criterion which includes the "front door criterion" and the "back door criterion".

2. Causal diagram and intervention: Review

2.1. Causal diagram and independencies

In statistical causal analysis, a directed acyclic graph which represents cause-effect relationships is called the causal diagram, and its mathematical definition is given in the following way.

In this paper, a joint distribution $f(x_1, x_2, \dots, x_n)$ is strictly positive, and $f(x_i|x_j)$ is a conditional distribution of x_i given x_j . Also, a graph is a pair $G = (V, E)$, where V is a finite set of vertices and the set E of arrows is a subset of the set $V \times V$ of ordered pairs of distinct vertices. In addition, graphtheoretic terminology used in this paper is provided in the Appendix.

DEFINITION 1. (*Causal diagram*)

Suppose that a directed acyclic graph $G = (V, E)$ and a set of variables $V =$

$\{X_1, \dots, X_n\}$ are given. When the joint distribution over a set V is factorized recursively according to the graph G as the following equation (2.1), the graph is called a Bayesian network.

$$(2.1) \quad f(x_1, \dots, x_n) = \prod_{i=1}^n f(x_i | pa(x_i)),$$

where $pa(x_i)$ is a set of parents of x_i in G . When $pa(x_i)$ is an empty set, $f(x_i | pa(x_i))$ is the marginal distribution over x_i . If, in addition, every parent-child family in the graph stands for an independent mechanism, the network is called a causal diagram. $\square$

For example, consider a causal diagram shown in Figure 2. The recursive factorization defined by Figure 2 is as follows:

$$(2.2) \quad f(x_1, x_2, x_3, x_4) = f(x_4 | x_1, x_2, x_3) f(x_3 | x_2) f(x_2 | x_1) f(x_1).$$

Figure 2. Causal Diagram (2)

If a joint distribution is factorized recursively according to G , then the independencies can be read off from the graph using the following test.

DEFINITION 2. (*D-separation*)

Let X , Y , and Z be three disjoint subsets of vertices in a causal diagram G . Then Z is said to *d-separate* X from Y , denoted $(X \perp\!\!\!\perp Y | Z)_G$, if along every path between a vertex in X and a vertex in Y there is a vertex w satisfying one of the following two conditions:

1. w has converging arrows and none of w or its descendants are in Z .
2. w does not have converging arrows and w is in Z .

If a path satisfies the conditions above, it is said to be blocked by Z . $\square$

In cases where the joint distribution is strictly positive, it can be shown that there is a one-to-one correspondence between the set of conditional independencies, $X \perp\!\!\!\perp Y | Z$, implied by the recursive factorization of equation (2.1) and the set of triples (X, Y, Z) that satisfies the d-separation criterion in G (Lauritzen [3]).

In Figure 2, since X_2 d-separates X_1 from X_3 , X_1 is conditionally independent of X_3 given X_2 .

2.2 Causal effect of single intervention

Under the causal diagram, Pearl [4] defines the effect of an external intervention as the causal effect, where “intervention” means that a variable is

forced to take on some fixed value. Pearl [4] gives its mathematical definition as follows:

DEFINITION 3. (*Causal effect of single intervention*)

Let $V = \{X, Y, Z_1, Z_2, \dots, Z_p\}$ be a set of vertices in causal diagram G . The causal effect of X on Y is defined as follows:

$$(2.3) \quad f(y|\tilde{x}) = \int_{z_1} \dots \int_{z_p} \frac{f(x, y, z_1, \dots, z_p)}{f(x|pa(x))} dz_1 \dots dz_p,$$

where $\tilde{x}$ indicates that X is fixed to x by an external intervention. $\square$

The graph corresponding to equation (2.3) is the subgraph of G from which all arrows entering X have been pruned. Such a subgraph is denoted by $G_{\tilde{x}}$. On the other hand, a path from X to Y in the graph G is said to be a back door path if it contains an arrow into X .

2.3 Identifiability criteria for single intervention

Given a causal diagram G , we seek to evaluate the causal effect from the joint distribution of the observed variables. In general, it is not easy to observe all covariates $Z_1, \dots, Z_p$. Hence, it is important to recognize sufficient sets of observed variables in order to evaluate the causal effect. Pearl [4] gives the identifiability criteria for the causal effects of a single intervention, where “identifiability” means that $f(y|\tilde{x})$ can be determined uniquely from the joint distribution of observed variables $X, Y, Z_1, Z_2, \dots$, and Z_n , namely, $f(y|\tilde{x})$ can be estimated consistently. The criteria are as follows.

DEFINITION 4. (*Back door criterion*)

Suppose there exist directed paths from X to Y in a causal diagram G . If a set of variables $\{Z_1, Z_2, \dots, Z_n\}$ satisfies the following conditions relative to an ordered pair of variables (X, Y) , then $\{Z_1, Z_2, \dots, Z_n\}$ is said to satisfy the back door criterion relative to (X, Y) .

1. No vertex in $\{Z_1, Z_2, \dots, Z_n\}$ is a descendant of X .
2. Every back door path from X to Y is blocked by $\{Z_1, Z_2, \dots, Z_n\}$. $\square$

THEOREM 1. (Pearl [4])

Suppose a set of variables $\{Z_1, Z_2, \dots, Z_n\}$ satisfies the back door criterion relative to (X, Y) in the causal diagram G . If $\{Z_1, Z_2, \dots, Z_n\}$ is observed with X and Y , then the causal effect of X on Y is identifiable and is given by the formula

$$(2.4) \quad f(y|\tilde{x}) = \int_{z_1} \int_{z_2} \dots \int_{z_n} f(y|x, z_1, z_2, \dots, z_n) f(z_1, z_2, \dots, z_n) dz_1 dz_2 \dots dz_n.$$

$\square$

In Figure 2, since the variable X_2 satisfies the back door criterion relative to (X_3, X_4) , the causal effect of X_3 on X_4 can be estimated consistently by observing X_2 with X_3 and X_4 .

DEFINITION 5. (*Front door criterion*)

Suppose there exist directed paths from X to Y in a causal diagram G . If a set of variables $\{Z_1, Z_2, \dots, Z_n\}$ satisfies the following conditions relative to (X, Y) , then $\{Z_1, Z_2, \dots, Z_n\}$ is said to satisfy the front door criterion relative to (X, Y) .

1. $\{Z_1, Z_2, \dots, Z_n\}$ intercepts all directed paths from X to Y .
2. There is no back door path from X to any vertex in $\{Z_1, Z_2, \dots, Z_n\}$.
3. Every back door path from any vertex in $\{Z_1, Z_2, \dots, Z_n\}$ to Y is blocked by X . □

In cases where a set of observed variables satisfies the front door criterion, Pearl [4] expressed a causal effect clearly. However, since an integral variable x' is mistaken for z by the misprint in Pearl [4], we give the correct equation as follows:

THEOREM 2. (*Pearl [4]*)

Suppose a set of variables $\{Z_1, Z_2, \dots, Z_n\}$ satisfies the front door criterion relative to (X, Y) in a causal diagram G . If $\{Z_1, Z_2, \dots, Z_n\}$ is observed with X and Y , then the causal effect of X on Y is identifiable and is given by the formula

$$(2.5) \quad f(y|\tilde{x}) = \int_{z_1} \int_{z_2} \dots \int_{z_n} f(z_1, z_2, \dots, z_n|x) \int_{x'} f(y|x', z_1, z_2, \dots, z_n) \times f(x') dx' dz_1 dz_2 \dots dz_n.$$

□

For example, consider the causal diagram shown by Figure 3.

Figure 3. Causal Diagram (3)

For intervention $\tilde{x}$, Z does not satisfy Definition 4. However, Z satisfies the front door criterion relative to (X, Y) . Therefore, the effect of the intervention $\tilde{x}$ is identifiable by observing Z from Theorem 2.

2.4. Causal effect of joint intervention

In section 2.3, we reviewed the criteria for recognizing indentifiability for the effect of a single intervention in causal diagrams. In practice, there are many cases in which we use a joint intervention in order to control the outcome and evaluate the causal effect. The results in section 2.3 can not be applied to these cases. Therefore, we define the effect of a joint intervention as follows and discuss its identifiability in this paper.

DEFINITION 6. (*Causal effect of joint intervention*)

Let $V = \{X_1, X_2, \dots, X_n, Y, Z_1, Z_2, \dots, Z_p\}$ be a set of vertices in causal diagram G . The causal effect of $\{X_1, X_2, \dots, X_n\}$ on Y is defined as follows:

$$(2.6) \quad f(y|\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) = \int_{z_1} \dots \int_{z_p} \frac{f(x_1, x_2, \dots, x_n, y, z_1, \dots, z_p)}{\prod_{i=1}^n f(x_i|pa(x_i))} dz_1 \dots dz_p.$$

□

2.5 Inference rules

Pearl [4] gives the following theorems regarding identifiability criteria for the effect of the intervention.

THEOREM 3. (*Inference rules: Pearl [4]*)

Let G be a causal diagram, $f(\cdot)$ stands for the distribution induced by the model, and the expression $f(y|\tilde{x}, z) \triangleq f(y, z|\tilde{x})/f(z|\tilde{x})$ stands for the distribution over y given that $Z=z$ is observed and X is fixed to x by an external intervention.

- Rule 1 (*Insertion/Deletions of observations*):

If every path between Y and Z is blocked by X and W in the graph $G_{\bar{X}}$, then

$$(2.7) \quad f(y|\tilde{x}, z, w) = f(y|\tilde{x}, w).$$

- Rule 2 (*Action/Observation exchange*):

If every back door path from Z to Y is blocked by X and W in the graph $G_{\bar{X}}$, then

$$(2.8) \quad f(y|\tilde{x}, \tilde{z}, w) = f(y|\tilde{x}, z, w).$$

- Rule 3 (*Insertion/Deletions of actions*):

If every path between Y and Z is blocked by X and W in the graph $G_{\bar{XZ}(W)}$, then

$$(2.9) \quad f(y|\tilde{x}, \tilde{z}, w) = f(y|\tilde{x}, w),$$

where $Z(W)$ is the set of Z -vertices that are not ancestors of any W -vertex in $G_{\bar{X}}$. □

THEOREM 4. (*Pearl [4]*)

A causal effect $f(y|\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ is identifiable in a causal diagram G if there exists a finite sequence of transformations, each conforming to one of the inference rules, which reduces $f(y|\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ into a distribution expression involving only observed variables. □

2.6. Identifiability criterion for joint intervention

Pearl and Robins [6] have discussed the identifiability of the causal effect of the joint intervention, and obtained the identifiability criteria and the evaluation of the effect of the joint intervention by using Inference Rules concretely. Their result is as follows.

DEFINITION 7. (*Admissible*)

Suppose there exist directed paths from any node in $\{X_1, X_2, \dots, X_n\}$ to Y in a causal diagram G , where X_i is an ancestor of X_{i+1} . If a set of variables $\{Z_1, Z_2, \dots, Z_n\}$ satisfies the following conditions, then $\{Z_1, Z_2, \dots, Z_n\}$ is said to be admissible relative to an ordered sequence of variables $(X_1, X_2, \dots, X_n, Y)$. For all $i(i=1, \dots, n)$,

1. Z_i is a nondescendant of X_i .
2. In the graph $G_{\overline{X_{i+1} \dots X_n}}$, every back door path from X_i to Y is blocked by $\{X_1, \dots, X_{i-1}, Z_1, \dots, Z_i\}$. $\square$

THEOREM 5. (*Pearl and Robins [6]*)

Suppose a set $\{Z_1, Z_2, \dots, Z_n\}$ of variables is admissible relative to $(X_1, X_2, \dots, X_n, Y)$ in a causal diagram G . If $\{Z_1, Z_2, \dots, Z_n\}$ is observed with $\{X_1, X_2, \dots, X_n\}$ and Y , then the causal effect of $\{X_1, X_2, \dots, X_n\}$ on Y is identifiable, and is given by the formula

$$(2.10) \quad f(y|\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) = \int_{z_1} \int_{z_2} \dots \int_{z_n} f(y|x_1, x_2, \dots, x_n, z_1, z_2, \dots, z_n) \\ \times \prod_{i=1}^n f(z_i|z_1, z_2, \dots, z_{i-1}, x_1, x_2, \dots, x_{i-1}) dz_1 dz_2 \dots dz_n.$$

 $\square$ **3. New identifiability criteria for joint intervention****3.1. The extension of the front door criterion**

Since the criterion developed by Pearl and Robins [6] is an extension of the back door criterion, the causal diagrams to which the criterion is applicable are limited. The identifiability criterion for causal effect of the joint intervention corresponding to the front door criterion has not been proposed.

DEFINITION 8. (*Extended front door criterion*)

Suppose there exist directed paths from any vertex in $\{X_1, X_2, \dots, X_n\}$ to Y in a causal diagram G , where X_i is an ancestor of X_{i+1} . If a non-empty set of variables $\{Z_1, Z_2, \dots, Z_n\}$ satisfies the following conditions, then $\{Z_1, Z_2, \dots, Z_n\}$ is said to satisfy the extended front door criterion relative to $(X_1, X_2, \dots, X_n, Y)$.

1. Z_i is a nondescendant of X_{i+1} and Z_{i+1} .
2. $\{X_1, X_2, \dots, X_n\}$ is admissible relative to $(Z_1, Z_2, \dots, Z_n, Y)$.
3. Every directed path from X_i to Y is intercepted by $\{X_{i+1}, \dots, X_n, Z_1, \dots, Z_n\}$.
4. Every back door path from $X_j(1 \leq j \leq i)$ to Z_i is blocked by $\{Z_1, \dots, Z_{i-1}, X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_i\}$ in the graph $G_{\overline{X_{i+1} \dots X_i}}$. $\square$

THEOREM 6.

Suppose a set of variables $\{Z_1, Z_2, \dots, Z_n\}$ satisfies the extended front door criterion relative to $(X_1, X_2, \dots, X_n, Y)$ in a causal diagram G . If $\{Z_1, Z_2, \dots, Z_n\}$ is observed with $X_1, X_2, \dots, X_n$ and Y , then the causal effect of $\{X_1, X_2, \dots, X_n\}$ on Y is identifiable and is given by the formula

$$\begin{aligned}
(3.1) \quad & f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n) \\
&= \int_{z_1} \int_{z_2} \cdots \int_{z_n} \prod_{i=1}^n f(z_i|z_1, z_2, \dots, z_{i-1}, x_1, x_2, \dots, x_i) \\
&\quad \times \int_{x_1} \int_{x_2} \cdots \int_{x_n} f(y|x'_1, x'_2, \dots, x'_n, z_1, z_2, \dots, z_n) \\
&\quad \times \prod_{i=1}^n f(x'_i|x'_1, x'_2, \dots, x'_{i-1}, z_1, z_2, \dots, z_{i-1}) dx'_1 dx'_2 \cdots dx'_n dz_1 dz_2 \cdots dz_n.
\end{aligned}$$

□

PROOF. From equations (2.1) and (2.3), we can obtain the following equation:

$$\begin{aligned}
(3.2) \quad & f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n) = \int_{z_1} \int_{z_2} \cdots \int_{z_n} f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, z_1, z_2, \dots, z_n) \\
&\quad \times \prod_{i=1}^n f(z_i|z_1, z_2, \dots, z_{i-1}, \check{x}_1, \check{x}_2, \dots, \check{x}_i) dz_1 dz_2 \cdots dz_n.
\end{aligned}$$

Since any element of $\{X_{i+1}, \dots, X_n\}$ is not an ancestor of Z_i from condition 1, the element is not an ancestor of $Z_1, \dots, Z_{i-1}$. Therefore, by using Rule 3, we can obtain $f(z_i|z_1, z_2, \dots, z_{i-1}, \check{x}_1, \check{x}_2, \dots, \check{x}_n) = f(z_i|z_1, z_2, \dots, z_{i-1}, \check{x}_1, \check{x}_2, \dots, \check{x}_i)$. Every back door path from X_1 to Z_i is blocked by $\{Z_1, \dots, Z_{i-1}, X_2, \dots, X_i\}$ in the graph $G_{\overline{X_2 \cdots X_i}}$ from condition 4. Therefore, using Rule 2,

$$(3.3) \quad f(z_i|z_1, z_2, \dots, z_{i-1}, \check{x}_1, \check{x}_2, \dots, \check{x}_i) = f(z_i|z_1, z_2, \dots, z_{i-1}, x_1, \check{x}_2, \dots, \check{x}_i).$$

Further, every back door path from X_2 to Z_i is blocked by $\{Z_1, \dots, Z_{i-1}, X_1, X_3, \dots, X_i\}$ in the graph $G_{\overline{X_3 \cdots X_i}}$ from condition 4. Therefore, using Rule 2,

$$(3.4) \quad f(z_i|z_1, z_2, \dots, z_{i-1}, x_1, \check{x}_2, \check{x}_3, \dots, \check{x}_i) = f(z_i|z_1, z_2, \dots, z_{i-1}, x_1, x_2, \check{x}_3, \dots, \check{x}_i).$$

Repeating this operation, the following results can be obtained:

$$\begin{aligned}
(3.5) \quad & \prod_{i=1}^n f(z_i|z_1, z_2, \dots, z_{i-1}, \check{x}_1, \check{x}_2, \dots, \check{x}_n) \\
&= \prod_{i=1}^n f(z_i|z_1, z_2, \dots, z_{i-1}, \check{x}_1, \check{x}_2, \dots, \check{x}_i) \\
&= \prod_{i=1}^n f(z_i|z_1, z_2, \dots, z_{i-1}, x_1, x_2, \dots, x_i).
\end{aligned}$$

On the other hand, since $\{X_1, X_2, \dots, X_n\}$ is admissible relative to $(Z_1, Z_2, \dots, Z_n, Y)$, every back door path from Z_n to Y is blocked by $\{X_1, \dots, X_n, Z_1, \dots, Z_{n-1}\}$. Therefore, we can obtain $f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, z_1, z_2, \dots, z_n) = f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, z_1, z_2, \dots, z_{n-1}, \check{z}_n)$. Furthermore, since every back door path from Z_{n-1} to Y is blocked by $\{X_1, \dots, X_n, Z_1, \dots, Z_{n-2}\}$ in $G_{\overline{Z_n}}$, we can obtain

$$\begin{aligned}
(3.6) \quad & f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, z_1, z_2, \dots, z_{n-1}, \check{z}_n) \\
&= f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, z_1, z_2, \dots, z_{n-2}, \check{z}_{n-1}, \check{z}_n).
\end{aligned}$$

Repeating this operation, the following results can be obtained:

$$(3.7) \quad f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, z_1, z_2, \dots, z_n) = f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, \check{z}_1, \check{z}_2, \dots, \check{z}_n).$$

Since every directed path from X_1 to Y is intercepted by $\{X_2, \dots, X_n, Z_1, \dots, Z_n\}$ from condition 3, we can obtain

$$(3.8) \quad f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, \check{z}_1, \check{z}_2, \dots, \check{z}_n) = f(y|\check{x}_2, \check{x}_3, \dots, \check{x}_n, \check{z}_1, \check{z}_2, \dots, \check{z}_n).$$

Furthermore, since all the directed paths from X_2 to Y are intercepted by $\{X_3, \dots, X_n, Z_1, \dots, Z_n\}$ from condition 3, we can obtain

$$(3.9) \quad f(y|\check{x}_2, \check{x}_3, \dots, \check{x}_n, \check{z}_1, \check{z}_2, \dots, \check{z}_n) = f(y|\check{x}_3, \check{x}_4, \dots, \check{x}_n, \check{z}_1, \check{z}_2, \dots, \check{z}_n).$$

Repeating this operation, the following results can be obtained:

$$(3.10) \quad f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, \check{z}_1, \check{z}_2, \dots, \check{z}_n) = f(y|\check{z}_1, \check{z}_2, \dots, \check{z}_n).$$

Therefore, since $\{X_1, X_2, \dots, X_n\}$ is admissible relative to $(Z_1, Z_2, \dots, Z_n, Y)$, $f(y|\check{z}_1, \check{z}_2, \dots, \check{z}_n)$ is equal to the following equation from Theorem 5:

$$\begin{aligned} (3.11) \quad & f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, z_1, z_2, \dots, z_n) \\ &= f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n, \check{z}_1, \check{z}_2, \dots, \check{z}_n) = f(y|\check{z}_1, \check{z}_2, \dots, \check{z}_n) \\ &= \int_{x_1} \int_{x_2} \dots \int_{x_n} f(y|x_1, x_2, \dots, x_n, z_1, z_2, \dots, z_n) \\ &\quad \times \prod_{i=1}^n f(x_i|x_1, x_2, \dots, x_{i-1}, z_1, z_2, \dots, z_{i-1}) dx_1 dx_2 \dots dx_n. \end{aligned}$$

Applying equations (3.5) and (3.11) to equation (3.2), we can obtain equation (3.1). Since the right hand side of equation (3.1) is a distribution expression involving only observed variables, $f(y|\check{x}_1, \check{x}_2, \dots, \check{x}_n)$ is identifiable. Q.E.D.

We demonstrate how Theorem 6 can be used to test the identifiability of the causal effect in Figure 1. First, note that $f(y|\check{x}_1)$ can not be identified even if both Z_1 and Z_2 are observed. However, if we have a joint intervention $(\check{x}_1, \check{x}_2)$, since (Z_1, Z_2) satisfies the extended front door criterion for the joint intervention $(\check{x}_1, \check{x}_2)$, $f(y|\check{x}_1, \check{x}_2)$ is found to be identifiable by using Theorem 6, and the causal effect is

$$(3.12) \quad f(y|\check{x}_1, \check{x}_2) = \int_{z_1} \int_{z_2} \int_{x_1} \int_{x_2} f(y|x'_1, x'_2, z_1, z_2) f(x'_2|x'_1, z_1) f(x'_1) \\ \times f(z_2|x_1, x_2) f(z_1|x_1) dx'_1 dx'_2 dz_1 dz_2.$$

3.2. Mixed door criterion

We can develop the identifiability criterion as follows:

DEFINITION 9. (*Mixed door criterion*)

Suppose there exist directed paths from $\{X_1, X_2, \dots, X_n\}$ to Y in a causal diagram G , where X_i is an ancestor of X_{i+1} . For all $i(i=1, \dots, n)$, if a vertex Z_i in $\{Z_1, Z_2, \dots, Z_n\}$ satisfies one of the following conditions, then $\{Z_1, Z_2, \dots, Z_n\}$ is said to satisfy the mixed door criterion relative to $(X_1, X_2, \dots, X_n, Y)$, where Z_i is a nondescendant of Z_{i+1} .

1. (a) Z_i is a nondescendant of X_i .
 (b) In the graph $G_{\overline{X_{i+1} \cdots X_n}}$, every back door path from X_i to Y is blocked by $\{X_1, \cdots, X_{i-1}, Z_1, \cdots, Z_i\}$.
2. (a) Z_i is a nondescendant of X_{i+1} , and X_i is a nondescendant of Z_i .
 (b) Every back door path from X_i to Z_i is blocked by $\{Z_1, \cdots, Z_{i-1}, X_1, \cdots, X_{i-1}\}$.
 (c) Every directed path from X_i to Y is intercepted by $\{X_{i+1}, \cdots, X_n, Z_1, \cdots, Z_i\}$.
 (d) In the graph $G_{\overline{X_{i+1} \cdots X_n}}$, every back door path from Z_i to Y is blocked by $\{X_1, \cdots, X_i, Z_1, \cdots, Z_{i-1}\}$. $\square$

THEOREM 7.

Suppose a set of variables $\{Z_1, Z_2, \cdots, Z_n\}$ satisfies the mixed door criterion relative to $(X_1, X_2, \cdots, X_n, Y)$ in a causal diagram G . If $\{Z_1, Z_2, \cdots, Z_n\}$ is observed with $\{X_1, X_2, \cdots, X_n\}$ and Y , then the causal effect of $\{X_1, X_2, \cdots, X_n\}$ on Y is identifiable. $\square$

PROOF. Without loss of generality, let a joint intervention be $(\check{x}_1, \check{x}_2)$. If every vertex in $\{Z_1, Z_2\}$ satisfies condition 1, the mixed door criterion implies that $\{Z_1, Z_2\}$ is admissible relative to (X_1, X_2, Y) . Hence, we consider the following three cases:

Case 1. Z_1 satisfies condition 1, and Z_2 satisfies condition 2.

Case 2. Z_1 satisfies condition 2, and Z_2 satisfies condition 1.

Case 3. Both Z_1 and Z_2 satisfy condition 2.

First, assume there exist directed paths from X_1 to Y which do not have X_2 , and consider the identifiability of the following causal effect.

$$(3.13) \quad f(y|\check{x}_1, \check{x}_2) = \int_{z_1} f(y|\check{x}_1, \check{x}_2, z_1) f(z_1|\check{x}_1, \check{x}_2) dz_1$$

Case 1: Since Z_1 is a nondescendant of X_1 and X_2 from condition 1-(a), $f(z_1|\check{x}_1, \check{x}_2) = f(z_1)$. Further, from condition 1-(b), $f(y|\check{x}_1, \check{x}_2, z_1) = f(y|x_1, z_1, \check{x}_2)$. Here, we consider the identifiability of the following equation:

$$(3.14) \quad f(y|x_1, z_1, \check{x}_2) = \int_{z_2} f(y|x_1, z_1, z_2, \check{x}_2) f(z_2|x_1, z_1, \check{x}_2) dz_2$$

Since X_1 and Z_1 are nondescendants of X_2 from condition 1-(a), a variable satisfying condition 2-(c) is Z_2 . From condition 2-(b), we can obtain $f(z_2|x_1, z_1, \check{x}_2) = f(z_2|x_1, x_2, z_1)$. Also, from condition 2-(d), $f(y|x_1, z_1, z_2, \check{x}_2) = f(y|x_1, z_1, \check{z}_2, \check{x}_2)$. Since every directed path from X_2 to Y is intercepted by Z_2 from condition 2-(c), we can obtain $f(y|x_1, z_1, \check{z}_2, \check{x}_2) = f(y|x_1, z_1, \check{z}_2)$. Here, we consider the identifiability of the following equation:

$$(3.15) \quad f(y|x_1, z_1, \check{z}_2) = \int_{x'_2} f(y|x_1, x'_2, z_1, \check{z}_2) f(x'_2|x_1, z_1, \check{z}_2) dx'_2.$$

Since X_2 is a nondescendant of Z_2 from condition 2-(a), $f(x'_2|x_1, z_1, \check{z}_2) = f(x'_2|x_1, z_1)$. Further, from condition 2-(d), we can obtain $f(y|x_1, x'_2, z_1, \check{z}_2) = f(y|x_1, x'_2,$

z_1, z_2). Hence, $f(y|\tilde{x}_1, \tilde{x}_2)$ can be expressed by X_1, X_2, Z_1 and Z_2 .

Case 2: Since X_1 is a nondescendant of Z_1 and Z_1 is a nondescendant of X_2 from condition 2-(a), $f(z_1|\tilde{x}_1, \tilde{x}_2)=f(z_1|\tilde{x}_1)$. Further, from condition 2-(b), we can obtain $f(z_1|\tilde{x}_1)=f(z_1|x_1)$. On the other hand, from condition 2-(d), we can obtain $f(y|\tilde{x}_1, \tilde{x}_2, z_1)=f(y|\tilde{x}_1, \tilde{x}_2, \tilde{z}_1)$. Every directed path from X_1 to Y is intercepted by X_2 and Z_1 . Therefore, we can obtain

$$(3.16) \quad f(y|\tilde{x}_1, \tilde{x}_2, \tilde{z}_1)=f(y|\tilde{x}_2, \tilde{z}_1)=\int_{x_1} f(y|x'_1, \tilde{x}_2, \tilde{z}_1)f(x'_1|\tilde{x}_2, \tilde{z}_1)dx'_1.$$

Since X_1 is a nondescendant of X_2 and Z_1 from condition 2-(a), $f(x'_1|\tilde{x}_2, \tilde{z}_1)=f(x'_1)$. Also, from condition 2-(d), we can obtain

$$(3.17) \quad f(y|x'_1, \tilde{x}_2, \tilde{z}_1)=f(y|x'_1, \tilde{x}_2, z_1)=\int_{z_2} f(y|x'_1, \tilde{x}_2, z_1, z_2)f(z_2|x'_1, \tilde{x}_2, z_1)dz_2.$$

From condition 1-(a), we can obtain $f(z_2|x'_1, \tilde{x}_2, z_1)=f(z_2|x'_1, z_1)$. Further, from condition 1-(b), we can obtain $f(y|x'_1, \tilde{x}_2, z_1, z_2)=f(y|x'_1, x_2, z_1, z_2)$.

From these results, we recognize that $f(y|\tilde{x}_1, \tilde{x}_2)$ can be identified.

Case 3: Since X_2 is not an ancestor of Z_1 from condition 2-(a), we can obtain $f(z_1|\tilde{x}_1, \tilde{x}_2)=f(z_1|\tilde{x}_1)$. Since every back door path from X_1 to Z_1 is blocked by ϕ from condition 2-(b), we can obtain $f(z_1|\tilde{x}_1)=f(z_1|x_1)$. On the other hand, every back door path from Z_1 to Y is blocked by X_1 in $G_{\bar{X}_1}$ from condition 2-(d). Therefore, we can obtain $f(y|\tilde{x}_1, \tilde{x}_2, z_1)=f(y|\tilde{x}_1, \tilde{x}_2, \tilde{z}_1)$. Also, since every directed path from X_1 to Y is intercepted by X_2 and Z_1 from condition 2-(c), we can obtain $f(y|\tilde{x}_1, \tilde{x}_2, \tilde{z}_1)=f(y|\tilde{x}_2, \tilde{z}_1)$. Hence, we consider the identifiability of the following equation:

$$(3.18) \quad f(y|\tilde{x}_2, \tilde{z}_1)=\int_{x_1} f(y|x'_1, \tilde{x}_2, \tilde{z}_1)f(x'_1|\tilde{x}_2, \tilde{z}_1)dx'_1.$$

Since X_1 is a nondescendant of Z_1 and X_2 from condition 2-(a), we can obtain $f(x'_1|\tilde{x}_2, \tilde{z}_1)=f(x'_1)$. Further, since every back door path from Z_1 to Y is blocked by X_1 in $G_{\bar{X}_1}$ from condition 2-(d), we can obtain

$$(3.19) \quad f(y|x'_1, \tilde{x}_2, \tilde{z}_1)=f(y|x'_1, \tilde{x}_2, z_1)=\int_{z_2} f(y|x'_1, \tilde{x}_2, z_1, z_2)f(z_2|x'_1, \tilde{x}_2, z_1)dz_2.$$

Since every back door path from X_2 to Z_2 is blocked by X_1 and Z_1 from condition 2-(b), we can obtain $f(z_2|x'_1, \tilde{x}_2, z_1)=f(z_2|x'_1, x_2, z_1)$. On the other hand, every back door path from Z_2 to Y is blocked by X_1, X_2 and Z_1 from condition 2-(d). Therefore, we can obtain $f(y|x'_1, \tilde{x}_2, z_1, z_2)=f(y|x'_1, \tilde{x}_2, z_1, \tilde{z}_2)$. Also, since every directed path from X_2 to Y is intercepted by Z_2 from condition 2-(c), we can obtain

$$(3.20) \quad f(y|x'_1, \tilde{x}_2, z_1, \tilde{z}_2)=f(y|x'_1, z_1, \tilde{z}_2)=\int_{x_2} f(y|x'_1, x'_2, z_1, \tilde{z}_2)f(x'_2|x'_1, z_1, \tilde{z}_2)dx'_2.$$

Since X_2 is not a nondescendant of Z_2 from condition 2-(a), we can obtain $f(x'_2|x'_1, z_1, \tilde{z}_2)=f(x'_2|x'_1, z_1)$. Further, since every back door path from Z_2 to Y

is blocked by X_1 , X_2 and Z_1 from condition 2-(d), we can obtain $f(y|x'_1, x'_2, z_1, \check{z}_2) = f(y|x'_1, x'_2, z_1, z_2)$. Hence, $f(y|\check{x}_1, \check{x}_2)$ can be expressed by X_1 , X_2 , Z_1 and Z_2 .

If all the directed paths from X_1 to Y have X_2 , the problem of the identifiability of interventions $(\check{x}_1, \check{x}_2)$ resolves into that of intervention $\check{x}_2$.

Q.E.D.

For example, consider the causal diagram shown by Figure 4.

Figure 4. Causal Diagram (4)

For interventions $(\check{x}_1, \check{x}_2)$, Z_2 satisfies neither Definition 7 nor Definition 8. However, $Z_1 = \phi$ satisfies condition 1 of Definition 9 and Z_2 satisfies condition 2 of Definition 9 relative to $(X_1, X_2, \dots, X_n, Y)$. Therefore, since (Z_1, Z_2) satisfies the mixed door criterion relative to $(X_1, X_2, \dots, X_n, Y)$, the effect of the joint intervention $(\check{x}_1, \check{x}_2)$ is identifiable by observing Z_2 , and the causal effect is given as follows.

$$(3.21) \quad f(y|\check{x}_1, \check{x}_2) = \int_{z_2} \int_{x_2} f(y|x_1, z_2, x'_2) f(x'_2|x_1) f(z_2|x_1, x_2) dx'_2 dz_2.$$

In cases where there exists a set $\{Z_1, Z_2, \dots, Z_n\}$ of observed variables satisfying the mixed door criterion, as seen from the proof of Theorem 7, a closed-form expression of causal effects will be given for proposed interventions.

Acknowledgements

The authors are grateful to the referees for their valuable suggestions.

Appendix

[Directed graph]

A directed graph is a pair $G=(V, E)$, where V is a finite set of vertices and the set E of arrows is a subset $V \times V$ of ordered pairs of distinct vertices. An arrow pointing α towards β indicates $(\alpha, \beta) \in E$ and $(\beta, \alpha) \notin E$. If there is an arrow pointing towards β , α is said to be a parent of β , and β a child of α . The set of parents of β is denoted as $pa(\beta)$, and the set of children of α as $ch(\alpha)$. A path from α to β is a sequence $\alpha = \alpha_0, \alpha_1, \dots, \beta = \alpha_n$ of distinct vertices such that $(\alpha_{i-1}, \alpha_i) \in E$ or $(\alpha_i, \alpha_{i-1}) \in E$ for all $i=1, 2, \dots, n$. A directed path from α towards β is a sequence of distinct vertices such that $(\alpha_{i-1}, \alpha_i) \in E$ for all $i=1, \dots, n$. A directed path which begins and ends in the same point is said to be cyclic. If a directed graph has no cycles, the graph is said to be acyclic. When there exists a directed path from α towards β , α is said to be the ancestor of β , and β the descendant of α . The set of ancestors of β is denoted as $an(\beta)$,

the set of descendants of α as $de(\alpha)$. The set of nondecendants of α is $nd(\alpha) = V \setminus (de(\alpha), \alpha)$. If there exists some vertex in a set of vertices γ along a directed path from α to β , the directed path from α to β is said to be intercepted by γ .

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