MONTE CARLO RESULTS ON THE SEMIPARAMETRIC NEAREST NEIGHBOR ESTIMATOR

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Robinson (1987) proposed to use a nearest neighbor approach to estimate the regression coefficient in a heteroskedastic linear model. While this estimator is asymptotically efficient, it has been said to be inefficient in small samples compared with other semiparametric estimators such as those using a kernel. Like other semiparametric methods, his estimators of variances, which are used to get the weighted least squares estimator of the regression coefficient, are constructed as a weighted sum of the squared ols residuals. As Robinson (1987) indicated himself, however, there exists a sample splitting problem in his estimator and this may cause the small-sample inefficiency. Therefore a slight modification improves the small sample property of the k-NN estimator. In this paper we report Monte Carlo experiments and show that the modified Nearest Neighbor estimator has a sufficient level of efficiency in small samples.

Key Words and Phrases: semiparametric regression, heteroskedasticity, nearest neighbor method, sample splitting.

1. Introduction

In a heteroskedastic linear model researchers have recently become interested in semiparametric methods. By using such methods we can get estimates whose efficiency does not depend on the validity of the assumed variance structure. Several semiparametric estimators have been proved to be asymptotically efficient.

To describe the semiparametric method simply we set up a heteroskedastic linear model as follows,

$$Y_i = X_i' \beta + U_i \quad (i=1, 2, \dots, n),$$

where $(X_1, Y_1), \dots, (X_n, Y_n)$ are random samples from (X, Y) , which is a $d+1$ -dimensional random variable with $E(U|X)=0$ and $Var(U|X)=\sigma^2(X)$. It should be noted that the concrete functional form of $\sigma^2(X)$ is unknown.

In the semiparametric setting we use the weighted least squares method to estimate the coefficient parameters,

$$\hat{\beta} = \left(\sum_{i=1}^n X_i X_i' \hat{\sigma}^{-2}(X_i) \right)^{-1} \left(\sum_{i=1}^n X_i Y_i \hat{\sigma}^{-2}(X_i) \right),$$

where $\hat{\sigma}^2(X)$ denotes the estimator of the conditional variance. The main difference among several semiparametric methods lies in the way to estimate

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$\sigma^2(X)$, i.e., which nonparametric method is employed. Carroll (1982) and Rose (1978) used a kernel method, while Robinson (1987) used a nearest neighbor method. Asymptotically, both estimators are equivalent, i.e., both of them are asymptotically equivalent as GLS

$$\hat{\beta}_G = \left(\sum_{i=1}^n X_i X_i' \sigma^{-2}(X_i) \right)^{-1} \left(\sum_{i=1}^n X_i Y_i \sigma^{-2}(X_i) \right),$$

but the kernel type estimate has proved to be asymptotically efficient only when the dimension of the independent variable that has a relation to the variance generating process is unity. In addition, the conditions necessary to prove the asymptotic efficiency of a kernel type estimate are stricter than those for Robinson's estimate in some senses. Therefore we can use Robinson's method in a wider class of models than the kernel type one.

On the other hand, according to some Monte Carlo works Robinson's estimator is generally inferior to the kernel type estimator in small samples. (See Rilstone (1991) for instance). Actually this small sample inefficiency of Robinson's estimator does not come from an essential reason but a technical one.

To see this we should realize the direct cause of the inefficiency. In either method the estimator of the variance is expressed by the weighted sum of the squared ordinal least squares residuals

$$\hat{\sigma}^2(X_i) = \sum_{j=1}^n W_{ij} \hat{U}_j^2,$$

where $\hat{\sigma}^2(X_i)$, W_{ij} , $\hat{U}_j^2$ denote the nonparametric estimator of the i -th variance, the j -th weight to construct the i -th estimator and the OLS residual of the j -th sample, respectively. When using a kernel method the weight is

$$W_{ij} = K((X_j - X_i)/b_n) / \sum_l K((X_l - X_i)/b_n),$$

where $K(\cdot)$ denotes a kernel function and b_n a bandwidth. From the ordinal property of the kernel function, the closer X_j is to X_i , the larger the value W_{ij} takes. Therefore W_{ii} always takes a maximum value when the estimand is σ_i^2 .

Unlike the kernel type estimation Robinson's nearest neighbor weight is not determined by the *absolute* distance between design points but the *relative* distance between them. In this method the value of the weights is pre-determined, i.e., we need to prepare a set of constants, $c_1, c_2, \dots, c_n$ which usually satisfy $c_1 \geq c_2 \geq \dots \geq c_n \geq 0$ and $\sum_j c_j = 1$. If X_j is the closest point to X_i among X_l ($l=1, \dots, n; l \neq i$), then W_{ij} gets the largest value c_1 as a weight and the second closest point is assigned c_2 , and so on. Note that X_i in the above definition does not include X_i itself and then W_{ii} is not defined through this process. In fact, W_{ii} is assigned to 0 for any i . Among all the ordinary least squares residuals, $\hat{U}_i^2$ is expected to have the most information about $\sigma^2(X_i)$ and so the estimation without it could cause a considerable loss in efficiency. This seems to be the main reason for the small sample inefficiency of Robinson's

estimator. He excluded $\hat{U}_i$ from the estimation of $\sigma^2(X_i)$ because of technical difficulty in proving the asymptotic efficiency. Hence it could be expected that his estimator remains asymptotically efficient and gets an improved small sample property by including $\hat{U}_i$ in the estimation of $\sigma^2(X_i)$, which is actually proved later.

In section 2 we will see the details of the modification and section 3 describes the necessary assumptions to prove asymptotic efficiency of our estimator. We report some Monte Carlo studies to see the property of several semiparametric estimators in small samples in section 4.

2. The modified estimator

Before describing the formal definition of the proposed estimator, we show its essence. At first we need to set the constant $c_i (i=1, 2, \dots, k)$ satisfying $c_1 \geq c_2 \geq \dots \geq c_k > 0$ and $\sum_{i=1}^k c_i = 1$, where k is a fixed number satisfying $k < n$. The nonincreasing condition is not necessarily needed in the formal definition but it may gain efficiency in small samples. Let $X_{i(j)}$ be the j -th closest point to X_i among $X_l (l=1, \dots, n; l \neq i)$ in the Euclidean norm and $\hat{U}_{i(j)}$ be an ordinary least squares residual corresponding to $X_{i(j)}$. It should be noted that the elements of X are normalized to eliminate the differences of unit and that elements of X which do not affect $\sigma^2(X)$ might be excluded. Using the notation above, we can express Robinson's estimate as

$$(2.1) \quad \hat{\sigma}^2(X_i) = \sum_{j=1}^k c_j \hat{U}_{i(j)}^2 \quad (i=1, 2, \dots, n).$$

If there are tied samples, that is, two or more samples happen to have the same value, the weights are equally divided. In this paper no such cases are considered to make the arguments clear. As mentioned in the previous section, $\hat{U}_{i(1)}^2, \hat{U}_{i(2)}^2, \dots, \hat{U}_{i(k)}^2$ do not contain $\hat{U}_i^2$ from the definition. This is the very point we want to improve and then the modified estimator is represented as

$$(2.2) \quad \hat{\delta}^2(X_i) = c_1 \hat{U}_i^2 + \sum_{j=2}^k c_j \hat{U}_{i(j-1)}^2 \quad (i=1, 2, \dots, n).$$

From (2.1) and (2.2) we can see the relation

$$\hat{\delta}^2(X_i) - \hat{\sigma}^2(X_i) = c_1 \hat{U}_i^2 + \sum_{j=1}^{k-1} (c_{j+1} - c_j) \hat{U}_{i(j)}^2 - c_k \hat{U}_{i(k)}^2,$$

and if $\{c_i\}$ is the uniform weight, i.e., $c_i = 1/k (i=1, 2, \dots, k)$, then

$$(2.3) \quad \hat{\delta}^2(X_i) - \hat{\sigma}^2(X_i) = (\hat{U}_i^2 - \hat{U}_{i(k)}^2)/k.$$

This relation tells us that the difference between the two estimators reflects the difference of residuals. Therefore the effect of the modification is small when the disturbances are fairly small. When we consider the asymptotic theory, we should increase k with increase of the sample size. To note this equation (2.3) converges to zero in probability as the sample size increases under the proper moment conditions. Hence the modified estimator is expect-

ed to be asymptotically equivalent to Robinson's.

Now we will write the formal definition of the modified weight and the corresponding estimators in accordance with Robinson. First we calculate

$$p_{ij} = 1 + \sum_{\substack{l=1 \\ l \neq i, j}}^n I(\rho_{il} < \rho_{ij}) \quad (i \neq j),$$

where $I(\cdot)$ represents the indicator function and

$$\rho_{ij} = \|X_i - X_j\|^2 \quad (i, j = 1, 2, \dots, n, i \neq j).$$

The scalar p_{ij} indicates that X_j is the p_{ij} -th closest to X_i . These index variables enable us to express Robinson's weights as

$$W_{ij} = \begin{cases} c_{p_{ij}} & (i \neq j) \\ 0 & (i = j) \end{cases} \quad (i, j = 1, 2, \dots, n)$$

and the modified ones as

$$W'_{ij} = \begin{cases} c_{p_{ij}+1} & (i \neq j) \\ c_1 & (i = j) \end{cases} \quad (i, j = 1, 2, \dots, n),$$

where $c_i (i = 1, 2, \dots, k)$ are the same constants used in (2.1), (2.2) and $c_i = 0$ for $i = k+1, \dots, n$. Using these weights, the estimators of $\sigma^2(X_i)$ and β are

$$\begin{aligned} \hat{\sigma}^2(X) &= \sum_{j=1}^n W'_{ij} \hat{U}_j^2 \quad (i = 1, 2, \dots, n) \\ \hat{\beta}_M &= \left(\sum_{i=1}^n X_i X_i' \hat{\sigma}^{-2}(X_i) \right)^{-1} \left(\sum_{i=1}^n X_i Y_i \hat{\sigma}^{-2}(X_i) \right). \end{aligned}$$

3. Asymptotic efficiency and consistency

The following are necessary assumptions to prove the asymptotic efficiency and the consistency of the modified estimator.

A1: $E(Y|X) = \beta'X$, a.s.

A2: $P(\text{Var}(U|X) \geq \varepsilon) = 1$, for some $\varepsilon > 0$

A3: $E(XX' / \text{Var}(U|X))$ is positive definite

A4: $E\|X\|^{2\nu/(\nu-2)} < \infty$

A5: $E|Y - E(Y|X)|^\nu = E|U|^\nu < \infty$

A6: $\lim_{n \rightarrow \infty} \max_{1 \leq i \leq k} k c_i < \infty$

A7: $k/n \rightarrow 0 (n \rightarrow \infty)$, $n^{4/\nu}/k \rightarrow 0 (n \rightarrow \infty)$,

where ν is some constant satisfying $4 < \nu$.

In fact, these assumptions are just the same as those of Robinson.

THEOREM: Under assumptions A1-A7, as $n \rightarrow \infty$

$$\sqrt{n}(\hat{\beta}_M - \beta) = \sqrt{n}(\hat{\beta}_G - \beta) + o_p(1) \rightarrow N(O, \Omega), \Omega = \{E(XX' \sigma^{-2}(X))\}^{-1}$$

and the covariance matrix Ω is consistently estimated by

$$\hat{\Omega}_M = \left(n^{-1} \sum_{i=1}^n X_i X_i' \hat{\sigma}_i^{-2} \right)^{-1}.$$

Robinson (1987) has completed his proof to show 6 propositions and 14 lemmas. As mentioned earlier, "sample splitting" in Robinson's estimation is for a technical reason. This means that with this property some parts in the proof become very simple. Of all the propositions and lemmas in Robinson (1987), only proposition 2 and lemma 1 make use of the property $W_{ii}=0$ in the proof. Therefore for the rest of them we can prove the corresponding results in the same way as Robinson; all we have to make is a slight change of the notation. Lemma 1 in Robinson (1987) is derived by applying proposition 1 in Stone (1977) and this causes the dependence on $W_{ii}=0$, while proposition 2 uses in the proof the property that $\hat{\sigma}_i$ is independent of X_i . Therefore we have to show the modified version of lemma 1 and proposition 2, which we henceforth denote lemma 1* and proposition 2*, and this completes the proof.

LEMMA 1*: Let $f(\cdot)$ be a Borel function such that $E|f(X)|^p < \infty$, for some $p \geq 1$, then

$$E\left\{\sum_j |f(X_j) - f(X_1)|^p W_{1j}\right\} \rightarrow 0$$

PROPOSITION 2*: $n^{-1/2} \sum_i X_i U_i (\hat{\sigma}_i^{-2} - \bar{\sigma}_i^{-2}) \xrightarrow{p} 0$, where $\hat{\sigma}_i^2 = \sum_j W_{ij} U_j^2$, $\bar{\sigma}_i^2 = \sum_j W_{ij} \sigma_j^2$ as in Robinson (1987).

4. Monte Carlo experiments

To compare the performance in small samples we have carried out a large number of Monte-Carlo experiments. In particular, we want to see if our modification surely improves the small sample efficiency of Robinson's estimator.

The model here is confined to simple linear regression

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i \sigma(X_i) \quad (i=1, 2, \dots, n),$$

where $\beta_0=5$ and $\beta_1=20$. The distribution of ε_i is independent standard normal. As regressor variables X_i , which are independently distributed of ε_i , we consider two kinds of distributions,

Model 1: $X_i \sim U(-0.5, 0.5)$

Model 2: $X_i \sim N(0, 0.25)$

For either model the following two functional forms of the conditional variances are considered.

(i) $\sigma^2(x) = \theta \exp(\gamma z^2)$, $z = 5 + 20x$

(ii) $\sigma^2(x) = \theta(1 + \gamma z^2)$, $z = 5 + 20x$

In both cases, we can get various levels of heteroskedasticity by changing the parameter γ and we use θ to normalize the expected variance in each setting to unity. In the above setting we try five types of estimation, i.e.,

- (a) ordinary least squares method (OLS)
- (b) kernel type method
- (c) Robinson's k -nearest neighbor method
- (d) modified k -nearest neighbor method

(e) generalized least squares method (GLS).

In using (b) we employ a normal kernel function (K1) and

$$K(z) = \begin{cases} 0.875(1 - |z|^{4/3}) & |z| \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

which is henceforth called K2.

In using (c) and (d) three types of weight functions are considered.

uniform weights (UNI) $c_i = 1/k$
 triangular weights (TRI) $c_i \propto k - i + 1, (i = 1, 2, \dots, k)$
 quadratic weights (QUD) $c_i \propto k^2 - (i - 1)^2, (i = 1, 2, \dots, k)$

A difficult problem in using the kernel method is the decision of proper bandwidth. As is well known, the bandwidth selection is a much more serious matter than the kernel selection. A very similar issue in the nearest neighbor method is the selection of k , the number of neighbors. The bandwidth plays almost the same role in the kernel method as k/n in the nearest neighbor method, i.e., to select the bandwidth or the nearest neighbor's k means to determine how many observations to use for local estimation. (In the kernel type method the average size is determined). The larger the value of the bandwidth is, the more observations we use in estimating the variance. Therefore to make a fair competition, we should give both parameters proper values. But the optimal selection of the bandwidth or the nearest neighbor's k is still very difficult. In the kernel estimation the cross-validation procedure is shown to have a desirable property in large samples. A similar procedure has been considered in a nonparametric nearest neighbor setting (See Li (1984)), but no corresponding result has been reported from the semiparametric setting we consider here. Further, this data-driven procedure has an inclination to behave poorly in small samples. In our experiments, therefore, we consider as many different values as possible for both parameters and select some appropriate values among them so that competitions are fair.

The outcomes of the experiments are given in Table 1 through Table 4 in Appendix B. All the figures in each table are mean squared errors of each estimator divided by those of GLS. Our intuition says that when heteroskedasticity is very small (γ is nearly 0), the OLS estimator does attain a sufficient level of efficiency. Our simulation result confirms it. Each semiparametric estimator, however, also shows a fairly good level of efficiency. It is when the degree of heteroskedasticity is fairly high that we realize the superiority of semiparametric methods. In each table semiparametric estimators show high levels of efficiency compared with the OLS estimator when γ is large. Concerning the most interesting question, that is, whether the k -nearest neighbor estimator is improved enough to use in practice, the result shows that the modification make a great deal of improvement. Further, after the modification the k -nearest neighbor estimator is nearly as efficient as the kernel type for small samples. In the k -nearest neighbor method we get little difference in results among three weight functions. The same result is true about the selection of the kernel function.

Appendix A

Proof of Theorem

To make the succeeding proof easy we define

$$w'_{ij} = \begin{cases} W'_{ij} & (i \neq j) \\ 0 & (i = j) \end{cases} \quad (i, j = 1, 2, \dots, n).$$

From the definition of W_{ij} and W'_{ij} we know that $W_{ij} = c_l$ implies $w'_{ij} = c_{l+1}$ for $i \neq j$ and $W_{ii} = w'_{ii} = 0$. This means that w'_{ij} makes the same permutation as Robinson's weight W_{ij} . Note that among $w_{ij} (j = 1, 2, \dots, n)$ there are only $k-1$ positive values for any i because c_1 is not used as a weight. If we define $w_{ij} = w'_{ij}/(1 - c_1)$, then this new weight is essentially the same as W_{ij} using $k-1$ residuals in the estimation. Therefore we write

$$(a.1) \quad \hat{\sigma}_{i,k-1}^2 = \sum_j w_{ij} \hat{U}_j^2, \quad \tilde{\sigma}_{i,k-1}^2 = \sum_j w_{ij} U_j^2, \quad \bar{\sigma}_{i,k-1}^2 = \sum_j w_{ij} \sigma_j^2,$$

which we consider as the same estimators as Robinson's. Hence these estimators satisfy a series of propositions and lemmas in Robinson (1987).

PROOF of LEMMA 1*: First we use the relation

$$\begin{aligned} E \left\{ \sum_{j=1}^n |f(X_j) - f(X_1)|^p W_{1j} \right\} &= E \left\{ \sum_{j=2}^n |f(X_j) - f(X_1)|^p w'_{1j} \right\} + E \{ |f(X_1) - f(X_1)|^p c_1 \} \\ &= (1 - c_1) E \left\{ \sum_j |f(X_j) - f(X_1)|^p w_{1j} \right\}. \end{aligned}$$

From A6 we have $c_1 = O(k^{-1})$ and

$$E \left\{ \sum_j |f(X_j) - f(X_1)|^p w_{1j} \right\} \rightarrow 0$$

is essentially the same as lemma 1 in Robinson (1987).

PROOF of PROPOSITION 2*:

We can write

$$\begin{aligned} n^{-1/2} \sum_i X_i U_i (\tilde{\delta}_i^{-2} - \bar{\delta}_i^{-2}) &= n^{-1/2} \sum_i X_i U_i (\tilde{\delta}_i^{-2} - \tilde{\phi}_i^{-2}) + n^{-1/2} \sum_i X_i U_i (\tilde{\phi}_i^{-2} - \bar{\phi}_i^{-2}) \\ &\quad + n^{-1/2} \sum_i X_i U_i (\bar{\phi}_i^{-2} - \bar{\delta}_i^{-2}), \end{aligned}$$

where $\tilde{\phi}_i^2 = \sum_j w'_{ij} U_j^2$, $\bar{\phi}_i^2 = \sum_j w'_{ij} \sigma_j^2$.

Then we shall see the convergence of each term;

$$(a.2) \quad \begin{aligned} n^{-1/2} \sum_i X_i U_i (\tilde{\delta}_i^{-2} - \tilde{\phi}_i^{-2}) &\xrightarrow{p} 0 & n^{-1/2} \sum_i X_i U_i (\tilde{\phi}_i^{-2} - \bar{\phi}_i^{-2}) &\xrightarrow{p} 0 \\ n^{-1/2} \sum_i X_i U_i (\bar{\phi}_i^{-2} - \bar{\delta}_i^{-2}) &\xrightarrow{p} 0. \end{aligned}$$

To prove the first part of (a.2), we first show

$$(a.3) \quad \max_i U_i^2 = \max\{U_1^2, U_2^2, \dots, U_n^2\} = O_p(n^\gamma) \quad \frac{2}{\nu} < \gamma \leq \frac{4}{\nu}.$$

By Bonferroni, Marcov inequality and A5,

$$P(\max_i U_i^2 > Cn^r) \leq nP(U_1^2 > Cn^r) \leq n(Cn^r)^{-\nu/2} E|U_1|^\nu = O(n^{\frac{\nu}{2}(\frac{2}{\nu}-r)}),$$

where $C > 0$ is some constant. We can easily show (a.3) by using A5 and the Marcov inequality.

We note that

$$\tilde{\phi}_i^2 - \tilde{\delta}_i^2 = \sum_j (w'_{ij} - W'_{ij}) U_j^2 = -c_1 U_i^2.$$

Then we have

$$\begin{aligned} n^{-1/2} \sum_i X_i U_i (\tilde{\delta}_i^{-2} - \tilde{\phi}_i^{-2}) &= n^{-1/2} \sum_i X_i U_i (\tilde{\phi}_i^2 - \tilde{\delta}_i^2) / (\tilde{\phi}_i^2 \tilde{\delta}_i^2) \\ &= -c_1 n^{-1/2} \sum_i X_i U_i U_i^2 / (\tilde{\phi}_i^2 \tilde{\delta}_i^2), \end{aligned}$$

and using this

$$\begin{aligned} \|n^{-1/2} \sum_i X_i U_i (\tilde{\delta}_i^{-2} - \tilde{\phi}_i^{-2})\| &= c_1 n^{-1/2} \|\sum_i X_i U_i U_i^2 (\tilde{\phi}_i^2 \tilde{\delta}_i^2)^{-1}\| \\ &\leq (\min_i \tilde{\phi}_i^2)^{-1} (\min_i \tilde{\delta}_i^2)^{-1} (\max_i U_i^2) c_1 \|n^{-1/2} \sum_i X_i U_i\|. \end{aligned}$$

$(\min_i \tilde{\phi}_i^2)^{-1} = O_p(1)$ and $(\min_i \tilde{\delta}_i^2)^{-1} = O_p(1)$ can be shown in the same way as lemma 13 in Robinson (1987) and $\|n^{-1/2} \sum_i X_i U_i\| = O_p(1)$ is a usual result. From (a.3) and A6

$$c_1 \max_i U_i^2 = O_p(k^{-1} n^r)$$

and

$$k^{-1} n^r = (n^{4/\nu} k^{-1}) (n^{r-4/\nu}) \rightarrow 0$$

by A7 and (a.3). We thus proved the first part of (a.2).

Next, we see the convergence of the second part.

From the definition (a.1) we have

$$n^{-1/2} \sum_i X_i U_i (\tilde{\phi}_i^{-2} - \bar{\phi}_i^{-2}) = (1 - c_1)^{-1} n^{-1/2} \sum_i X_i U_i (\tilde{\delta}_{i,k-1}^{-2} - \bar{\delta}_{i,k-1}^{-2}).$$

The convergence of $1 - c_1 \rightarrow 1$ holds from A6. As seen earlier,

$$n^{-1/2} \sum_i X_i U_i (\tilde{\delta}_{i,k-1}^{-2} - \bar{\delta}_{i,k-1}^{-2}) \xrightarrow{p} 0$$

is considered to be equivalent to proposition 2 in Robinson (1987) and so has already been proved by him.

Last, we show the third part of (a.2).

As in the proof of the first term

$$\|n^{-1/2}\sum_i X_i U_i(\bar{\psi}_i^{-2} - \bar{\delta}_i^{-2})\| \leq (\min_i \bar{\psi}_i^2)^{-1} (\min_i \bar{\delta}_i^2)^{-1} c_1 (\max_i \sigma_i^2) \|n^{-1/2}\sum_i X_i U_i\|.$$

$(\min_i \bar{\psi}_i^2)^{-1} = O_p(1)$ is easily deduced as Robinson's lemma 12 and

$$\max_i \sigma_i^2 = O_p(n^\gamma)$$

can be also shown as (a.3) by the Liapounov inequality. The rest of the proof is the same as the proof of the first part of (a.2). This completes the proof.

Appendix B.

Simulation Results

In Table 1-Table 4 the following abbreviations are used.

RUNI : Robinson's k-NN estimator with UNIFORM weights

RTRI : Robinson's k-NN estimator with TRIangular weights

RQUD : Robinson's k-NN estimator with QUDratic weights

MUNI : Modified k-NN estimator with UNIFORM weights

MTRI : Modified k-NN estimator with TRIangular weights

MQUD: Modified k-NN estimator with QUDratic weights

K1 : Kernel estimator with K1 kernel function (normal kernel)

K2 : Kernal estimator with K2 kernel function

Table 1. $X \sim U(-0.5, 0.5)$ $\sigma^2(x) = \theta(1 + \gamma z^2)$ $z = \beta_0 + \beta_1 x$

sz=30	$\hat{\beta}_0$						$\hat{\beta}_1$					
γ	0.00	0.01	0.20	0.50	1.00	2.00	0.00	0.10	0.20	0.50	1.00	2.00
θ	1.00	0.15	0.08	0.03	0.02	0.01	1.00	0.15	0.08	0.03	0.02	0.01
OLS	1.00	1.21	1.33	1.50	1.61	1.72	1.00	1.54	1.72	1.92	2.05	2.16
RUNI												
k=5	1.69	1.72	1.76	1.84	1.94	2.04	1.42	1.59	1.71	1.90	2.05	2.17
k=9	1.22	1.31	1.36	1.46	1.55	1.63	1.15	1.29	1.39	1.56	1.69	1.80
k=13	1.08	1.17	1.22	1.31	1.39	1.46	1.09	1.23	1.30	1.41	1.50	1.58
k=17	1.06	1.23	1.31	1.42	1.51	1.59	1.04	1.30	1.37	1.48	1.56	1.63
RTRI												
k=5	1.70	1.71	1.75	1.83	1.92	2.02	1.44	1.59	1.71	1.90	2.04	2.15
k=9	1.25	1.33	1.38	1.46	1.54	1.63	1.16	1.29	1.38	1.55	1.68	1.80
k=13	1.12	1.19	1.23	1.31	1.38	1.44	1.11	1.23	1.30	1.43	1.53	1.61
k=17	1.07	1.18	1.23	1.31	1.38	1.44	1.06	1.22	1.28	1.37	1.44	1.50
RQUD												
k=5	1.74	1.75	1.78	1.86	1.95	2.05	1.49	1.64	1.75	1.94	2.07	2.19
k=9	1.27	1.33	1.38	1.46	1.54	1.62	1.18	1.29	1.39	1.56	1.68	1.80
k=13	1.14	1.20	1.24	1.31	1.37	1.44	1.12	1.23	1.31	1.44	1.53	1.62
k=17	1.09	1.17	1.21	1.29	1.35	1.41	1.08	1.21	1.27	1.36	1.43	1.50
MUNI												
k=5	1.15	1.16	1.19	1.24	1.28	1.33	1.15	1.25	1.28	1.34	1.38	1.41
k=9	1.07	1.12	1.15	1.21	1.26	1.30	1.05	1.14	1.19	1.26	1.31	1.36
k=13	1.02	1.07	1.11	1.18	1.23	1.29	1.03	1.14	1.19	1.26	1.31	1.36
k=17	1.01	1.10	1.15	1.24	1.30	1.37	1.01	1.17	1.22	1.30	1.36	1.41

Table 1 (continued).

MTRI												
k=5	1.14	1.15	1.18	1.22	1.26	1.31	1.15	1.25	1.28	1.33	1.37	1.40
k=9	1.07	1.10	1.13	1.18	1.23	1.27	1.07	1.16	1.20	1.26	1.30	1.34
k=13	1.03	1.07	1.10	1.16	1.20	1.25	1.04	1.13	1.18	1.24	1.30	1.34
k=17	1.01	1.07	1.10	1.17	1.22	1.28	1.02	1.13	1.18	1.25	1.30	1.35
MQUD												
k=5	1.13	1.15	1.18	1.23	1.27	1.31	1.15	1.26	1.30	1.35	1.39	1.42
k=9	1.07	1.10	1.13	1.18	1.22	1.26	1.08	1.17	1.21	1.26	1.31	1.35
k=13	1.04	1.07	1.10	1.15	1.20	1.25	1.05	1.14	1.18	1.25	1.30	1.34
k=17	1.02	1.06	1.10	1.16	1.21	1.26	1.03	1.13	1.18	1.24	1.29	1.34
K1												
bw=0.10	1.06	1.06	1.09	1.14	1.18	1.23	1.08	1.15	1.18	1.23	1.28	1.32
bw=0.20	1.02	1.07	1.12	1.20	1.28	1.34	1.04	1.14	1.20	1.28	1.34	1.39
bw=0.30	1.00	1.12	1.21	1.33	1.43	1.52	1.02	1.18	1.25	1.34	1.41	1.47
bw=0.40	1.00	1.16	1.27	1.41	1.52	1.61	1.01	1.24	1.33	1.44	1.52	1.59
K2												
bw=0.10	1.12	1.16	1.19	1.24	1.29	1.33	1.16	1.28	1.31	1.36	1.40	1.43
bw=0.20	1.08	1.08	1.10	1.15	1.20	1.24	1.09	1.16	1.19	1.24	1.28	1.32
bw=0.30	1.05	1.06	1.09	1.14	1.20	1.25	1.07	1.14	1.18	1.24	1.29	1.33
bw=0.40	1.03	1.06	1.10	1.18	1.24	1.30	1.04	1.14	1.19	1.27	1.32	1.38
sz=60												
	$\hat{\beta}_0$						$\hat{\beta}_1$					
γ	0.00	0.10	0.20	0.50	1.00	2.00	0.00	0.10	0.20	0.50	1.00	2.00
θ	1.00	0.15	0.08	0.03	0.02	0.01	1.00	0.15	0.08	0.03	0.02	0.01
OLS	1.00	1.29	1.42	1.61	1.74	1.85	1.00	1.57	1.76	1.99	2.14	2.25
RUNI												
k=8	1.35	1.41	1.42	1.45	1.48	1.53	1.33	1.41	1.46	1.52	1.58	1.63
k=13	1.15	1.20	1.22	1.26	1.31	1.35	1.13	1.24	1.31	1.43	1.52	1.60
k=18	1.06	1.12	1.15	1.20	1.25	1.30	1.07	1.21	1.29	1.44	1.55	1.64
k=23	1.04	1.10	1.13	1.18	1.23	1.29	1.06	1.20	1.28	1.41	1.51	1.59
RTRI												
k=8	1.40	1.49	1.50	1.53	1.56	1.59	1.40	1.47	1.51	1.57	1.62	1.67
k=13	1.15	1.20	1.22	1.26	1.29	1.33	1.16	1.24	1.29	1.39	1.45	1.51
k=18	1.09	1.14	1.16	1.19	1.23	1.28	1.10	1.21	1.28	1.40	1.49	1.56
k=23	1.06	1.10	1.12	1.16	1.20	1.25	1.07	1.19	1.26	1.39	1.48	1.56
RQUD												
k=8	1.45	1.54	1.56	1.58	1.61	1.64	1.44	1.51	1.55	1.62	1.66	1.70
k=13	1.17	1.22	1.24	1.27	1.30	1.34	1.18	1.26	1.31	1.39	1.45	1.50
k=18	1.10	1.15	1.17	1.20	1.23	1.27	1.12	1.21	1.28	1.38	1.46	1.53
k=23	1.07	1.11	1.12	1.16	1.20	1.24	1.08	1.19	1.26	1.37	1.46	1.53
MUNI												
k=8	1.12	1.17	1.19	1.22	1.24	1.26	1.14	1.17	1.19	1.22	1.25	1.28
k=13	1.06	1.11	1.13	1.16	1.19	1.22	1.04	1.11	1.15	1.20	1.24	1.27
k=18	1.02	1.08	1.10	1.14	1.17	1.21	1.03	1.12	1.18	1.25	1.31	1.36
k=23	1.02	1.08	1.10	1.15	1.19	1.24	1.02	1.14	1.20	1.29	1.36	1.41
MTRI												
k=8	1.12	1.17	1.19	1.22	1.24	1.26	1.13	1.17	1.20	1.23	1.26	1.28
k=13	1.06	1.11	1.13	1.16	1.18	1.20	1.06	1.11	1.13	1.17	1.20	1.23
k=18	1.04	1.08	1.10	1.13	1.16	1.19	1.03	1.11	1.14	1.20	1.24	1.28
k=23	1.02	1.07	1.09	1.13	1.16	1.20	1.02	1.12	1.16	1.24	1.29	1.34

Table 1 (continued).

MQUD												
k=8	1.12	1.18	1.20	1.23	1.25	1.27	1.12	1.18	1.21	1.25	1.28	1.30
k=13	1.07	1.12	1.14	1.16	1.19	1.21	1.07	1.12	1.14	1.18	1.20	1.23
k=18	1.04	1.09	1.11	1.14	1.16	1.19	1.04	1.11	1.14	1.19	1.22	1.26
k=23	1.03	1.07	1.09	1.12	1.16	1.19	1.03	1.11	1.15	1.21	1.26	1.30
K1												
bw=0.10	1.03	1.08	1.10	1.13	1.17	1.21	1.04	1.10	1.13	1.18	1.22	1.26
bw=0.20	1.01	1.12	1.18	1.27	1.34	1.41	1.02	1.14	1.20	1.30	1.36	1.42
bw=0.30	1.00	1.20	1.31	1.45	1.56	1.65	1.01	1.19	1.27	1.38	1.46	1.52
bw=0.40	1.00	1.25	1.37	1.54	1.66	1.77	1.01	1.25	1.35	1.49	1.57	1.65
K2												
bw=0.10	1.10	1.17	1.19	1.22	1.24	1.27	1.10	1.16	1.18	1.22	1.24	1.27
bw=0.20	1.04	1.08	1.10	1.13	1.16	1.19	1.05	1.10	1.13	1.17	1.21	1.24
bw=0.40	1.02	1.10	1.15	1.23	1.29	1.35	1.02	1.13	1.19	1.28	1.35	1.41

Table 2. $X \sim U(-0.5, 0.5)$ $\sigma^2(x) = \theta \exp(\gamma x^2)$ $z = \beta_0 + \beta_1 x$

sz=30	$\hat{\beta}_0$					$\hat{\beta}_1$				
γ	0.00	0.01	0.01	0.02	0.02	0.00	0.01	0.01	0.02	0.02
θ	1.00	0.43	0.30	0.20	0.13	1.00	0.43	0.30	0.20	0.13
OLS	1.00	1.11	1.28	1.57	2.07	1.00	1.42	1.82	2.46	3.50
RUNI										
k=5	1.69	1.69	1.71	1.74	1.79	1.42	1.50	1.55	1.61	1.69
k=9	1.22	1.27	1.34	1.44	1.57	1.15	1.25	1.32	1.41	1.52
k=13	1.08	1.17	1.29	1.47	1.71	1.09	1.26	1.38	1.55	1.77
k=17	1.06	1.23	1.43	1.75	2.22	1.04	1.33	1.56	1.89	2.35
RTRI										
k=5	1.70	1.70	1.72	1.75	1.79	1.44	1.51	1.56	1.63	1.71
k=9	1.25	1.29	1.34	1.41	1.50	1.16	1.24	1.29	1.36	1.45
k=13	1.12	1.18	1.26	1.38	1.53	1.11	1.24	1.32	1.42	1.56
k=17	1.07	1.18	1.31	1.51	1.78	1.06	1.25	1.39	1.58	1.83
RQUD										
k=5	1.74	1.73	1.75	1.78	1.82	1.49	1.57	1.62	1.68	1.76
k=9	1.27	1.30	1.34	1.40	1.48	1.18	1.25	1.30	1.36	1.44
k=13	1.14	1.19	1.26	1.35	1.48	1.12	1.23	1.30	1.39	1.51
k=17	1.09	1.17	1.28	1.43	1.64	1.08	1.24	1.35	1.50	1.69
MUNI										
k=5	1.15	1.14	1.17	1.21	1.27	1.15	1.23	1.29	1.37	1.46
k=9	1.07	1.09	1.12	1.17	1.23	1.05	1.13	1.18	1.24	1.32
k=13	1.02	1.05	1.11	1.19	1.28	1.03	1.14	1.22	1.30	1.41
k=17	1.01	1.07	1.16	1.30	1.48	1.01	1.17	1.29	1.44	1.63
MTRI										
k=5	1.14	1.13	1.15	1.20	1.27	1.15	1.23	1.30	1.38	1.50
k=9	1.07	1.08	1.10	1.14	1.19	1.07	1.14	1.19	1.25	1.32
k=13	1.03	1.05	1.09	1.15	1.21	1.04	1.13	1.19	1.25	1.33
k=17	1.01	1.05	1.10	1.18	1.28	1.02	1.14	1.21	1.31	1.42
MQUD										
k=5	1.13	1.12	1.15	1.20	1.28	1.15	1.25	1.32	1.43	1.57
k=9	1.07	1.07	1.10	1.14	1.19	1.08	1.16	1.21	1.28	1.36
k=13	1.04	1.05	1.09	1.13	1.19	1.05	1.13	1.19	1.26	1.34
k=17	1.02	1.04	1.09	1.15	1.23	1.03	1.13	1.20	1.28	1.38

Table 2 (continued).

K1										
bw=0.10	1.06	1.05	1.06	1.09	1.13	1.08	1.15	1.18	1.24	1.30
bw=0.20	1.02	1.03	1.07	1.15	1.26	1.04	1.13	1.19	1.29	1.41
bw=0.30	1.00	1.04	1.13	1.29	1.54	1.02	1.15	1.26	1.42	1.66
bw=0.40	1.00	1.07	1.19	1.41	1.77	1.01	1.20	1.36	1.61	1.99
K2										
bw=0.10	1.12	1.13	1.15	1.20	1.28	1.16	1.26	1.34	1.44	1.58
bw=0.20	1.08	1.06	1.07	1.10	1.13	1.09	1.15	1.19	1.24	1.31
bw=0.30	1.05	1.04	1.06	1.10	1.14	1.07	1.13	1.18	1.23	1.30
bw=0.40	1.03	1.03	1.07	1.13	1.22	1.04	1.13	1.18	1.26	1.36
sz=60	$\hat{\beta}_0$					$\hat{\beta}_1$				
γ	0.00	0.01	0.01	0.02	0.02	0.00	0.01	0.01	0.02	0.02
θ	1.00	0.43	0.30	0.20	0.13	1.00	0.43	0.30	0.20	0.13
OLS	1.00	1.17	1.35	1.66	2.17	1.00	1.38	1.77	2.40	3.43
RUNI										
k=8	1.35	1.37	1.39	1.41	1.42	1.33	1.35	1.37	1.39	1.41
k=13	1.15	1.17	1.20	1.23	1.26	1.13	1.17	1.19	1.22	1.24
k=18	1.06	1.11	1.15	1.20	1.25	1.07	1.13	1.17	1.21	1.24
k=23	1.04	1.11	1.18	1.27	1.37	1.06	1.14	1.21	1.29	1.37
RTRI										
k=8	1.40	1.43	1.45	1.46	1.47	1.40	1.40	1.42	1.44	1.46
k=13	1.15	1.18	1.20	1.22	1.24	1.16	1.19	1.20	1.22	1.24
k=18	1.09	1.12	1.15	1.17	1.20	1.10	1.14	1.16	1.18	1.20
k=23	1.06	1.10	1.14	1.18	1.23	1.07	1.12	1.16	1.20	1.24
RQUD										
k=8	1.45	1.48	1.49	1.50	1.51	1.44	1.44	1.46	1.47	1.50
k=13	1.17	1.19	1.21	1.23	1.26	1.18	1.20	1.22	1.24	1.25
k=18	1.10	1.13	1.15	1.17	1.20	1.12	1.15	1.17	1.19	1.21
k=23	1.07	1.10	1.13	1.17	1.20	1.08	1.13	1.16	1.19	1.22
MUNI										
k=8	1.12	1.14	1.16	1.18	1.22	1.14	1.14	1.16	1.19	1.23
k=13	1.06	1.09	1.11	1.13	1.16	1.04	1.07	1.10	1.12	1.15
k=18	1.02	1.06	1.09	1.12	1.15	1.03	1.07	1.10	1.13	1.16
k=23	1.02	1.07	1.11	1.17	1.23	1.02	1.09	1.13	1.19	1.24
MTRI										
k=8	1.12	1.14	1.16	1.19	1.22	1.13	1.14	1.16	1.21	1.26
k=13	1.06	1.08	1.10	1.13	1.15	1.06	1.08	1.10	1.12	1.16
k=18	1.04	1.06	1.08	1.11	1.13	1.03	1.06	1.09	1.11	1.14
k=23	1.02	1.06	1.08	1.11	1.15	1.02	1.07	1.10	1.13	1.17
MQUD										
k=8	1.12	1.14	1.16	1.20	1.24	1.12	1.14	1.18	1.24	1.32
k=13	1.07	1.09	1.11	1.13	1.17	1.07	1.09	1.11	1.14	1.18
k=18	1.04	1.07	1.09	1.11	1.13	1.04	1.07	1.09	1.12	1.15
k=23	1.03	1.06	1.08	1.11	1.14	1.03	1.06	1.09	1.12	1.16
K1										
bw=0.10	1.03	1.06	1.07	1.09	1.12	1.04	1.06	1.08	1.10	1.13
bw=0.20	1.01	1.07	1.13	1.22	1.34	1.02	1.08	1.13	1.21	1.31
bw=0.30	1.00	1.11	1.23	1.41	1.69	1.01	1.12	1.21	1.36	1.56
bw=0.40	1.00	1.14	1.29	1.54	1.93	1.01	1.16	1.31	1.54	1.89

Table 2 (continued).

K2										
bw=0.10	1.10	1.13	1.14	1.17	1.20	1.10	1.11	1.14	1.18	1.23
bw=0.20	1.04	1.06	1.08	1.09	1.11	1.05	1.06	1.08	1.10	1.12
bw=0.30	1.02	1.06	1.09	1.12	1.16	1.04	1.06	1.09	1.12	1.16
bw=0.40	1.02	1.07	1.12	1.19	1.28	1.02	1.08	1.12	1.19	1.27

Table 3. $X \sim N(0, 0.25)$ $\sigma^2(x) = \theta(1 + \gamma x^2)$ $z = \beta_0 + \beta_1 x$

sz=30	$\hat{\beta}_0$					$\hat{\beta}_1$				
γ	0.00	0.05	0.20	0.60	1.00	0.00	0.05	0.20	0.60	1.00
θ	1.00	0.29	0.09	0.03	0.02	1.00	0.29	0.09	0.03	0.02
OLS	1.00	1.21	1.42	1.57	1.62	1.00	1.54	2.12	2.57	2.76
RUNI										
k=5	1.65	1.71	1.78	1.88	1.93	1.43	1.53	1.74	2.05	2.21
k=9	1.18	1.26	1.34	1.44	1.50	1.13	1.32	1.62	1.97	2.13
k=13	1.09	1.20	1.29	1.37	1.41	1.06	1.27	1.54	1.83	1.95
k=17	1.07	1.21	1.35	1.45	1.49	1.01	1.30	1.60	1.88	2.00
RTRI										
k=5	1.72	1.77	1.83	1.91	1.96	1.46	1.56	1.76	2.05	2.20
k=9	1.25	1.33	1.40	1.50	1.56	1.13	1.31	1.59	1.94	2.10
k=13	1.11	1.21	1.28	1.35	1.39	1.07	1.25	1.52	1.82	1.96
k=17	1.07	1.19	1.28	1.36	1.39	1.04	1.25	1.52	1.80	1.92
RQUD										
k=5	1.79	1.82	1.87	1.95	1.99	1.49	1.58	1.78	2.06	2.20
k=9	1.29	1.36	1.43	1.53	1.58	1.14	1.31	1.59	1.92	2.08
k=13	1.13	1.23	1.29	1.37	1.41	1.08	1.25	1.51	1.82	1.96
k=17	1.09	1.20	1.28	1.35	1.38	1.05	1.24	1.50	1.78	1.90
MUNI										
k=5	1.15	1.24	1.27	1.31	1.32	1.14	1.29	1.42	1.55	1.61
k=9	1.09	1.17	1.22	1.26	1.29	1.06	1.23	1.40	1.57	1.64
k=13	1.03	1.14	1.20	1.25	1.28	1.04	1.22	1.40	1.59	1.67
k=17	1.04	1.16	1.25	1.33	1.36	1.02	1.24	1.47	1.68	1.77
MTRI										
k=5	1.14	1.22	1.26	1.30	1.32	1.14	1.29	1.42	1.55	1.61
k=9	1.09	1.17	1.21	1.25	1.27	1.06	1.23	1.38	1.52	1.59
k=13	1.04	1.13	1.18	1.23	1.25	1.04	1.21	1.38	1.55	1.63
k=17	1.03	1.13	1.19	1.25	1.28	1.03	1.21	1.40	1.58	1.67
MQUD										
k=5	1.13	1.21	1.26	1.30	1.32	1.13	1.30	1.45	1.58	1.64
k=9	1.09	1.17	1.21	1.25	1.27	1.07	1.24	1.38	1.52	1.59
k=13	1.05	1.14	1.19	1.23	1.25	1.05	1.22	1.38	1.54	1.61
k=17	1.03	1.13	1.19	1.24	1.26	1.04	1.21	1.39	1.56	1.64
K1										
bw=0.10	1.07	1.15	1.20	1.26	1.28	1.12	1.25	1.38	1.52	1.59
bw=0.20	1.02	1.10	1.19	1.27	1.31	1.05	1.20	1.37	1.55	1.64
bw=0.30	1.00	1.13	1.26	1.37	1.41	1.02	1.23	1.48	1.71	1.81
bw=0.40	1.00	1.15	1.32	1.44	1.49	1.01	1.29	1.60	1.88	2.00
K2										
bw=0.10	1.14	1.22	1.28	1.34	1.37	1.12	1.35	1.56	1.74	1.82
bw=0.20	1.09	1.17	1.23	1.29	1.32	1.14	1.28	1.42	1.56	1.63
bw=0.30	1.05	1.13	1.20	1.26	1.29	1.11	1.23	1.36	1.50	1.57
bw=0.40	1.02	1.11	1.19	1.26	1.29	1.07	1.20	1.35	1.52	1.60

Table 3 (continued).

sz=60	$\hat{\beta}_0$					$\hat{\beta}_1$				
γ	0.00	0.05	0.02	0.60	1.00	0.00	0.05	0.20	0.60	1.00
θ	1.00	0.29	0.09	0.03	0.02	1.00	0.29	0.09	0.03	0.02
OLS	1.00	1.20	1.42	1.59	1.65	1.00	1.56	2.17	2.65	2.85
RUNI										
k=8	1.44	1.50	1.53	1.61	1.67	1.23	1.40	1.70	2.06	2.23
k=13	1.15	1.23	1.32	1.46	1.54	1.13	1.34	1.76	2.26	2.49
k=18	1.08	1.16	1.24	1.37	1.44	1.07	1.29	1.72	2.24	2.48
k=23	1.04	1.12	1.19	1.30	1.35	1.04	1.29	1.70	2.16	2.37
RTRI										
k=8	1.49	1.54	1.58	1.66	1.71	1.29	1.46	1.74	2.05	2.20
k=13	1.19	1.26	1.33	1.46	1.52	1.13	1.33	1.70	2.13	2.32
k=18	1.11	1.17	1.24	1.37	1.44	1.08	1.29	1.70	2.18	2.40
k=23	1.07	1.13	1.21	1.32	1.39	1.06	1.28	1.69	2.19	2.41
RQUD										
k=8	1.51	1.56	1.61	1.69	1.74	1.34	1.51	1.77	2.06	2.20
k=13	1.22	1.28	1.35	1.46	1.52	1.14	1.34	1.68	2.07	2.25
k=18	1.13	1.19	1.26	1.37	1.44	1.09	1.29	1.67	2.12	2.32
k=23	1.08	1.15	1.21	1.33	1.39	1.06	1.28	1.67	2.14	2.35
MUNI										
k=8	1.18	1.21	1.21	1.23	1.24	1.06	1.18	1.30	1.41	1.46
k=13	1.09	1.14	1.17	1.22	1.24	1.06	1.19	1.36	1.53	1.60
k=18	1.05	1.11	1.15	1.21	1.24	1.04	1.20	1.41	1.64	1.74
k=23	1.03	1.09	1.14	1.20	1.23	1.02	1.20	1.44	1.69	1.80
MTRI										
k=8	1.15	1.19	1.20	1.22	1.23	1.08	1.21	1.33	1.43	1.47
k=13	1.11	1.16	1.18	1.21	1.23	1.04	1.17	1.31	1.45	1.51
k=18	1.07	1.12	1.15	1.19	1.21	1.03	1.17	1.34	1.52	1.60
k=23	1.04	1.10	1.14	1.19	1.21	1.02	1.18	1.38	1.59	1.68
MQUD										
k=8	1.14	1.18	1.20	1.22	1.23	1.08	1.22	1.35	1.45	1.49
k=13	1.11	1.16	1.18	1.21	1.22	1.04	1.18	1.31	1.43	1.49
k=18	1.08	1.13	1.15	1.19	1.21	1.03	1.17	1.32	1.47	1.54
k=23	1.05	1.10	1.14	1.18	1.20	1.02	1.17	1.35	1.53	1.61
K1										
bw=0.10	1.05	1.09	1.13	1.18	1.20	1.04	1.13	1.24	1.36	1.42
bw=0.20	1.01	1.09	1.18	1.27	1.30	1.01	1.13	1.33	1.55	1.64
bw=0.30	1.00	1.13	1.27	1.39	1.43	1.00	1.20	1.48	1.74	1.86
bw=0.40	1.00	1.16	1.33	1.47	1.53	1.00	1.27	1.61	1.91	2.04
K2										
bw=0.10	1.12	1.17	1.22	1.28	1.30	1.12	1.26	1.42	1.56	1.62
bw=0.20	1.06	1.10	1.13	1.18	1.20	1.06	1.15	1.25	1.36	1.42
bw=0.30	1.02	1.08	1.14	1.20	1.23	1.04	1.13	1.26	1.40	1.47
bw=0.40	1.01	1.09	1.17	1.25	1.29	1.03	1.14	1.31	1.51	1.60

Table 4. $X \sim N(0, 0.25)$ $\sigma^2(x) = \theta \exp(\gamma z^2)$ $z = \beta_0 + \beta_1 x$

sz=30	$\hat{\beta}_0$						$\hat{\beta}_1$					
γ	0.00	0.01	0.01	0.01	0.01	0.01	0.00	0.01	0.01	0.01	0.01	0.01
θ	1.00	0.68	0.56	0.43	0.30	0.17	1.00	0.68	0.56	0.43	0.30	0.17
OLS	1.00	1.08	1.17	1.33	1.59	2.05	1.00	1.27	1.60	2.18	3.27	5.32
RUNI												
k=5	1.65	1.70	1.75	1.83	1.93	2.05	1.43	1.50	1.55	1.65	1.86	2.26
k=9	1.18	1.23	1.31	1.42	1.57	1.80	1.13	1.24	1.35	1.53	1.82	2.34
k=13	1.09	1.16	1.24	1.38	1.60	1.95	1.06	1.20	1.35	1.58	1.97	2.65
k=17	1.07	1.14	1.24	1.41	1.71	2.24	1.01	1.20	1.41	1.76	2.35	3.39
RTRI												
k=5	1.72	1.77	1.81	1.89	1.98	2.09	1.46	1.53	1.58	1.68	1.88	2.24
k=9	1.25	1.31	1.37	1.46	1.59	1.76	1.13	1.22	1.32	1.48	1.74	2.20
k=13	1.11	1.18	1.25	1.37	1.54	1.80	1.07	1.18	1.30	1.49	1.81	2.37
k=17	1.07	1.14	1.23	1.37	1.60	1.96	1.04	1.19	1.35	1.60	2.02	2.77
RQUD												
k=5	1.79	1.83	1.88	1.95	2.03	2.12	1.49	1.55	1.60	1.70	1.90	2.25
k=9	1.29	1.34	1.40	1.49	1.60	1.76	1.14	1.23	1.32	1.47	1.72	2.15
k=13	1.13	1.20	1.27	1.38	1.53	1.75	1.08	1.18	1.29	1.46	1.75	2.27
k=17	1.09	1.16	1.23	1.36	1.56	1.86	1.05	1.18	1.32	1.54	1.90	2.56
MUNI												
k=5	1.15	1.21	1.24	1.27	1.31	1.36	1.14	1.24	1.31	1.40	1.51	1.65
k=9	1.09	1.14	1.19	1.24	1.30	1.36	1.06	1.17	1.25	1.35	1.46	1.60
k=13	1.03	1.10	1.15	1.22	1.29	1.39	1.04	1.16	1.25	1.37	1.50	1.68
k=17	1.04	1.10	1.16	1.24	1.35	1.48	1.02	1.17	1.29	1.45	1.64	1.90
MTRI												
k=5	1.14	1.19	1.22	1.26	1.30	1.35	1.14	1.23	1.30	1.40	1.52	1.67
k=9	1.09	1.14	1.18	1.22	1.27	1.32	1.06	1.17	1.25	1.34	1.44	1.57
k=13	1.04	1.10	1.14	1.20	1.26	1.33	1.04	1.16	1.24	1.34	1.45	1.60
k=17	1.03	1.09	1.14	1.20	1.28	1.37	1.03	1.15	1.25	1.37	1.51	1.69
MQUD												
k=5	1.13	1.18	1.21	1.25	1.30	1.36	1.13	1.23	1.31	1.42	1.55	1.73
k=9	1.09	1.14	1.18	1.22	1.26	1.31	1.07	1.18	1.26	1.34	1.45	1.58
k=13	1.05	1.11	1.15	1.20	1.25	1.31	1.05	1.16	1.24	1.34	1.44	1.57
k=17	1.03	1.09	1.14	1.19	1.26	1.33	1.04	1.16	1.24	1.35	1.47	1.62
K1												
bw=0.10	1.07	1.11	1.13	1.14	1.16	1.18	1.12	1.19	1.22	1.27	1.32	1.38
bw=0.20	1.02	1.05	1.07	1.10	1.13	1.17	1.05	1.13	1.17	1.22	1.28	1.34
bw=0.30	1.00	1.05	1.09	1.14	1.20	1.28	1.02	1.13	1.20	1.29	1.40	1.54
bw=0.40	1.00	1.06	1.11	1.19	1.30	1.45	1.01	1.15	1.26	1.41	1.61	1.89
K2												
bw=0.10	1.14	1.19	2.16	1.24	1.28	1.33	1.12	1.24	1.40	1.48	1.62	1.78
bw=0.20	1.09	1.13	2.09	1.16	1.18	1.21	1.14	1.21	1.31	1.30	1.36	1.43
bw=0.30	1.05	1.09	1.11	1.13	1.15	1.17	1.11	1.17	1.21	1.25	1.29	1.34
bw=0.40	1.02	1.06	1.07	1.10	1.12	1.15	1.07	1.14	1.18	1.22	1.26	1.32
sz=60	$\hat{\beta}_0$						$\hat{\beta}_1$					
γ	0.00	0.01	0.01	0.01	0.01	0.01	0.00	0.01	0.01	0.01	0.01	0.01
θ	1.00	0.68	0.56	0.43	0.30	0.17	1.00	0.68	0.56	0.43	0.30	0.17

Table 4 (continued).

OLS	1.00	1.07	1.15	1.30	1.55	2.00	1.00	1.29	1.63	2.27	3.49	5.85
RUNI												
k=8	1.44	1.48	1.51	1.55	1.60	1.63	1.23	1.32	1.38	1.46	1.58	1.80
k=13	1.15	1.20	1.25	1.32	1.41	1.54	1.13	1.24	1.33	1.44	1.62	1.92
k=18	1.08	1.14	1.20	1.29	1.41	1.60	1.07	1.21	1.32	1.49	1.73	2.13
k=23	1.04	1.09	1.16	1.27	1.44	1.69	1.04	1.21	1.35	1.57	1.88	2.39
RTRI												
k=8	1.49	1.53	1.56	1.60	1.64	1.71	1.29	1.37	1.43	1.51	1.61	1.81
k=13	1.19	1.24	1.28	1.34	1.41	1.51	1.13	1.23	1.30	1.39	1.54	1.79
k=18	1.11	1.16	1.21	1.28	1.38	1.51	1.08	1.20	1.29	1.41	1.60	1.92
k=23	1.07	1.12	1.17	1.26	1.38	1.56	1.06	1.19	1.31	1.46	1.70	2.09
RQUD												
k=8	1.51	1.55	1.58	1.61	1.66	1.72	1.34	1.42	1.48	1.55	1.65	1.84
k=13	1.22	1.27	1.31	1.35	1.42	1.51	1.14	1.24	1.30	1.39	1.52	1.76
k=18	1.13	1.18	1.22	1.29	1.37	1.49	1.09	1.20	1.28	1.39	1.55	1.84
k=23	1.08	1.13	1.18	1.26	1.36	1.51	1.06	1.19	1.29	1.42	1.62	1.96
MUNI												
k=8	1.18	1.20	1.21	1.23	1.25	1.28	1.06	1.13	1.18	1.23	1.30	1.39
k=13	1.09	1.13	1.15	1.17	1.20	1.23	1.06	1.13	1.17	1.22	1.28	1.34
k=18	1.05	1.09	1.12	1.15	1.19	1.22	1.04	1.13	1.19	1.26	1.32	1.39
k=23	1.03	1.06	1.10	1.14	1.19	1.24	1.02	1.14	1.22	1.30	1.38	1.47
MTRI												
k=8	1.15	1.17	1.19	1.21	1.23	1.26	1.08	1.16	1.21	1.27	1.36	1.46
k=13	1.11	1.14	1.16	1.18	1.21	1.23	1.04	1.11	1.16	1.21	1.27	1.34
k=18	1.07	1.10	1.13	1.15	1.18	1.21	1.03	1.11	1.16	1.21	1.27	1.34
k=23	1.04	1.08	1.11	1.14	1.17	1.20	1.02	1.12	1.18	1.24	1.30	1.37
MQUD												
k=8	1.14	1.16	1.18	1.20	1.23	1.27	1.08	1.16	1.22	1.30	1.41	1.54
k=13	1.11	1.14	1.16	1.18	1.21	1.24	1.04	1.12	1.17	1.22	1.29	1.38
k=18	1.08	1.11	1.13	1.15	1.18	1.21	1.03	1.11	1.16	1.21	1.27	1.34
k=23	1.05	1.09	1.11	1.14	1.17	1.19	1.02	1.11	1.16	1.22	1.28	1.35
K1												
bw=0.10	1.05	1.07	1.07	1.07	1.07	1.08	1.04	1.07	1.09	1.12	1.15	1.21
bw=0.20	1.01	1.03	1.05	1.07	1.08	1.11	1.01	1.06	1.08	1.11	1.15	1.18
bw=0.30	1.00	1.04	1.07	1.11	1.16	1.22	1.00	1.08	1.14	1.20	1.28	1.37
bw=0.40	1.00	1.05	1.10	1.16	1.25	1.38	1.00	1.12	1.21	1.33	1.48	1.67
K2												
bw=0.10	1.12	1.15	1.15	1.17	1.19	1.23	1.12	1.16	1.21	1.30	1.45	1.65
bw=0.20	1.06	1.08	1.08	1.08	1.08	1.09	1.06	1.08	1.10	1.14	1.19	1.27
bw=0.30	1.02	1.05	1.05	1.06	1.07	1.07	1.04	1.07	1.09	1.11	1.14	1.18
bw=0.40	1.01	1.04	1.05	1.06	1.07	1.09	1.03	1.07	1.08	1.11	1.14	1.17

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