Research Note

Rigorous Evaluation of the Series Solution of Integral Equation for Effective Emissivity of a Semi-Infinitely Long Circular Blackbody Cavity in Order of Reflectivity

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ABSTRACT

Aiming at rigorous evaluation of a semi-infinitely long circular blackbody cavity, which is also a representative element of micro-cavity or carbon-nano-tube (CNT) array radiator, we propose that the series solution of the integral equation for the effective emissivity at the interior of the circular cavity should be reduced to an explicit form only with wall reflectivity. Applying that, we can get enough accuracy of calculating the effective emissivity with the approximation in the 2nd power of wall reflectivity.

KEYWORDS: blackbody cavity, integral equation, semi-infinite cylinder, effective emissivity, and series method

1. Introduction

Blackbody cavities have been powerful as reference sources in photometry, radiometry, and radiation thermometry. In addition, most recently, a plane radiator that is arrayed in micro-cavities¹⁾ or carbon-nano-tubes (CNT) is proposed and investigated^{2,3)}. Its elemental cavity or tube can be represented by an apparent circular cylindrical cavity aligned with the vertical axis of its plane surface.

In general, as the difficulty of experimental evaluation of radiation characteristics of these cavities, theoretical and numerical methods of calculation are important⁴⁾. Conventional scheme of calculation is to solve effective emissivity on the surface of the cavity's interior wall, based on the mutual internal reflection between gray walls. For diffusive and continuous case, the basic equation that should be solved is an integral equation with respect to the effective emissivity.

Regarding as the simplest model to approximate the cylindrical tube or hall because of the important geometry, Buckley treated a semi-infinite circular cylindrical cavity with a unique approximation⁵⁾. He derived the effective emissivity as the solution including exponential functions. His analytical method is now called the kernel approximation method⁶⁾. It is difficult to estimate the accuracy of its solution because of sequence of the transcendental functions. It has been a representative one and seemed to be moderately accurate over all the amount of the wall emissivity, however. In most recent years, with the aid of high performance of computers, exact effective emissivities have been able to be obtained for various type cavities. The accuracies of these calculations are usually

estimated by the convergence of the calculated effective emissivity directly or indirectly, not by the deviation to the true solution. In either method however, it cannot be inferred self-containedly whether the calculation is in some local minimum leading to be false.

In other words, we have not known how the true solution is or how far from it our calculation is even for the simplest case until now. We also have not known the accurate dependence of the effective emissivity with the interval from the mouth of the cavity.

Next, let us describe on the cavity shape. As mentioned in Bedford's literature⁴⁾, a diffuse sphere is theoretically important because an exact analytical expression for its emissivity can be derived. Its construction is difficult, however. The second simpler case is a cone or a cylinder. For the surface with circular symmetry, we can take the angle factor to be an infinitely narrow circular band on the wall, or a circular annulus on the base or the lid, if there is. Hence we can reduce each integral to a dependence on a single coordinate. In general, the more symmetry an analytical model has, the better the reduction of its calculation is. For a cylinder, the kernel of the integral equation is commutative with respect to coordinates, or keeping minimal symmetry, but not keeping for a cone. In additional notice, it is found that a semi-infinite cylinder is a limit shape of a cone as its apex being infinitely far from its mouth, comparing the kernel of a semi-infinite cylinder with that of a cone.

Let us describe from another viewpoint. If there is a single surface in a cavity model, the integral equation should be scalar. Otherwise, it should be simultaneous, which is difficult to solve. Unfortunately, only an infinite

long cylinder or a both ends opened one has a single surface, which is far apart from presently used cavities.

After all, most simple and fundamental cavity model is a semi-infinite circular cylindrical cavity, in all that approximates the present with mathematical treatment. According to Sparrow's prior numerical quadrature of the integral equation⁷⁸⁾, the solution of a cylinder with a base to the aspect ratio of over 4 can be well approximated by that of a semi-infinite one for the wall emissivity beyond 0.5. As the aspect ratio of a carbon-nano tube element is about three figures in Yang's literature²⁾, hence, it can be evaluated sufficiently by a semi-infinite cylinder model.

In spite of schematic simplicity however, its rigorous quadrature is considerably complex or is in need of some a priori conditions⁵⁷⁾. Main purpose of this monograph is to propose an analytic formula that is convenient for revealing the typical radiation characteristics of a circular semi-infinite cavity in some quantitative reliability with an adequate approximation of the series solution from its integral equation, trying to lighten various properties of a typical cavity that have been overlooked through its statement.

2. Improving representation of the series solution

If the circular cavity has the interior wall which is uniform, gray, isothermal, and lambertian, whose wall emissivity and reflectivity are ε and ρ respectively, then the effective emissivity ϕ at a point x_0 in that cavity's wall is dominated by the following integral equation

$$\phi(x_0) = \varepsilon + \rho \int_s \phi(x) \underline{K}'(x_0, x) dx \quad (1)$$

where $\underline{K}'(x_0, x)$ denotes the angle factor from the infinitesimal circular area including x_0 to that including x . In addition, $\int_s$ denotes the integration over its whole surface.

In order to solve the equation (1), successive substitution (or approximation) is mostly used⁹⁾¹⁰⁾¹¹⁾. As the result of its iterative process, we obtain

$$\phi(x_0) = \varepsilon [1 + \rho \varepsilon_1 + \rho^2 \varepsilon_2 + \cdots + \rho^n \varepsilon_n + \cdots] \quad (2)$$

Many researchers treat ε_n as the n^{th} term's coefficient of series solution (2) with respect to ρ in their basic⁹⁾¹⁰⁾ or applicative¹²⁾ discussions.

If wall emissivity ε and reflectivity ρ are independent of each other, that is mathematically right, however, a physical law, Kirchhoff's law, is lying between ε and ρ . As $\varepsilon + \rho = 1$ on condition that the interior wall is non-transmitted, ε depends upon ρ in the 1st order. Consequently, ε_n is not equivalent to the n^{th} term's coefficient of the series solution in power of ρ . As a fact, for the case of a semi-infinite cylindrical cavity¹²⁾, the 1st successive approximated effective emissivity which is

derived by the truncation of (2), seen in (2'), is not convergent to 1 as being far from the mouth and is less than $\varepsilon^{1/2}$ near the mouth. Therefore, it seems not so good as first order of the effective emissivity nevertheless the 1st term's coefficient $\varepsilon_{n=1}$ can easily be clarified with analytical technique shown by the following detail:

$$\begin{aligned} \phi(x_0) &= \varepsilon + \rho \varepsilon \int_s \underline{K}'(x_0, x) dx \\ &= \varepsilon [1 + \rho \{1 + x_0 - (2x_0^2 + 1)^{1/2}\}] \quad (2') \end{aligned}$$

In the following, we improve the n^{th} successive series solution with substituting ε by ρ .

To start with, we rewrite the integral equation (1) so that

$$\phi(x_0) = 1 + \rho \{ \int_s \phi(x) \underline{K}'(x_0, x) dx - 1 \}. \quad (3)$$

And its series solution is also reduced to

$$\phi(x_0) = 1 + \rho \rho_1 + \rho^2 \rho_2 + \cdots + \rho^n \rho_n + \cdots, \quad (4)$$

where ρ_n is the n^{th} term's coefficient with respect to ρ , which is a function only of x_0 , respectively. Putting (4) into (3), the formula (4) is found to become the rigorous solution of the equation (3) under the conditions

$$\rho_1(x_0) = \int_s \underline{K}'(x_0, x) dx - 1 \quad (5a)$$

and

$$\rho_{n+1}(x_0) = \int_s \rho_n(x) \underline{K}'(x_0, x) dx \quad (5b)$$

Namely, the quantities of all ρ_n can be fixed uniquely by recurrent definition. It is noted that the form of (4) is similar to that treated by Gouffe¹³⁾, and that by DeVos with Quinn's following¹⁴⁾, nevertheless from another viewpoint. It is also noted that ρ_n depend only upon the angular factor of the cavity, denoted by $\underline{K}'(x^*, x)$. Similar discussion with numerical manner can be seen in Ohwada's literature¹⁵⁾. With analogous to Sydner's literature⁹⁾, except all ρ_n are non-positive in our scheme, we can readily find that the effective emissivity denoted by the formula (4) uniformly converges over all amount of wall reflectivity ρ because it is always in the open interval (0, 1).

Second, let us discuss physical meaning of the formula (4) with the aid of perturbation theory. According to the theory, physical system is composed of an unperturbed state and of perturbed states. The unperturbed state has no interaction to any subjects in the system. In emission system, interaction is induced by reflection. Namely, the unperturbed state in emission system has no reflectance, that is ideal blackbody whose emissivity must be unity, and can be seen at 1st term in the formula (4).

Last, to the contrary, we notice the fact that successive substitution in power of ε cannot lead to a fixed solution but to another integral equation. It suggests that, in emission system, its dominant quantity is not emissivity but reflectivity.

3. Derivation of the 2nd successive order coefficient

Let us consider applying the present solution to a concrete canonical problem. At the beginning, its model is illustrated as the cylindrical cavity whose length is L . Both ends of the cavity are opened; namely, there are no bottoms. Its interior wall emissivity is ε . The coordinate system has an origin at one of the ends, and has positive direction toward another end. All coordinates are normalized by interior diameter of the cavity. In the following, the integral $\int_s$ such as to be seen in (1), (3), and (5) means to integrate from 0 to L , $\int_0^L$.

Then, the cavity's angle factor is

$$\begin{aligned} \underline{K}'(x_0, x) &= \underline{K}'(x_0 \sim x) \\ &= 1 - |x_0 - x| \{2(x_0 - x)^2 + 3\} [2\{(x_0 - x)^2 + 1\}^{3/2}]^{-1}. \end{aligned} \quad (6)$$

It should be remarked that (6) is equal to be the limit of the angle factor of a conical cavity as its height tends to be infinite, hence that a semi-infinite circular cavity also can be said to be a canonical model for evaluating the limit of amounts of a conical cavity because its dominant integral equation also tends to be in accordance with that of a semi-infinite circular cavity¹⁵⁾ as already mentioned in introduction.

Applying (6) to the formula (4), we can take a concrete calculation, especially, with analytical scheme in the 2nd successive order term.

According to (5a), as the 1st successive order coefficient,

$$\begin{aligned} \rho_1 &= x_0 - (2x_0^2 + 1) \{2(x_0^2 + 1)^{1/2}\}^{-1} \\ &\quad + L_0 - (2L_0^2 + 1) \{2(L_0^2 + 1)^{1/2}\}^{-1}, \end{aligned} \quad (7)$$

where $L_0 = L - x_0$.

Putting the formula (7) into the integrand of (5b), as the 2nd successive order coefficient,

$$\begin{aligned} \rho_2 &= L \rho_1 - L(L^2 + 1)^{1/2} + L \\ &\quad + \int_0^{x_0} \{J^+(u; x_0) - J_-(u; x_0)/2\} du + \int_0^{L_0} \{J^+(u; x_0) - J_+(u; x_0)/2\} du \\ &\quad + \int_0^{L_0} \{J^-(u; L_0) - J_-(u; L_0)/2\} du + \int_0^{x_0} \{J^+(u; L_0) - J_+(u; L_0)/2\} du, \end{aligned} \quad (8)$$

where

$$J^\pm(u; v) = \{u(u^2 + 1)^{-1/2} + u(u^2 + 1)^{-3/2}/2\} \{v \pm u\}^2 + 1\}^{-1/2},$$

and

$$J_\pm(u; v) = \{u(u^2 + 1)^{-1/2} + u(u^2 + 1)^{-3/2}/2\} \{v \pm u\}^2 + 1\}^{-1/2}. \quad (9)$$

Unfortunately, the integrals in the coefficient ρ_2 are elliptical and cannot be solved analytically. In the following, we discuss with the use of numerical integration to equation (8), on condition that semi-infinite cavity can be simulated by much long cavity, for example, $L = 500$ times of the diameter.

4. Numerical discussion of the 2nd order approximation

We turn to comparing the previous results with the current 2nd successive order approximated result. According to the way of Sparrow's first investigation⁷⁾ where an integral equation including the angle factor (6) explicitly has been solved directly and has been still rather reliable to such a simple model with 4 significant figures for over moderate gray wall, the maximum deviation of Buckley's⁵⁾ and that of the current result respectively from the Sparrow's result are given for each ε in Table 1. The numerical integration in Sparrow's quadrature, however, is presently improved from simple Simpson's rule to adaptive one; AVINT¹⁶⁾.

Note the following: In our present work, each coefficient of the power series solution with respect to ρ can take as much accuracy as we need respectively because all the functions and their bounded integrals in the coefficients are primarily known. Only the truncation of powers with respect to ρ is substantial. In the following discussion, numerical data are shown to the decimal places that the deviations are obvious respectively.

Table 1 Maximum deviation of effective emissivity from the results in Sparrow's scheme with AVINT

$\varepsilon \rightarrow$	0.9	0.81	0.75	0.5	0.25	0.1
Buckley's	0.00141*		0.00224	0.0058	0.0124	0.022
Present	0.00007	0.00060	0.00158	0.0178	0.0900	0.237

* 1 term approximation

Comparing the Buckley's deviation in the table with the prior one¹⁷⁾, the current calculation with AVINT is found to be rather reliable. Above all, for high ε , the deviation of the present work is rather small. On the contrary, the deviation becomes explosively great for low ε because of the truncated error of the effective emissivity with respect to ρ , detailed in the following section.

Next, let us discuss the sufficient condition with respect to ρ that the deviation is so guaranteed as to be in some error bound, for example less than 1 %, nearly corresponding to 2 decimal places. True effective emissivity as the formula (4) is smaller than the truncated one because all coefficients ρ_n are non-positive

as already mentioned. On the other hand, as $\rho_n > -1$, that is greater than the one whose coefficients are all -1 except some initial terms. Now, for the current case that truncated number is 2, we have

$$1 + \rho\rho_1 + \rho^2\rho_2 - \rho^3 \dots - \rho^m \dots < \phi(x_0) < 1 + \rho\rho_1 + \rho^2\rho_2. \quad (10)$$

On the left hand, it is readily found that the series converges, as

$$-\rho^3 - \rho^4 \dots - \rho^m \dots = -\rho^3(1-\rho)^{-1}. \quad (11)$$

Taking account of Kirchhoff's law, then

$$1 + \rho\rho_1 + \rho^2\rho_2 - \rho^3/\varepsilon < \phi(x_0) < 1 + \rho\rho_1 + \rho^2\rho_2. \quad (12)$$

With the fact that the true effective emissivity ϕ is always between $\varepsilon^{1/2}$ and 1, the truncated error E which is defined as follows;

$$E = (1 + \rho\rho_1 + \rho^2\rho_2) \phi^{-1} - 1, \quad (13)$$

is clearly obeying

$$E \approx \rho^3 (\varepsilon\phi)^{-1} < \rho^3 \varepsilon^{-3/2}. \quad (14)$$

If $\rho^3 \varepsilon^{-3/2} < 1\%$, the truncated error is always less 1 %. Hence, the sufficient condition that the truncated error should be less 1 % is found to be $\varepsilon > 0.807$. A more detailed presentation for the slightly higher case $\varepsilon = 0.81$ is made in Table 2, as we compare with Sparrow's scheme. From the table, present results are seen to be of good accuracy, almost 3 effective decimal places, over various positions x_0 .

Table 2 Results of effective emissivity for $\varepsilon = 0.81$

$x_0 \rightarrow$	0	1	2	3	4	5
Sparrow's	0.9000	0.9852	0.9968	0.9990	0.9997	0.9999
Present	0.9005	0.9856	0.9969	0.9990	0.9996	0.9998

Finally, we attempt to formulate the amount ρ_2 analytically.

5. Analytical discussion of the 2nd order coefficient of the solution for semi-infinite shape cavity

Note that both ρ_1 and ρ_2 have commutative law between x_0 and L_0 , respectively. It suggests that the effective emissivity in the cylindrical cavity with both ends opened is symmetric to its half point $L/2$. Nevertheless, we can deduce the effective emissivity of the semi-infinite circular cavity, which is with both ends opened on condition that L_0 is not finite as $L \rightarrow \infty$, to the bounded function only with respect to x_0 . Namely, ρ_1 is reduced to

$$\rho_1 = x_0 - (2x_0^2 + 1)[2(x_0^2 + 1)^{1/2}]^{-1}. \quad (15)$$

It is clear that $\rho_1 \rightarrow 0$ as $x_0 \rightarrow \infty$. Hence the effective emissivity converges to 1 at least in the 1st order.

In the same way as ρ_1 , the coefficient ρ_2 also can be reduced to the function only with respect to x_0 by more complicate calculation as seen in Appendix 1, using Maclaurin's series in power of x_0 and asymptotic series as $L \rightarrow \infty$, not L_0 remarkably:

$$\rho_2 \approx -1/8 - (15\pi/64 - 1/2)x_0 + 5x_0^2/8 \dots \quad (16)$$

Especially it should be noticed $\rho_1 = -1/2$ and $\rho_2 = -1/8$, as $x_0 = 0$, respectively. They are equal to the 1st and 2nd order coefficients of Maclaurin series of $(1-\rho)^{1/2}$ in power of ρ respectively, which is the effective emissivity at the mouth of a semi-infinite circular cavity, known as equal as $\varepsilon^{1/2}$ in numerical⁽⁴⁾⁷⁾ or analytical⁽⁵⁾¹⁸⁾ manner. Moreover, we can comprehend the reason why the maximum deviation of effective emissivity is explosively great for low ε as follows. The deviation of effective emissivity at the mouth can be bounded analytically from above discussion:

$$\begin{aligned} dev.(x_0 = 0) &= 1 - (1/2)\rho - (1/8)\rho^2 - (1-\rho)^{1/2}, \\ &> (1/16)\rho^3, \end{aligned} \quad (17)$$

which is always less than the maximum deviation of effective emissivity as its definition. According to equation (17), if $\varepsilon = 0.9, 0.5$, and 0.1 , then the maximum deviation should be larger than 0.00006, 0.0078, and 0.0455 respectively, which is found to be so explosively increasing as ρ increasing, especially noticed, greater than the Buckley's deviation as seen in Table 1 for the latter two cases.

As additional notice, we can derive the gradient of ϕ at the mouth of the cavity in the 2nd order of ρ from (15) and (16). That is:

$$(\partial/\partial x_0)\phi(x_0 = 0) \approx \rho - (15\pi/64 - 1/2)\rho^2. \quad (18)$$

Ideally, the higher power of x_0 in the formula (16) is calculated, the more precise evaluation of ρ_2 can be done. In actual calculation, however, an explosion of a numerical error by truncating the power of x_0 is unavoidable. Then we need some asymptotic technique for evaluating ρ_2 when the point x_0 is far from the origin. Evaluating equation (8) numerically with IMSL package¹⁹⁾ for integration as $L = 500$, it is found that $\rho_2 \approx -10^{0.8}/x_0^3$ as $x_0 \gg 1$. This is followed by theoretical result $\rho_2 \rightarrow -1/8x_0^3$ as $x_0 \rightarrow \infty$ from equations (5), whose detailed account is given in Appendix 2. According to above discussions, we propose an analytic formula for fitting ρ_2 over all positive x_0 , using a logistic curve between $x_0 = 0$ and ∞ . In the current analysis, we use Hill equation²⁰⁾ denoted by $H(x_0)$,

because its asymptotic behavior is algebraic. Besides, the analytic formula must be equated to the formula (16) in a finite power of x_0 as $x_0 \ll 1$. Choosing an adequate elementary transcendental function, the analytic formula is to be supposed as

$$\rho_2 \approx p(x_0) \exp(-2x_0) + H(x_0), \quad (19)$$

where $p(x_0)$ is a polynomial, in the present research, provided to be quadratic. Considering that

$$\exp(-2x_0)^{-1} = \exp(+2x_0) = 1 + 2x_0 + (1/2!)(2x_0)^2 + \dots, \quad (20)$$

as x_0 is small enough, we have

$$\begin{aligned} p(x_0) \sim & \{-1/8 - (15\pi/64 - 1/2)x_0 + 5x_0^2/8 \dots\} \{1 + 2x_0 + (1/2!)(2x_0)^2 \dots\} \\ \approx & -1/8 - (15\pi/64 - 1/4)x_0 - (15\pi/32 - 11/8)x_0^2, \quad (21) \end{aligned}$$

with omitting $H(x_0)$. As the 1st term of (19) is fixed to be quadratic in power of x_0 , the order of $H(x_0)$ must be at least 3 as $x_0 \ll 1$.

Consequently, $H(x_0)$ is fixed as follows:

$$H(x_0) = -(x_0^3/8)(10^q + x_0^8)^{-1}. \quad (22)$$

If we have $q = 2$, maximum deviation of the amount of the approximated effective emissivity with (19) from the numerical one mentioned in the previous section is found to be under 0.068 %, sufficient to 3 effective decimal places for the lowest case of $\varepsilon = 0.81$.

6. Conclusion

The rigorous solution of an integral equation for a semi-infinite cylindrical cavity, which is a canonical model not only cylindrical but also of conical one, can be formulated to an infinite series with respect to wall reflectivity. Each coefficient of the solution, however, cannot always be solved analytically. We propose that the approximated series solution in the 2nd order of wall reflectivity is in sufficient of its evaluation. Based on this research, we will also be under development of the series formulation of its normal emittance in the 2nd order of wall emissivity, which is convenient to actual design of ordinary blackbody cavities or its arraying plane radiator in high emissive media such as CNT.

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Appendix 1. Procedure of deriving formula (11) from equation (8)

At the beginning, we generalize integrals in equation (8) as:

$$I(Y, Z) = \int_0^Y \{J^+(u; Z) - J^-(u; Z)/2\} du \quad (A1)$$

where Z is a parameter in the integrand function. Note that $J^-(u; Z) = -J^+(-u; Z) = J^+(u; -Z)$, and so on.

Using notation (A1), equation (8) is rewritten so that

$$\rho_2 = L\rho_1 - L(L^2 + 1)^{1/2} + L + I(x_0, -x_0) + I(L_0, x_0) + I(L_0, -L_0) + I(x_0, L_0). \quad (A2)$$

As already mentioned, we assume the semi-infinite cavity is that with both ends opened, whose length L is infinite. Then our problem is how to reduce equation (A2), equal to equation (8), under the condition $L \rightarrow \infty$.

First, we consider $I(L_0, x_0)$ and $I(x_0, -x_0)$, which are derived from (A1) as $Z = \pm x_0$, respectively. Because x_0 is small enough, the integrand of (A1) can be represented as Maclaurin's sum in power of x_0 , or Z . Therefore, provided that Y is large but finite amount in (A1), the term by term integrals to the Maclaurin's sum are possible, and we have²¹⁾

$$\begin{aligned} I(Y, Z) &= Y^2/2 + YZ - (15/32)YZ(Y^2 + 1)^{-1} - (15/32)Z \tan^{-1} Y \\ &- YZ\{16(Y^2 + 1)^2\}^{-1} - (3/16)Z^2(Y^2 + 1)^{-2} - Z^2\{16(Y^2 + 1)^2\}^{-1} \\ &+ Z^2/4 + \{8(Y^2 + 1)\}^{-1} - 1/8. \end{aligned} \quad (A3)$$

On $I(x_0, -x_0)$, as $Y = -Z$ and $Y \ll 1$, in the 2nd power of Y , we have

$$I(Y, -Y) \approx (5/8)Y^2. \quad (A4)$$

On the other case, $I(L_0, x_0)$, as $Y + Z = L \gg 1$ and $Z \ll 1$, we obtain

$$\begin{aligned} I(Y, Z) &\approx Y^2/2 + YZ - (15\pi/64)Z + Z^2/4 - 1/8 \\ &\quad (A5) \end{aligned}$$

in the 2nd power of Z .

Second, we consider $I(x_0, L_0)$ and $I(L_0, -L_0)$. In these

cases, we equate Z to L_0 , which is positive and large enough. Then, $1/Z \ll 1$ as $Z \gg 1$, and we can also represent the integrand of (A1) as Maclaurin's sum in power of $1/Z$. With the term by term integrals to the sum, we have²¹⁾

$$\begin{aligned} I(Y, \pm Z) &= \pm Z(Z^2 + 1)^{1/2}/2 + Y\{(Z^2 + 1)^{1/2} - 1\} \\ &- [\pm Z(Z^2 + 1)^{-1/2}/2 - Y/2\{(Z^2 + 1)^{-1/2} - 1\}]. \end{aligned} \quad (A6)$$

If $Y \ll 1$,

$$I(Y, Z) \approx 3YZ^2/4. \quad (A7)$$

On the other hand, if $Y \sim Z \gg 1$,

$$I(Y, -Z) \approx Y^2/2 - Y/2 + \text{constant} \quad (A8)$$

Unfortunately, the constant term in (A8) cannot be derived because the convergence of Maclaurin's sum to the integrand of (A1) is violated as $u \sim Z$. We evaluate the constant term from the integral (A1) by numerical scheme directly. Then (A8) is derived as

$$I(Y, -Z) \approx Y^2/2 - Y/2 + 1/2 \quad (A8')$$

By putting Maclaurin's sum of (7) in the 2nd power of x_0 , (A4), (A5), (A7), and (A8') into (A2), with replacing (Y, Z) with (x_0, L_0) respectively, all terms depending upon L are omitted. Then Maclaurin's sum of ρ_2 is deduced to (15) in the 2nd power of x_0 . On the same way, we can get higher order terms of ρ_2 in succession, and from them we can deduce ρ_2 to Maclaurin's series as rigorous expression.

Appendix 2. Proof of $\rho_2 \rightarrow -0.125/x_0^3$ as $x_0 \rightarrow \infty$

For the case of semi-infinite cavity, the integral (5b) is reduced to

$$\rho_2 = \int_0^\infty \rho_1(x) \underline{K}'(x_0 \sim x) dx \quad (A9)$$

Taking account of (6), we have

$$\int_{-\infty}^\infty \underline{K}'(x_0 \sim x) dx = 1. \quad (A10)$$

And we derive as $x_0 \rightarrow \infty$:

$$\underline{K}'(x_0 \sim x) \approx (3/8)(x_0 - x)^{-4} \rightarrow 0 \quad (A11)$$

for all "finite" x .

Obeying the conditions (A10) and (A11), the asymptotical performance of the angle factor $\underline{K}'(x_0 \sim x)$ is like a performance of Dirac's delta function $\delta(x_0 \sim x)$. With the

definition of Dirac's delta function, as $x_0 \rightarrow \infty$, (A9) is

$$\rho_2 \approx \int_0^{\infty} \rho_1(x) \delta(x_0 - x) dx = \rho_1(x_0). \quad (\text{A12})$$

On the other hand, the asymptotic performance of the 1st successive order coefficient of the effective emissivity is easily derived from (10) as

$$\rho_1(x_0) \approx -(1/\theta) x_0^{-3}. \quad (\text{A13})$$

In conclusion, from (A12) and (A13), we can get

$$\rho_2(x_0) \approx -(1/\theta) x_0^{-3}. \quad (\text{A14})$$

Besides, with recursive usage of (A12) for all ρ_n , we can also get

$$\rho_n(x_0) \approx -(1/\theta) x_0^{-3}. \quad (\text{A15})$$

Therefore, by substituting all ρ_n in equation (4), we can derive asymptotic formula of $\phi(x_0)$ as follows:

$$\begin{aligned} \phi(x_0) &\approx 1 - (1/\theta) x_0^{-3} \rho(1 + \rho + \rho^2 + \dots + \rho^n + \dots) \\ &\approx 1 - (1/\theta) x_0^{-3} \rho(1 - \rho)^{-1} \\ &\approx 1 - (1/\theta) x_0^{-3} \rho \varepsilon^{-1}. \end{aligned} \quad (\text{A16})$$

Note that the formula (A16) is quite rigorous with respect to the reflectivity ρ . We have found both the numerical calculations by AVINT and by Bedford's zonal approximation^{4, 22)} for a long cylindrical cavity with a bottom respectively tend to obeying the formula (A16) when $\phi(x_0) > 0.997$, nevertheless with the influence of various error factors such as the truncation of asymptotical terms, the existence of the bottom, and so on.