■ Paper

Focusing Properties of Bicylindrical Acoustic Lenses

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Abstract:

This theoretical report concerns the focusing of a plane progressive acoustic wave by a bicylindrically-curved lens. It was assumed that the lens was made of polymethylmethacrylate (PMMA) and was submerged in water. The analytical approach that is based on full-wave theory began with expanding an incident plane wave into cylindrical waves to satisfy appropriate boundary conditions at an interface between the water and the lens. The continuity conditions of displacement and stress at the interface may determine the generation of longitudinal and shear waves in the lens. Lens-aperture effects on focused field performance were taken into account by confining the incident wave to the region within the aperture. The Rayleigh integral associated with the particle velocity on the exit interface of the lens enabled us to evaluate the entire sound field behind the lens. Numerical examples demonstrated no significant influence on the fields near the focus, even when the shear wave generation was included in the theory. Wave diffraction due to a finite lens aperture, however, became a major effective factor in beam focusing.

Classification: Physical acoustics, Wave propagation

Keyword: Acoustic lens, Bicylindrical lens, Focusing, Resolution

1. Introduction

Acoustic lenses are widely used in ultrasonic-imaging systems to display high spatial resolution images and to allow ultrasonic equipment in industrial technology to reach a high acoustic power at a focal point. The lens, ordinarily made of plastic or solid material, conveys sound at a higher speed than the surrounding medium, e.g., water. Therefore, the lens configuration should be concave.

Geometric ray theory is a powerful predictor like optics and provides a first approximation in evaluating lens performance. The ray model, however, breaks down in principle near the focus, where wave diffrac-

tion becomes a significant factor in sound field formation. Additionally, it does not take account of the shear wave that may be generated at the interface between the lens and surrounding fluid and may have potential effects on field profiles. In contrast, wave acoustics can accurately analyze fields, even for mode conversion, although some tedious numerical computation is needed.

This study discusses problems of a single lens whose shape is bicylindrical and which is made of polymethylmethacrylate (PMMA). The lens is submerged in water. The principle concerns are twofold. The first is to investigate the effects of shear wave

generation on focusing. To this end, we solve wave equations, subject to appropriate boundary conditions, for the general situation where a plane wave is incident obliquely on the lens. This situation includes the generation not only of longitudinal but also of shear waves through the process of mode conversion at the boundary. To predict the focused field behind the lens, we use the Rayleigh integral and the particle velocity on the exit surface of the lens. Several researchers have pointed out that the mode conversion is of no importance in lens performance and that only slight differences occur even if the presence of shear waves is accounted for¹⁻⁴⁾. If their opinion is correct, it considerably simplifies computational problems⁵⁻⁷⁾. Unfortunately, however, their analyses still lack quantitative evaluation. Some numerical examples in this report demonstrate that sound pressure profiles near the focus are less affected by the shear wave generation like several researchers' speculation.

The second aim of this study is to clarify the effect of lens aperture on field formation. An actual lens has a finite aperture size that determines the focal length and pressure peak at the focus. In some cases, various lens aberrations, especially cylindrical aberration, may crucially affect lens performance. Numerical simulations show that decreasing the aperture size widens the beam pattern at the focus. It does not necessarily, however, reduce the pressure peak in a smooth fashion. Other interesting findings are presented in the present demonstrations.

2. Theory for a bicylindrical lens

Let us imagine that a plane wave with infinite length in the y axis perpendicular to the present paper surface is obliquely incident on a cylindrically curved acoustic lens, as shown in **Fig. 1**. The lens, made of solid material with density ρ_0 , has radius d and a finite aperture of $2a$. The longitudinal and shear wave speeds in the lens are c_l and c_s , respectively. The lens

Fig. 1 Geometry of a cylindrical lens with infinite length in the y axis.

is surrounded by water, whose density is ρ and which has the sound speed c . An additional assumption is that all the sound absorptions in the lens and water can be neglected for *ad hoc* simplicity. We introduce the velocity potential ϕ_i with a unity amplitude to express the incident plane wave that is propagating in the z direction at some angle θ_i . In this case, ϕ_i is given by

$$\phi_i = e^{-jkr \cos(\theta - \theta_i)}, \quad (1)$$

where $k = \omega / c$ is the wavenumber and ω is the angular frequency of the wave. r in eq. (1) is the distance between the cylinder's origin O and a receiving point P , then $\sqrt{z^2 + x^2}$. Additionally, the point P makes an angle $\theta (= \tan^{-1}(x/z))$ with the z axis.

Incidentally, the following identical equation holds for arbitrary variables ξ and φ :

$$e^{-j\xi \cos \varphi} = \sum_{n=-\infty}^{\infty} (-j)^n J_n(\xi) e^{-jn\varphi}, \quad (2)$$

with $J_n(\cdot)$ being the Bessel function. Now, suppose that the incident waves inside the aperture are all gathered by the lens and the waves outside the aperture are absorbed perfectly without any reflection. This means that ϕ_i must satisfy $\exp[-jkd \cos(\theta - \theta_i)]$ for $|x| \leq a$ and vanish for $|x| > a$. Then

$$\phi_i = \sum_{n=-\infty}^{\infty} a_n H_n^{(2)}(kd) e^{-jn\theta} = \begin{cases} \sum_{n=-\infty}^{\infty} (-j)^n J_n(kd) e^{-jn(\theta-\theta_0)} & -\theta_0 \leq \theta \leq \theta_0 \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where $H_n^{(2)}(\cdot)$ is the Hankel function of the second kind, and θ_0 is the opening angle of the inlet aperture that is given by $\sin^{-1}(a/d)$. The arbitrary coefficients a_n in eq. (3) are determined using the orthogonality property of trigonometric functions. Multiplying both sides by $\exp[jm\theta]$ and integrating the resultant equation with respect to θ over $-\pi \leq \theta \leq \pi$ result in

$$a_n = \frac{1}{H_n^{(2)}(kd)} \sum_{m=-\infty}^{\infty} (-j)^m e^{jm\theta_0} J_m(kd) \times \frac{\sin[(n-m)\theta_0]}{n-m}. \quad (4)$$

When the incident wave reaches the inlet interface l where the properties of the medium or acoustic impedance change abruptly, we can expect both a reflection and transmission of the wave to occur. These basic phenomena generate at least three waves; i.e., a reflected wave, which travels back to the water, and longitudinal and shear waves, which travel forward in the lens as elastic bulk waves. In the same manner as in the previous report⁸⁾, the analytical solutions of the three waves are obtained. For the reflected wave, the velocity potential ϕ_r takes the form:

$$\phi_r = \sum_{n=-\infty}^{\infty} a_n A_n H_n^{(1)}(kr) e^{-jn\theta} \quad (r \leq d), \quad (5)$$

where $H_n^{(1)}(\cdot)$ is the Hankel function of the first kind. A_n will be determined explicitly by boundary conditions a short time later.

The next problem is to derive wave potentials for the longitudinal and shear waves in the lens. The displacement vector $\mathbf{u}$ in a solid is given by

$$\mathbf{u} = \nabla\psi + \nabla \times \mathbf{A}. \quad (6)$$

The vector potential $\mathbf{A}$ is always solenoidal; i.e., $\nabla \cdot \mathbf{A} = 0$. In this case, only the component A_y in the y axis is essential for the displacement generation. Another scalar potential ψ that describes the longitudinal wave is also generated, and these two components take the following forms:

$$\psi = \sum_{n=-\infty}^{\infty} a_n B_n H_n^{(2)}(k_l r) e^{-jn\theta} \quad (r \geq d), \quad (7)$$

$$A_y = \sum_{n=-\infty}^{\infty} a_n C_n H_n^{(2)}(k_t r) \frac{d}{d\theta} e^{-jn\theta} \quad (r \geq d). \quad (8)$$

Here, $k_l = \omega/c_l$ and $k_t = \omega/c_t$ are the wavenumbers of the longitudinal and shear waves, respectively. Furthermore, the coefficients B_n and C_n with the participation of A_n are completely determined by specifying three appropriate boundary conditions on the inlet interface l : one is the continuity of the components of the particle velocity normal to the interface, and the other is the continuity of the stress component normal to the interface. The remaining condition is obtained by speculating that the tangential component of the stress vanishes because of the water being inviscid. Thus, we can determine A_n , B_n , and C_n completely from the three boundary conditions, although some perseverance is needed to derive the solutions. To help the readers, in Appendix A we provide the coefficients A_n to C_n and the consequently obtained components u_r and u_θ .

An actual lens has an exit interface l_2 , as shown in **Fig. 2**. When waves travel through the lens and reach

Fig. 2 Geometry of a bicylindrically focused lens.

the exit interface, they are partially reflected owing to the change in acoustic impedance. The thus reflected waves travel back to the inlet, and wave reflection occurs again at l_1 . As a first approximation, we neglect such multiple reflections among the inlet, exit, and lateral faces of the lens because the non-planar configuration of the bicylindrical interfaces is likely to weaken any reflections. According to our recent investigation, the transmission coefficient of a plane wave for a plano-concave lens is formulated using a simple model based on geometrical acoustics, even for relatively wide-angle opening⁹⁾:

$$v_n = T_v(\mathbf{v} \cdot \mathbf{n}), \quad (9)$$

$$T_v = \frac{2Z_l}{Z_w + Z_l}, \quad (10)$$

where v_n is the normal component of the particle velocity on the interface, and $\mathbf{v}$ is the particle velocity when the interface is hypothetically absent, expressed as

$$\mathbf{v} = j\omega(u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta), \quad (11)$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors along the positive r and θ directions. Furthermore, $\mathbf{n}$ is the outer unit normal vector on l_2 , T_v is the transmission coefficient of the particle velocity at normal incidence on a plane boundary, and the specific acoustics impedances for the longitudinal wave is Z_w in water, and Z_l in the lens. To evaluate the total sound field behind the lens, we employ the two-dimensional Green function, which has the form with the Hankel function of order zero. Namely,

$$\phi = -j \frac{1}{2} \int_{l_2} v_n H_0^{(2)}(kR) ds. \quad (12)$$

Here R is the distance between a source point $Q(x_0, z_0)$ on l_2 and a receiving point $P(x, z)$ and is expressed as $R = \sqrt{(x - x_0)^2 + (z - z_0)^2}$ from Fig. 2.

In addition, ds in eq. (12) is the length element on l_2 and is given by $ds = d_2 d\theta_2$. Therefore, eq. (12) yields

$$\phi = -j \frac{d_2}{2} \int_{-\theta_m}^{\theta_m} v_n H_0^{(2)}(kR) d\theta_2, \quad (13)$$

where θ_m is the opening angle of the exit aperture, being given by $\sin^{-1}(a/d_2)$.

3. Numerical examples

To know how much the shear wave and lens parameters such as the aperture radius affect focused fields behind the lens, we now demonstrate lens performance numerically. Just like the previous report⁸⁾, we choose acrylic resin or polymethylmethacrylate (PMMA) as the lens material, listing the physical quantities of PMMA in **Table 1**. The wave speed and sound absorption were measured at a frequency of 1 MHz and at room temperature using longitudinal and shear-wave transducers in direct contact with PMMA plates of different thicknesses in a through-transmission experiment. These values were almost same as data in the literature¹⁰⁻¹²⁾. The lens radii d_1 and d_2 may be changed variously. However, we confined ourselves to preparing two lenses with different radii but almost the same focal lengths. One of them is $d_1 = 60$ mm and $d_2 = 60$ mm (hereinafter simply called the "60/60-mm lens"), and the other is $d_1 = 100$ mm and $d_2 = 35.5$ mm (the "100/35.5-mm lens"). The thickness h is the same,

Table 1 Physical quantities of a PMMA lens. Subscript "l" refers to longitudinal waves, and "s" refers to shear waves. The sound-absorption coefficients and speeds are measured at 1 MHz and at room temperature.

Lens density ρ_0 [kg/m ³]	1180
Sound velocity c_l [m/s]	2760
Sound velocity c_s [m/s]	1340
Absorption coefficient α_l [neper/m]	15
Absorption coefficient α_s [neper/m]	39
Water density ρ [kg/m ³]	998
Sound velocity c [m/s]	1483
Sound absorption	neglected

2 mm, for the two lenses. Because of a relatively great sound absorption in the lens, we must modify our theory. Fortunately, we can accomplish this modification by approximating the wave numbers as $k_t = \omega/c_t - j\alpha_t$ and $k_l = \omega/c_l - j\alpha_l$ in place of $k_t = \omega/c_t$ and $k_l = \omega/c_l$, eqs. (7) and (8).

Fig. 3 shows pressure-amplitude profiles along the beam axis when an incoming plane wave is normally incident on the 60/60-mm lens ($\theta_i = 0$). To examine only the effect of shear waves on the fields, the sound absorptions in the lens are neglected in the first stage; i.e., $\alpha_t = \alpha_l = 0$. Additionally, the lens aperture radius a is assigned to be 37.5-mm under the constraint that the incident waves would be confined within the critical angle θ_c . Incidentally, the angle, defined as $\theta_c = \sin^{-1}(c_l / c_t)$, has a value of 32.5° , so that a is greater than $d_l \sin \theta_c = 32.2$ -mm beam. The abscissa denotes the axial distance beginning at the lens center 0, as indicated in Fig. 2. The ordinate denotes the pressure amplitude relative to the incoming pressure amplitude of $|p| = |j\omega\rho\phi_i| = \omega\rho$. This amplitude level corresponds to the focusing gain.

It is apparent from these data that the wave is focused by the effect of the lens. The pressure increases

Fig. 3 On-axis pressure curves behind the 60/60-mm lens with aperture radius $a = 37.5$ mm (solid curve). A 1-MHz plane wave is impinging normally on the lens ($\theta_i = 0$). Sound absorption in the lens is not taken into account. The data are plotted in dots for the neglected generation of the shear wave.

es in amplitude with peaks and dips during propagation, attaining a maximum amplitude of about 13 dB at the focus near 55 mm and then decreasing gradually. In the figure, the propagation curve is shown in dots when the shear wave generation is neglected by setting all the coefficients C_n and shear terms to zero. Interestingly, the two curves overlap almost over a long range, from 20 mm to 150 mm, that includes the focus. To be precise, there is some difference in the pre-focal and post-focal regions, but since this difference is slight, we may actually neglect the shear-wave generation in the design of lenses even when the aperture is too large to include the critical angle.

A similar numerical simulation is executed for the 100/35.5-mm lens. The results are shown in **Fig. 4**. The aperture radius is changed to $a = 30$ mm, being smaller than $d_l \sin \theta_c = 53.7$ mm. Therefore the mode conversion seems not to occur actively. In fact, the two pressure curves have no noticeable difference in amplitude except for the post-focal region greater than 30 mm from the focus. Due to there being less aberration, the 100/35.5-mm lens leads to the generation of a high acoustic pressure compared with the 60/60-mm lens, although the aperture of the former lens is smaller than that of the latter.

Fig. 4 On-axis pressure curves behind the 100/35.5-mm lens with aperture radius $a = 30$ mm (solid curve). A 1-MHz plane wave is impinging normally on the lens ($\theta_i = 0$). The data are plotted in dots for the neglected generation of the shear wave.

Fig. 5 shows pressure curves for different angles of the wave incidence from $\theta_i = 0$ to 15° . The abscissa denotes the distance from the starting point 0 along the new axis z' . The new Cartesian coordinates (x', z') is related with the old ones (x, z) in a familiar form:

$$\left. \begin{aligned} x &= x' \cos \theta_i + z' \sin \theta_i \\ z &= -x' \sin \theta_i + z' \cos \theta_i \end{aligned} \right\} \quad (14)$$

Strictly speaking, the beam axis with an amplitude peak is slightly shifted from the z' axis in a positive

Fig. 5 Pressure curves along the z' axis for the 100/35.5-mm lens with aperture radius $a = 30$ mm. A 1-MHz plane wave is impinging obliquely on the lens from $\theta_i = 0$ to 15° . The curve indicating the neglected generation of the shear wave is plotted in dots for $\theta_i = 15^\circ$ only.

direction of the x' axis because of the finite lens thickness $h = 2$ mm. The shift is given approximately by $\Delta x' = h \times (c_l/c) \sin \theta_i$ for small incident angles and is generally quite small. For example, the shift takes a value of only 0.66 mm for $\theta_i = 10^\circ$. Actually, the larger the incident angle becomes, the more the amplitude peak moves towards the lens, and the more the amplitude decreases. These tendencies of the peak movement and amplitude reduction are the same as the fields in a single lens system with a spherically curved configuration⁸⁾. An additional curve in dots indicates the data from neglecting the generation of the shear wave. Importantly, the shear-wave generation somewhat influences the pressure profile in comparison with the normal-incidence case. However, the profile around the focus is unchanged whether the shear wave is included or excluded.

Beam patterns or pressure amplitude profiles along the lines perpendicular to the z' axis are shown in **Fig. 6**, where the incident angle is changed from $\theta_i = 0$ to 15° every 5° . Observational positions are determined by amplitude peaks in **Fig. 5**; i.e., $z' = 55.5$ mm for $\theta_i = 0$, $z' = 54.7$ mm for $\theta_i = 5^\circ$, $z' = 51.9$ mm for $\theta_i = 10^\circ$, and $z' = 48.8$ mm for $\theta_i = 15^\circ$. The Rayleigh's

Fig. 6 Beam patterns at various incident angles from 0 to 15° for the 100/35.5-mm lens with aperture radius $a = 30$ mm. The receiving points on the beam axis are chosen at the pressure peaks in **Fig. 5**; i.e., $z' = 55.5$ mm for $\theta_i = 0$, $z' = 54.7$ mm for $\theta_i = 5^\circ$, $z' = 51.9$ mm for $\theta_i = 10^\circ$, and $z' = 48.8$ mm for $\theta_i = 15^\circ$. The curve indicating the neglected generation of the shear wave is plotted in dots for $\theta_i = 15^\circ$ only.

criterion¹³⁾ that is regarded as a measure of image resolution and is defined as $\Delta\theta = \lambda/2a$ for a long and narrow aperture or slit (λ is the wavelength) becomes 1.4° by substituting $\lambda = 1.48$ mm and $a = 30$ mm. Since the present numerical data of the angle resolution indicate about 1.5° , which is a little larger than the theoretical value that is up to at least 10° , this single lens system may serve roughly as the upper limit of the angle resolution. At $\theta_i = 15^\circ$, however, the beam seems to broaden. Furthermore, the amplitude peak is decreased by several decibels in comparison with those at other angles, and the patterns of the sidelobes are pronouncedly asymmetric with respect to the beam axis. The pressure levels of the sidelobes are increased as well. A dotted line in the figure denotes the data for $\theta_i = 15^\circ$ only for the case assuming that the shear wave is absent. Like in Fig. 5, the generation of the shear wave hardly influences the focused field in the paraxial region within $\pm 2^\circ$.

It is expected that decreasing the aperture size would move the acoustic focus towards the lens because wave diffraction appears prominently in small apertures. **Fig. 7** shows axial pressure profiles for normal incidence to the 60/60-mm lens. The aperture radius is changed from $a = 50$ mm to 15 mm. Actually,

Fig. 7 Pressure-amplitude profiles along the axis by varying the aperture radius for the 60/60-mm lens. Plane waves are normally incident on the lens. Sound absorptions in the lens are included in the data.

decreasing the aperture makes the pressure peak low and moves the focus toward the lens, indicated, for example, by comparing the data between the 30-mm radius aperture and the 15-mm one. In detail, we note that first the focus moves somewhat away from the lens and that it then slightly approaches the lens. Furthermore, the focal zone, which is usually defined as the distance between the starting and ending axial points where the amplitude drops to -6 dB of the amplitude at the focus, lies over a wide region when a is decreased. Interestingly, there are no significant changes in pressure profiles for $a = 40$ mm and 50 mm. This is because the critical angle θ_c for the 60/60-mm lens is 32.5° and an aperture radius greater than 32.2 mm does not in a sensitive way affect the field formation behind the lens.

Beam patterns on the focal lines are shown in **Fig. 8** for the 60/60-mm lens. The aperture radius is changed in the same way as in Fig. 7. The beam widens obviously when the aperture is decreased. For example, the total -6 -dB beam width is 2.3 mm for $a = 30$ mm and almost the same as it is for the lens when $a = 40$ mm and 50 mm. By contrast, the beam width becomes

Fig. 8 Beam patterns for various aperture radii of the 60/60-mm lens. Plane waves are normally incident on the lens. The receiving lines are chosen at the pressure peaks in Fig. 7; i.e., $z = 56$ mm for $a = 50, 40$ mm, $z = 56.7$ mm for $a = 30$ mm, $z = 58.3$ mm for $a = 20$ mm, and $z' = 55.5$ mm for $a = 15$ mm. Sound absorptions in the lens are included in the data.

somewhat broad at 2.9 mm for $a = 20$ mm. As for the Rayleigh's criterion, $\Delta\theta$ at $a = 30$ mm has the same value, 1.4° , as in Fig. 6. However, the actual resolution is $1.7^\circ (= \tan^{-1}(1.65/56.7))$, slightly larger than the resolution of the 100/35.5-mm lens. Sidelobe pressure levels are relatively high for the lenses with the 40-mm and larger apertures, probably due to the aberration. Of the five lenses, the 30-mm radius lens has the lowest level in the first sidelobe. When the radius becomes small, the aberration is weakened, and the side-lobe levels are decreased. A reduction in levels of about 3-dB reduction is found at the first sidelobe by comparing the data of $a = 40$ mm and 30 mm. Overall, the sidelobe levels range from -10 dB to -15 dB. These levels are a few decibels higher than those of the corresponding spherically curved lens⁸⁾.

4. Conclusions

The present theoretical report has examined numerically the opinion that the generation of shear waves in an acoustic solid lens has no great effect on focused fields, even for relatively thick lenses with widely opening apertures that include the critical angle. The fields near the focus never change significantly their features. The result thus obtained definitely supports the validity of the previous presumption or opinion that the presence of shear waves, caused by mode conversion, is not important in predicting lens performance. This means that in place of the fluid/solid model presented here the fluid/fluid model that does not need to include the generation of shear waves is in principle acceptable for analytical predictions of lens performances. Consequent simplification of the theory and numerical computation will help considerably in the further development of ultrasonic lens design. Needless to say, it should be noted that wave diffraction due to a finite lens aperture becomes a major effective factor in beam focusing.

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Appendix A

Using eqs. (6) ~ (8), the displacement components u_r and u_θ are given by

$$\begin{aligned} u_r &= \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial A_y}{\partial \theta} \\ &= \sum_{n=-\infty}^{\infty} a_n \left\{ k_l B_n H_n^{(2)'}(k_l r) - \frac{n^2}{r} C_n H_n^{(2)}(k_l r) \right\} e^{-jn\theta} \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} u_\theta &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} - \frac{\partial A_y}{\partial r} \\ &= -j \sum_{n=-\infty}^{\infty} n a_n \left\{ \frac{B_n}{r} H_n^{(2)}(k_l r) - k_l C_n H_n^{(2)'}(k_l r) \right\} e^{-jn\theta} \end{aligned} \quad (\text{A.2})$$

Taking account of the boundary conditions described in the body, we obtain the following three equations:

$$\begin{aligned} H_n^{(1)'}(z_1) A_n + \frac{z_2}{z_1} H_n^{(2)'}(z_2) B_n' \\ - \frac{n^2}{z_1} H_n^{(2)}(z_3) C_n' = -H_n^{(2)'}(z_1), \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} H_n^{(1)}(z_1) A_n + \frac{1}{\rho c_l^2} \left\{ \lambda H_n^{(2)}(z_2) - 2\mu H_n^{(2)''}(z_2) \right\} B_n' \\ - \frac{2\mu n^2}{\rho z_3^2 c_l^2} \left\{ H_n^{(2)}(z_3) - z_3 H_n^{(2)'}(z_3) \right\} C_n' \\ = -H_n^{(2)}(z_1), \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} 2 \left\{ z_2 H_n^{(2)'}(z_2) - H_n^{(2)}(z_2) \right\} B_n' \\ + \left\{ z_3 H_n^{(2)'}(z_3) - z_3^2 H_n^{(2)''}(z_3) - n^2 H_n^{(2)}(z_3) \right\} C_n' \\ = 0, \end{aligned} \quad (\text{A.5})$$

where $z_1 = kd$, $z_2 = k_l d$, and, $z_3 = k_l d$. Additionally, $B_n' = j\omega B_n$ and $C_n' = j\omega C_n$. The coefficients A_n to C_n are completely determined by solving eqs. (A.3) to (A.5), and the resultant solutions yield:

$$A_n = \frac{\beta_1 (\alpha_{22} \alpha_{33} - \alpha_{23} \alpha_{32}) + \beta_2 (\alpha_{13} \alpha_{32} - \alpha_{12} \alpha_{33})}{\alpha_{11} (\alpha_{22} \alpha_{33} - \alpha_{23} \alpha_{32}) + \alpha_{21} (\alpha_{13} \alpha_{32} - \alpha_{12} \alpha_{33})}, \quad (\text{A.6})$$

$$B_n = \frac{B'_n}{j\omega} = -\frac{4}{\pi\omega z_1} \times \frac{1}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21} + \alpha_{32} (\alpha_{21} \alpha_{13} - \alpha_{11} \alpha_{23}) / \alpha_{33}}, \quad (\text{A.7})$$

$$C_n = \frac{C'_n}{j\omega} = -\frac{\alpha_{32}}{\alpha_{33}} B_n, \quad (\text{A.8})$$

where $\alpha_{11} = H_n^{(1)'}(z_1)$, $\alpha_{12} = (z_2/z_1) H_n^{(2)'}(z_2)$, $\alpha_{13} = -(n^2/z_1) H_n^{(2)}(z_3)$, $\alpha_{21} = H_n^{(1)}(z_1)$, $\alpha_{22} = \{\lambda H_n^{(2)}(z_2) - 2\mu H_n^{(2)'}(z_2)\} / \rho c_t^2$, $\alpha_{23} = -2\mu n^2 \{H_n^{(2)}(z_3) - z_3 H_n^{(2)'}(z_3)\} / \rho z_3^2 c_t^2$, $\alpha_{32} = 2\{z_2 H_n^{(2)'}(z_2) - H_n^{(2)}(z_2)\}$, $\alpha_{33} = z_3 H_n^{(2)'}(z_3) - z_3^2 H_n^{(2)''}(z_3) - n^2 H_n^{(2)}(z_3)$, $\beta_1 = -H_n^{(2)'}(z_1)$, and $\beta_2 = -H_n^{(2)}(z_1)$.

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