Ineffability of $\mathcal{P}_{\kappa}\lambda$ for $\lambda$ with small cofinality

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Abstract. We study ineffability, the Shelah property, and indescribability of $\mathcal{P}_{\kappa}\lambda$ when $\text{cf}(\lambda) < \kappa$. We prove that if $\lambda$ is a strong limit cardinal with $\text{cf}(\lambda) < \kappa$ then the ineffable ideal, the Shelah ideal, and the completely ineffable ideal over $\mathcal{P}_{\kappa}\lambda$ are the same, and that it can be precipitous. Furthermore we show that $\Pi^1_1$-indescribability of $\mathcal{P}_{\kappa}\lambda$ is much stronger than ineffability if $2^\lambda = \lambda^{<\kappa}$.

1. Introduction.

Combinatorial principles for a cardinal, ineffability, and weak compactness were studied thoroughly in Baumgartner [4]. First we review some definitions:

Definition 1.1. For a regular uncountable cardinal $\kappa$,

1. $\kappa$ is weakly compact if, for all $\langle a_\alpha : \alpha < \kappa \rangle$ with $a_\alpha \subseteq \alpha$, there exists $A \subseteq \kappa$ such that $\{ \alpha < \kappa : A \cap \alpha = a_\alpha \cap \beta \}$ is unbounded in $\kappa$ for all $\beta < \kappa$,

2. $\kappa$ is ineffable (respectively almost ineffable) if, for all $\langle a_\alpha : \alpha < \kappa \rangle$ with $a_\alpha \subseteq \alpha$, there exists $A \subseteq \kappa$ such that $\{ \alpha < \kappa : A \cap \alpha = a_\alpha \}$ is stationary in $\kappa$ (respectively unbounded in $\kappa$).

The definition of ineffability and almost ineffability is due to Jensen and Kunen. Weak compactness originated from the study of compactness of infinitary logic (see section 4 in Kanamori [18]). The above combinatorial definition (1) was found by Baumgartner [4]. Afterward ineffability was translated into $\mathcal{P}_{\kappa}\lambda$-structures by Jech [13], where $\kappa$ is a regular uncountable cardinal, $\lambda \geq \kappa$ is a cardinal, and $\mathcal{P}_{\kappa}\lambda = \{ x \subseteq \lambda : |x| < \kappa \}$. Carr [8] defined the Shelah property, mild ineffability, and indescribability of $\mathcal{P}_{\kappa}\lambda$ as a generalization of weak compactness of a cardinal. These properties of $\mathcal{P}_{\kappa}\lambda$ have been widely studied when $\text{cf}(\lambda) \geq \kappa$, and it has been shown that ineffability, almost ineffability, and the Shelah property form a proper hierarchy. For instance, if $\kappa$ is almost $\kappa^+$-ineffable then there are stationary many $\alpha < \kappa$ such that $\alpha$ is $\alpha^+$-Shelah.

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On the other hand, Abe [3] showed that ineffability of $\mathcal{P}_\kappa \lambda$ coincides with almost ineffability if $2^\lambda = \lambda^{< \kappa}$. Hence the above mentioned hierarchy can be collapsed if $\text{cf}(\lambda) < \kappa$. We will investigate ineffability, the Shelah property, and indescribability of $\mathcal{P}_\kappa \lambda$ when $\text{cf}(\lambda) < \kappa$.

We know $\lambda^{< \kappa}$ is the size of $\mathcal{P}_\kappa \lambda$. We also try to decide the size of $\mathcal{P}_\kappa \lambda$ under weaker assumptions than before. Solovay [20] proved $\lambda^{< \kappa} = \lambda^+$ if $\kappa$ is $\lambda$-(super)compact and $\text{cf}(\lambda) < \kappa$, where $\lambda^+$ denotes the minimal cardinal greater than $\lambda$, and Johnson [15] showed that $\lambda^{< \kappa} = \lambda$ holds if $\kappa$ is $\lambda$-Shelah and $\text{cf}(\lambda) \geq \kappa$. We extend this to the following:

**Theorem 1.2.**

1. If $\kappa$ is mildly $\lambda$-ineffable and $\text{cf}(\lambda) \geq \kappa$, then $\lambda^{< \kappa} = \lambda$, and
2. if $\kappa$ is $\lambda$-Shelah and $\text{cf}(\lambda) < \kappa$ then $\lambda^{< \kappa} = \lambda^+$.

The following theorem can be seen as an extension of a theorem of Abe in [3]. This shows that ineffability, the Shelah property, and complete ineffability of $\mathcal{P}_\kappa \lambda$ can be the same when $\text{cf}(\lambda) < \kappa$, and the corresponding ideals can be precipitous. This contrasts with the fact that the completely ineffable ideal is not precipitous if $\text{cf}(\lambda) \geq \kappa$.

**Theorem 1.3.** Assume $\lambda$ is a strong limit cardinal with $\text{cf}(\lambda) < \kappa$. Then

1. $\text{NSh}_{\kappa \lambda} = \text{NAIn}_{\kappa \lambda} = \text{NIn}_{\kappa \lambda} = \text{NCIn}_{\kappa \lambda}$, and
2. if $\kappa$ is $\lambda$-ineffable and $\mu > \lambda$ is a Woodin cardinal, then, in $V^{\text{Col}(\lambda^+, < \mu)}$, $\kappa$ remains $\lambda$-ineffable and $\text{NSh}_{\kappa \lambda} = \text{NAIn}_{\kappa \lambda} = \text{NIn}_{\kappa \lambda} = \text{NCIn}_{\kappa \lambda}$ is precipitous.

$\text{NSh}_{\kappa \lambda}, \text{NAIn}_{\kappa \lambda}, \text{NIn}_{\kappa \lambda}$, and $\text{NCIn}_{\kappa \lambda}$ are ideals corresponding to the Shelah property, almost ineffability, ineffability, and complete ineffability respectively. To prove Theorem 1.2, we give a simple characterization of $\text{NIn}_{\kappa \lambda}$. Using this, we have the consistency of the statement that $\text{cf}(\lambda) < \kappa$ and $\kappa$ is completely $\lambda$-ineffable but not mildly $\lambda^{< \kappa}$-ineffable.

Baumgartner defined indescribability of $\mathcal{P}_\kappa \lambda$ and Carr [8] showed that $\Pi_1^1$-indescribability is equivalent to the Shelah property if $\text{cf}(\lambda) \geq \kappa$. The next theorem shows that, if $\text{cf}(\lambda) < \kappa$, this equivalence can be false. Moreover $\Pi_1^1$-indescribability can be much stronger than ineffability.

**Theorem 1.4.** Assume $2^\lambda = \lambda^{< \kappa}$. Then $\text{NIn}_{\kappa \lambda} \subseteq \Pi_{\kappa \lambda}$ holds, and if $\kappa$ is $\lambda$-ineffable then $\text{NIn}_{\kappa \lambda} \not\subseteq \Pi_{\kappa \lambda}$.

$\Pi_{\kappa \lambda}$ is the ideal corresponding to $\Pi_1^1$-indescribability.

Part (2) of Theorem 1.2 and Theorem 1.4 are answers to questions of Abe in [2].
2. Preliminaries.

We refer the reader to Kanamori [18] for general background and basic notation. Throughout this paper, \( \kappa \) denotes an inaccessible cardinal and \( \lambda \) denotes a cardinal equal to or greater than \( \kappa \). In fact, the properties mentioned in this paper imply the inaccessibility of \( \kappa \).

Recall that \( \P_{\kappa \lambda} = \{x \subseteq \lambda : |x| < \kappa\} \).

In this paper, an ideal (respectively a filter) over \( \P_{\kappa \lambda} \) means a \( \kappa \)-complete fine ideal (respectively filter) over \( \P_{\kappa \lambda} \). That is, \( I \subseteq \P(\P_{\kappa \lambda}) \) is called an ideal over \( \P_{\kappa \lambda} \) if the following hold:

1. \( \forall X \in I \forall Y \subseteq X (Y \in I) \),
2. \( \forall \gamma < \kappa \forall \{X_\xi : \xi < \gamma\} \subseteq I (\bigcup_{\xi < \gamma} X_\xi \in I) \),
3. \( \forall a \in \P_{\kappa \lambda} (\{x \in \P_{\kappa \lambda} : a \nsubseteq x\} \in I) \).

For an ideal \( I \) over \( \P_{\kappa \lambda} \), \( I^* \) denotes the dual filter of \( I \), and \( I^+ = \P(\P_{\kappa \lambda}) \setminus I \). An element of \( I^+ \) is called an \( I \)-positive set. For \( X \in I^+ \), let \( I|X = \{Y \subseteq \P_{\kappa \lambda} : Y \cap X \in I\} \). \( I|X \) is the restriction of \( I \) to \( X \).

An ideal \( I \) over \( \P_{\kappa \lambda} \) is called normal if for every \( X \in I^+ \) and function \( f : X \to \P_{\kappa \lambda} \) with \( \forall x \in X (f(x) \in x) \), there exists \( \alpha < \lambda \) such that \( \{x \in X : f(x) = \alpha\} \in I^+ \). In a trivial sense, the non-proper ideal is normal.

For a set \( X \subseteq \P_{\kappa \lambda} \), \( X \) is unbounded if \( \forall x \in \P_{\kappa \lambda} \exists y \in X (x \subseteq y) \). \( X \) is closed if for every \( \gamma < \kappa \) and \( \subseteq \)-increasing sequence \( \langle x_\xi : \xi < \gamma\rangle \) in \( X \), \( \bigcup_{\xi < \gamma} x_\xi \in X \). A closed and unbounded set is called club. A set \( S \subseteq \P_{\kappa \lambda} \) is stationary if \( S \) intersects any club set.

The following fact is well-known:

**Fact 2.1.** For \( X \subseteq \P_{\kappa \lambda} \), the following are equivalent:

1. \( X \) is stationary in \( \P_{\kappa \lambda} \),
2. for every \( f : \lambda \times \lambda \to \lambda \), there exists \( x \in X \) such that \( x \cap \kappa \in \kappa \) and \( f^{\omega}(x \times x) \subseteq x \), and
3. for every \( f : \lambda \times \lambda \to \P_{\kappa \lambda} \), there exists \( x \in X \) such that \( \bigcup f^{\omega}(x \times x) \subseteq x \).

The non-stationary ideal over \( \P_{\kappa \lambda} \), \( \NS_{\kappa \lambda} \), is the set of all \( X \subseteq \P_{\kappa \lambda} \) such that \( X \) is non-stationary in \( \P_{\kappa \lambda} \).

**Fact 2.2.** \( \NS_{\kappa \lambda} \) is the minimal normal ideal over \( \P_{\kappa \lambda} \).

**Definition 2.3.** For \( x, y \in \P_{\kappa \lambda} \), we define \( x < y \) if \( x \subseteq y \) and \( |x| < |y \cap \kappa| \).

For \( X \subseteq \P_{\kappa \lambda} \), a function \( f : X \to \P_{\kappa \lambda} \) is said to be \( < \)-regressive if \( f(x) < x \) for every \( x \in X \) with \( x \cap \kappa \neq \emptyset \).

An ideal \( I \) over \( \P_{\kappa \lambda} \) is strongly normal if the following condition is satisfied:
For every $X \in I^+$ and $\prec$-regressive function $f : X \to \mathcal{P}_\kappa \lambda$, there exists $y \in \mathcal{P}_\kappa \lambda$ such that \{ $x \in X : f(x) = y$ \} $\in I^+$.

The non-proper ideal is trivially strongly normal.
For $x \in \mathcal{P}_\kappa \lambda$, we denote the set \{ $y \in \mathcal{P}_\kappa \lambda : y < x$ \} by $\mathcal{P}_{\kappa \cap x}$. If $x \cap \kappa$ is a regular cardinal, then properties of $\mathcal{P}_\kappa \lambda$ correspond to the properties of $\mathcal{P}_{\kappa \cap x}$. For example, $X \subseteq \mathcal{P}_{\kappa \cap x}$ is stationary if for all $f : x \times x \to \mathcal{P}_{\kappa \cap x}$ there exists $y \in X$ such that $\bigcup f^\sim(y \times y) \subseteq y$.

For $f : \mathcal{P}_\kappa \lambda \to \mathcal{P}_\kappa \lambda$, we let $C_f = \{ x \in \mathcal{P}_\kappa \lambda : f^\sim \mathcal{P}_{\kappa \cap x} \subseteq \mathcal{P}_{\kappa \cap x} \}$.

**Definition 2.4.** WNS$_{\kappa \lambda}$ = \{ $X \subseteq \mathcal{P}_\kappa \lambda : \exists f : \mathcal{P}_\kappa \lambda \to \mathcal{P}_\kappa \lambda (C_f \cap X = \emptyset)$ \}.

**Fact 2.6** (Abe [1]).
\begin{enumerate}
  \item \{ $x \in \mathcal{P}_\kappa \lambda : e(x) \cap \lambda = x$ \} $\in$ WNS$_{\kappa \lambda}$.
  \item \{ $x \in \mathcal{P}_\kappa \lambda^{<\kappa} : e(x) \cap \lambda = x$ \} $\in$ WNS$_{\kappa \lambda^{<\kappa}}$.
  \item For $X \subseteq \mathcal{P}_\kappa \lambda$, $X \in$ WNS$_{\kappa \lambda}$ if and only if $e^\sim X \in$ WNS$_{\kappa \lambda^{<\kappa}}$.
\end{enumerate}

Ineffability and the Shelah property of $\mathcal{P}_\kappa \lambda$ are defined in the following.

**Definition 2.7** (Carr [8], [9], Jech [13]). Let $X$ be a subset of $\mathcal{P}_\kappa \lambda$.
\begin{enumerate}
  \item $X$ is **ineffable** (respectively *almost ineffable*) if, for all $\langle a_x : x \in X \rangle$ with $a_x \subseteq x$, there exists $A \subseteq \lambda$ such that \{ $x \in X : A \cap x = a_x$ \} is stationary (respectively unbounded).
  \item $X$ has the **Shelah property**, or simply $X$ is Shelah if, for all $\langle f_x : x \in X \rangle$ with $f_x : x \to \lambda$, there exists $f : \lambda \to \lambda$ such that, for all $y \in \mathcal{P}_\kappa \lambda$, the set \{ $x \in X : f^\sim y \cap = f_x^\sim y$ \} is unbounded.
  \item $X$ is **mildly ineffable** if, for all $\langle a_x : x \in X \rangle$ with $a_x \subseteq x$, there exists $A \subseteq \lambda$ such that, for all $y \in \mathcal{P}_\kappa \lambda$, the set \{ $x \in X : A \cap y = a_x \cap y$ \} is unbounded.
\end{enumerate}
We say that $\kappa$ is $\lambda$-ineffable (almost $\lambda$-ineffable, $\lambda$-Shelah, mildly $\lambda$-ineffable respectively) if $\mathcal{P}_\kappa \lambda$ is ineffable (almost ineffable, Shelah, mildly ineffable respectively).

Notice that the Shelah property implies mildly ineffability,

$\text{NIn}_{\kappa \lambda}$ (respectively $\text{NAIn}_{\kappa \lambda}$, $\text{NSh}_{\kappa \lambda}$) is the set of all $X \subseteq \mathcal{P}_\kappa \lambda$ such that $X$ is not ineffable (respectively almost ineffable, Shelah).

**FACT 2.8** (Carr [8], [9]).

1. $\kappa$ is weakly compact $\iff$ $\kappa$ is $\kappa$-Shelah $\iff$ $\kappa$ is mildly $\kappa$-ineffable.
2. $\kappa$ is ineffable (respectively almost ineffable) $\iff$ $\kappa$ is $\kappa$-ineffable (respectively almost $\kappa$-ineffable).
3. $\text{NSh}_{\kappa \lambda}$, $\text{NAIn}_{\kappa \lambda}$, and $\text{NIn}_{\kappa \lambda}$ are normal ideals over $\mathcal{P}_\kappa \lambda$. Moreover these are strongly normal if $\text{cf}(\lambda) \geq \kappa$.
4. If $\kappa$ is mildly $\lambda$-ineffable, then, for $X \subseteq \mathcal{P}_\kappa \lambda$, $X$ is mildly ineffable if and only if $X$ is unbounded.

**FACT 2.9** (Carr [9]). For $X \subseteq \mathcal{P}_\kappa \lambda$, $X$ is ineffable (almost ineffable) if and only if, for all $\langle f_x : x \in X \rangle$ with $f_x : x \to x$, there exists $f : \lambda \to \lambda$ such that $\{x \in X : f|y = f_x\} \subseteq X$ is stationary (unbounded). Hence $\text{NSh}_{\kappa \lambda} \subseteq \text{NAIn}_{\kappa \lambda} \subseteq \text{NIn}_{\kappa \lambda}$ holds.

The next fact follows from the normality of $\text{NSh}_{\kappa \lambda}$ and a standard coding argument.

**FACT 2.10.** For $X \subseteq \mathcal{P}_\kappa \lambda$, $X$ is Shelah if and only if, for any $\langle f_x : x \in X \rangle$ with $f_x : x \to x$ and $\langle g_x : x \in X \rangle$ with $g_x : x \to x$, there exists $f : \lambda \to \lambda$ and $g : \lambda \to \lambda$ such that $\{x \in X : f|y = f_x|y, g|y = g_x|y\}$ is unbounded for all $y \in \mathcal{P}_\kappa \lambda$.

For an infinite set $X$, let $[X]^\omega$ be the set of all $x \subseteq X$ such that $|x| = \omega$. $F : [X]^\omega \to X$ is called an $\omega$-Jonsson function for $X$ if the following holds: There is no $Y \subsetneq X$ such that $F^\omega[Y]^\omega \subseteq Y$ and $|Y| = |X|$. It is well-known that every infinite set $X$ has an $\omega$-Jonsson function for $X$ (see Erdős-Hajnal [11]).

**FACT 2.11** (Abe [2], Johnson [16]). Let $\mu$ be a cardinal with $\mu \leq \lambda$.

1. If $F : [\mu]^\omega \to \mu$ is an $\omega$-Jonsson function for $\mu$, then $\{x \in \mathcal{P}_\kappa \lambda : F^\omega[x \cap \mu]^\omega \subseteq x \cap \mu$ and $F|[x \cap \mu]^\omega$ is $\omega$-Jonsson for $x \cap \mu \in \text{NSh}_{\kappa \lambda}^*$
2. If $\mu$ is regular, then $\{x \in \mathcal{P}_\kappa \lambda : \text{ot}(x \cap \mu)$ is regular $\} \subset \text{NSh}_{\kappa \lambda}^*$, where $\text{ot}(x)$ denotes the order type of $x$.

### 3. Basic properties of ineffabilities.

In this section, we will show some basic properties of ineffabilities of $\mathcal{P}_\kappa \lambda$. 


First we prove the strong normality of $\text{NSh}_{\kappa \lambda}$, $\text{NAIn}_{\kappa \lambda}$, and $\text{NIn}_{\kappa \lambda}$ without the condition that $\text{cf}(\lambda) \geq \kappa$.

**Proposition 3.1.** $\text{NSh}_{\kappa \lambda}$, $\text{NAIn}_{\kappa \lambda}$, and $\text{NIn}_{\kappa \lambda}$ are strongly normal ideals.

**Proof.** We will only show the strong normality of $\text{NSh}_{\kappa \lambda}$. The others can be verified by a similar argument. Let $X \subseteq \text{NSh}_{\kappa \lambda}$ and let $g : X \to \mathcal{P}_{\kappa \lambda}$ be a $<\text{-regressive}$ function. By the normality of $\text{NSh}_{\kappa \lambda}$, we may assume that there exists $\mu < \kappa$ such that $\text{ot}(g(x)) = \mu$ for all $x \in X$. Furthermore we may assume $\mu \subseteq x$ for all $x \in X$. For each $x \in X$, let $h_x : \mu \to x$ be an increasing enumerating map of $g(x)$.

Let $X_a = \{x \in X : g(x) = a\}$. Suppose $X_a \in \text{NSh}_{\kappa \lambda}$ for all $a \in \mathcal{P}_{\kappa \lambda}$. For each $a \in \mathcal{P}_{\kappa \lambda}$, let $\langle f^a_x : x \in X_a \rangle$ be a counterexample to the Shelah property of $X_a$. Consider the sequences $\langle f^g_x : x \in X \rangle$ and $\langle h_x : x \in X \rangle$. By the Shelah property of $X$, there exist $f : \lambda \to \lambda$ and $h : \mu \to \lambda$ such that $\{x \in X : f|y = f^g_x|y, h|y = h_x|y\}$ is unbounded for all $y \in \mathcal{P}_{\kappa \lambda}$. Let $b = h \upharpoonright \mu \in \mathcal{P}_{\kappa \lambda}$. We will prove that $\{x \in X_b : f|y = f^g_x|y\}$ is unbounded for all $y \in \mathcal{P}_{\kappa \lambda}$, which is a contradiction. Let $y \in \mathcal{P}_{\kappa \lambda}$. We may assume that $\mu \subseteq y$. Then $\{x \in X : f|y = f^g_x|y, h|y = h_x|y\}$ is unbounded. Let $x \in X$ be such that $y \subseteq x$, $h|y = h_x|y$, and $f|y = f^g_x|y$. Since $\mu \subseteq y$, we have $h = h|y = h_x|y = h_x$, and this means that $g(x) = b$. Therefore $f|y = f^g_x|y = f^g_b|y$ holds. □

Next we show a variation of $(\text{UP})_{\kappa \lambda X}$ in Carr [8] from mild ineffability. We will use this in the next section.

Recall that a filter over $\mathcal{P}_{\kappa \lambda}$ means a $\kappa$-complete fine filter.

For a regular uncountable cardinal $\theta$, $H_\theta$ denotes the set of all $x$ such that $|TC(x)| < \theta$ where $TC(x)$ is the minimal transitive set containing $x$. It is known that $H_\theta$ is a model of ZFC–Power Set Axiom.

**Proposition 3.2.** Let $\theta$ be a sufficiently large regular cardinal, and let $N$ be any expansion of $(H_\theta, \in, \kappa, \lambda)$. Let $X \subseteq \mathcal{P}_{\kappa \lambda}$ and $M < N$ be such that $X \in M$ and $|M| = \lambda \subseteq M$. Then $X$ is mildly ineffable if and only if there exists a proper filter $F$ over $\mathcal{P}_{\kappa \lambda}$ such that $X \in F$ and $F$ is an $M$-ultrafilter. Here “$F$ is an $M$-ultrafilter” means that, for all $X \in M \cap \mathcal{P}(\mathcal{P}_{\kappa \lambda})$, either $X \in F$ or $\mathcal{P}_{\kappa \lambda} \setminus X \in F$.

**Proof.** Assume $X$ is mildly ineffable. We will construct an $M$-ultrafilter. Let $\langle X_\alpha : \alpha < \lambda \rangle$ be an enumeration of $\mathcal{P}(\mathcal{P}_{\kappa \lambda}) \cap M$. For each $x \in X$, let $a_x = \{\alpha \in x : x \in X_\alpha\}$. Then, by the mild ineffability of $X$, there exists $A \subseteq \lambda$ such that $\{x \in X : a_x \cap y = A \cap y\}$ is unbounded for all $y \in \mathcal{P}_{\kappa \lambda}$. Let $F$ be the filter over $\mathcal{P}_{\kappa \lambda}$ generated by $\{X \cap \bigcap_{\alpha \in y} X_\alpha : y \in \mathcal{P}_{\kappa A}\}$, that is $Y \in F$ if and only if $X \cap \bigcap_{\alpha \in y} X_\alpha \subseteq Y$ for some $y \in \mathcal{P}_{\kappa A}$. It is clear that $F$ is a $\kappa$-complete
filter over $\mathcal{P}_\kappa \lambda$ and $X \in F$. Notice that $X_\alpha \in F$ for all $\alpha \in A$. We check that $F$ is a proper fine filter and an $M$-ultrafilter.

**Finess.** Let $\alpha < \lambda$. Since $\alpha \in \lambda \subseteq M$, there exists $\beta < \lambda$ such that $X_\beta = \{ x \in \mathcal{P}_\kappa \lambda : \alpha \in x \}$. Take $x \in X$ such that $\alpha, \beta \in x$ and $A \cap \{ \beta \} = a_x \cap \{ \beta \}$. Since $\alpha \in x$, we have $x \in X_\beta$, so $\beta \in a_x$ and $\beta \in A$.

**Properness.** It is enough to show that $X \cap \bigcap_{\alpha \in y} X_\alpha \neq \emptyset$ for all $y \in \mathcal{P}_\kappa A$. For $y \in \mathcal{P}_\kappa A$, we can pick $x \in X$ such that $y \subseteq x$ and $a_x \cap y = A \cap y = y$. Then $x \in \bigcap_{\alpha \in a_x} X_\alpha \subseteq \bigcap_{\alpha \in y} X_\alpha$, thus $X \cap \bigcap_{\alpha \in y} X_\alpha \neq \emptyset$.

Now we check that $F$ is an $M$-ultrafilter. Let $Y \in \mathcal{P}(\mathcal{P}_\kappa \lambda) \cap M$. Then there are $\alpha, \beta < \lambda$ such that $X_\alpha = Y$ and $X_\beta = \mathcal{P}_\kappa \lambda \setminus Y$. Take $x \in \mathcal{P}_\kappa \lambda$ such that $\alpha, \beta \in x$ and $A \cap \{ \alpha, \beta \} = a_x \cap \{ \alpha, \beta \}$. Then either $x \in X_\alpha$ or $x \in X_\beta$ hold, hence we have $\alpha \in a_x$ or $\beta \in a_x$. Thus $\alpha \in A$ or $\beta \in A$.

To show the converse, assume that there exists a proper $M$-ultrafilter $F$. By the elementarity of $M$, it is enough to show that, for all $\langle a_x : x \in \mathcal{P}_\kappa \lambda \rangle \in M$ with $a_x \subseteq x$, there exists $A \subseteq \lambda$ such that $\{ x \in X : a_x \cap y = A \cap y \}$ is unbounded for all $y \in \mathcal{P}_\kappa \lambda$. Fix $\langle a_x : x \in \mathcal{P}_\kappa \lambda \rangle \in M$. Since $\lambda \subseteq M$ and $F$ is an $M$-ultrafilter with $X \in F$, for each $\alpha < \lambda$, either $\{ x \in X : \alpha \in a_x \} \in F$ or $\{ x \in \mathcal{P}_\kappa \lambda : \alpha \notin a_x \} \in F$. Let $A = \{ \alpha < \lambda : \{ x \in X : \alpha \in a_x \} \in F \}$. Then it is not hard to see that $\{ x \in X : a_x \cap y = A \cap y \} \in F$, so the set is unbounded for all $y \in \mathcal{P}_\kappa \lambda$. $\square$

4. The Shelah property, mild ineffability, and the size of $\mathcal{P}_\kappa \lambda$.

Johnson [16] showed that $\lambda^{<\kappa} = \lambda$ holds if $\kappa$ is $\lambda$-Shelah and $\text{cf}(\lambda) \geq \kappa$. We see that the same result holds for mild ineffability, and moreover $\lambda^{<\kappa} = \lambda^+$ holds if $\kappa$ is $\lambda$-Shelah and $\text{cf}(\lambda) < \kappa$.

**Proposition 4.1.** Assume $\kappa$ is mildly $\lambda$-ineffable and $\text{cf}(\lambda) \geq \kappa$. Then $\lambda^{<\kappa} = \lambda$.

**Proof.** Mild ineffability is downward closed, that is, if $\mathcal{P}_\kappa \lambda$ is mildly ineffable and $\kappa \leq \lambda' < \lambda$ then $\mathcal{P}_\kappa \lambda'$ is mildly ineffable. Thus it is enough to prove the case when $\lambda$ is regular. We will show that there exists an unbounded subset $X$ of $\mathcal{P}_\kappa \lambda$ such that $|X| = \lambda$. If this can be shown, then $\mathcal{P}_\kappa \lambda = \bigcup \{ \mathcal{P}(x) : x \in X \}$, which proves $\lambda^{<\kappa} \leq \lambda \cdot \kappa^{<\kappa} = \lambda$.

Let $\theta$ be a sufficiently large regular cardinal. Let $M < \langle H_\theta, \in, \kappa, \lambda \rangle$ be such that $\lambda \subseteq M$ and $|M| = \lambda$. Then, by Proposition 3.2, we can find a proper $\kappa$-complete fine $M$-ultrafilter $F$ over $\mathcal{P}_\kappa \lambda$. $M$ is not transitive, but we can take an ultrapower $M$ by $F$ in the usual way. Moreover it is not hard to see that Los’s theorem holds between $M$ and $\text{Ult}(M, F)$: For any formula $\varphi$ and $f_1, \ldots, f_n \in M \cap \mathcal{P}_\kappa \lambda M$, $\{ x \in \mathcal{P}_\kappa \lambda : M \vDash \varphi(f_1(x), \ldots, f_n(x)) \} \in F$ if and only if $\text{Ult}(M, F)$
\[
\varphi([f_1], \ldots, [f_n]), \text{ where } [f] \text{ is an equivalence class of } f. \text{ Since } F \text{ is } \kappa\text{-complete in } V, \text{ Ult}(M, F) \text{ is well-founded. Let } N \text{ be the transitive collapse of Ult}(M, F). \text{ Now we identify } N \text{ with Ult}(M, F). \text{ Let } j : M \to N \text{ be the corresponding elementary embedding. Since } F \text{ is fine, we have that } j^{\ast} \iota \subseteq [f_\text{id}], \text{ where } f_\text{id} \text{ is the identity map on } \mathcal{P}_\kappa \lambda. \text{ Furthermore } F \text{ is } \kappa\text{-complete and } ||f_\text{id}||^N < j(\kappa), \text{ hence the critical point of } j \text{ is } \kappa. \text{ Since } \sup(j^{\ast} \lambda) \leq \sup([f_\text{id}]) \text{ and } \{x \in \mathcal{P}_\kappa \lambda : \sup(x) < \lambda \} \in F, \text{ we have } \sup(j^{\ast} \lambda) < j(\lambda). \text{ Notice that we do not require that } j^{\ast} \lambda \subseteq N, \text{ but we have } j^{\ast} x \in N \text{ for all } x \in \mathcal{P}_\kappa \lambda \cap M.

We check that } j^{\ast} \lambda \text{ is } < \kappa\text{-closed, that is, for all } c \subseteq j^{\ast} \lambda, \sup(c) \in j^{\ast} \lambda \text{ if } \text{ot}(c) < \kappa. \text{ Let } \alpha < \lambda \text{ be the minimal ordinal such that } \sup(c) \leq j(\alpha). \text{ Then } \sup(c) = \sup(j^{\ast} \alpha). \text{ Hence } \text{cf}(\alpha) < \kappa. \text{ Take } d \in M \text{ such that } \text{ot}(d) = \text{cf}(\alpha) \text{ and } d \text{ is unbounded in } \alpha. \text{ Then } j(\alpha) = \sup(j(d)) = \sup(j^{\ast} d) = \sup(j^{\ast} \alpha) = \sup(c). \text{ Therefore we have } \sup(c) \in j^{\ast} \lambda.

Now take an arbitrary stationary subset } S \text{ of } \{\alpha < \lambda : \text{cf}(\alpha) < \kappa\} \text{ with } S \in M.

**Claim 4.2.** } j(S) \cap \sup(j^{\ast} \lambda) \text{ is stationary in } \sup(j^{\ast} \lambda) \text{ in } V.

**Proof of the Claim 4.2.** \text{ Let } C \text{ be a } < \kappa\text{-club subset of } \sup(j^{\ast} \lambda). \text{ Since } j^{\ast} \lambda \text{ is also } < \kappa\text{-closed, we may assume that } C \subseteq j^{\ast} \lambda. \text{ Let } D = j^{\ast}^{-1} C. \text{ Then } D \text{ is unbounded in } \lambda. \text{ Thus there exists } \alpha \in S \text{ such that } D \cap \alpha \text{ is unbounded in } \alpha. \text{ Since } \alpha \in M, \text{ we can take an unbounded subset } b \text{ of } \alpha \text{ such that } b \in M \text{ and } \text{ot}(b) = \text{cf}(\alpha). \text{ Then } j(\alpha) = \sup(j(b)) = \sup(j^{\ast} b) = \sup(j^{\ast} \alpha). \text{ Since } D \cap \alpha \text{ is unbounded in } \alpha, \text{ hence } j^{\ast}(D \cap \alpha) = j^{\ast} D \cap j(\alpha) \text{ is unbounded in } j(\alpha). \text{ Since } j^{\ast} D \subseteq C, \text{ we have } j(\alpha) \in C. \text{ Hence we have } j(\alpha) \in j(S) \cap C. \qed

Now fix pairwise disjoint stationary subsets } \{S_\alpha : \alpha < \lambda\} \text{ of } \{\beta < \lambda : \text{cf}(\beta) < \kappa\} \text{ with } \{S_\alpha : \alpha < \lambda\} \subseteq M. \text{ For } \beta < \lambda \text{ with } \omega < \text{cf}(\beta) < \kappa, \text{ let } c_\beta = \{\alpha < \beta : S_\alpha \cap \beta \text{ is stationary in } \beta\}. \text{ Since the } S_\alpha \text{'s are pairwise disjoint, we have } \text{cf}(\beta) \leq \text{cf}(\beta) < \kappa. \text{ Now let } X = \{c_\beta : \beta < \lambda, \omega < \text{cf}(\beta) < \kappa\}. \text{ Then } X \text{ is a subset of } \mathcal{P}_\kappa \lambda \text{ with } |X| = \lambda. \text{ Finally we show that } X \text{ is unbounded to complete the proof.}

\text{Let } f \text{ be a function on } \mathcal{P}_\kappa \lambda \text{ such that } f \in M \text{ and } [f] = \sup(j^{\ast} \lambda). \text{ Since } j^{\ast} \lambda \subseteq [f_\text{id}], [f_\text{id}] \cap [f] \text{ is unbounded in } [f]. \text{ Because } ||f_\text{id}||^N < j(\kappa), \text{ cf}^N([f]) < j(\kappa) \text{ and so } \{x \in \mathcal{P}_\kappa \lambda : \text{cf}(f(x)) < \kappa\} \subseteq F. \text{ Take an arbitrary } y \in \mathcal{P}_\kappa \lambda. \text{ Let } \alpha \in y. \text{ By Claim 4.2, } j(S_\alpha) \cap \sup(j^{\ast} \lambda) \text{ is stationary. Hence } \{x \in \mathcal{P}_\kappa \lambda : S_\alpha \cap f(x) \text{ is stationary in } f(x)\} \subseteq F. \text{ By the } \kappa\text{-completeness of } F, \text{ we have } \{x \in \mathcal{P}_\kappa \lambda : \forall \alpha \in y (S_\alpha \cap f(x) \text{ is stationary in } f(x)), c_f(x) < \kappa\} \subseteq F. \text{ Therefore we can take } x \in \mathcal{P}_\kappa \lambda \text{ such that } \omega < \text{cf}(f(x)) < \kappa \text{ and } y \subseteq c_f(x) \in X. \text{ This shows } X \text{ is unbounded.} \qed

\text{The proof of the above proposition shows that a simultaneous stationary reflection principle of } \{\alpha < \lambda : \text{cf}(\alpha) < \kappa\} \text{ follows from mild } \lambda\text{-ineffability. The}
following is an extension of Johnson’s result [15]:

**Proposition 4.3.** Assume \( \lambda \) is regular and \( \kappa \) is mildly \( \lambda \)-ineffable. Let \( \delta < \kappa \) and \( \langle S_\alpha : \alpha < \delta \rangle \) be stationary subsets of \( \{ \beta < \lambda : \text{cf}(\beta) < \kappa \} \). Then, for every \( \gamma < \kappa \), there exists \( \beta < \lambda \) such that \( \gamma < \text{cf}(\beta) \) and \( S_\alpha \cap \beta \) is stationary in \( \beta \) for all \( \alpha < \delta \).

Now we prove that the Shelah property of \( \mathcal{P}_\kappa \lambda \) with \( \text{cf}(\lambda) < \kappa \) implies that \( \lambda^{<\kappa} = \lambda^+ \).

**Proposition 4.4.** Assume \( \kappa \) is \( \lambda \)-Shelah and \( \text{cf}(\lambda) < \kappa \). Then \( \lambda^{<\kappa} = \lambda^+ \).

**Proof.** This proof is based on an argument of Tryba [21]. First we introduce a notion of *scale*. Fix an increasing sequence of regular cardinals \( \langle \lambda_i : i < \text{cf}(\lambda) \rangle \) which converges to \( \lambda \). We denote \( \Pi_i^{<\text{cf}(\lambda)} \lambda_i \) by \( \Pi_i \). For \( f, g \in \Pi_i \), let \( f <^* g \) if and only if \( \{ i < \text{cf}(\lambda) : f(i) \geq g(i) \} \) is bounded in \( \text{cf}(\lambda) \). We say that \( \langle f_\xi : \xi < \lambda^+ \rangle \) is a *scale* for \( \Pi_i \) if the following hold:

1. \( f_\xi \in \Pi_i \) for all \( \xi < \lambda^+ \),
2. for \( \xi < \eta < \lambda^+ \), \( f_\xi <^* f_\eta \), and
3. for all \( f \in \Pi_i \), there exists \( \xi < \lambda^+ \) such that \( f <^* f_\xi \).

It is a basic fact of Shelah’s PCF-theory that there exists a sequence of regular cardinals \( \langle \lambda_i : i < \text{cf}(\lambda) \rangle \) and a scale \( \langle f_\xi : \xi < \lambda^+ \rangle \) for \( \Pi_i \) (see Burke-Magidor [6] or Shelah [19]).

Now fix an increasing sequence of regular cardinals \( \langle \lambda_i : i < \text{cf}(\lambda) \rangle \) which converges to \( \lambda \) and a scale \( \langle f_\alpha : \alpha < \lambda^+ \rangle \) for \( \Pi_i \). For each \( \lambda_i \), fix an \( \omega \)-Jonsson function \( h_i : [\lambda_i]^{<\omega} \to \lambda_i \). Let \( e : \mathcal{P}_\kappa \lambda \to \mathcal{P}_\kappa \lambda^{<\kappa} \) be a canonical map. Let \( X \subseteq \mathcal{P}_\kappa \lambda \) be the set of all \( x \in \mathcal{P}_\kappa \lambda \) such that:

- \( x \cap \kappa \) is an inaccessible > \( \text{cf}(\lambda) \),
- \( \text{ot}(x \cap \lambda_i) \) is regular for all \( i < \text{cf}(\lambda) \),
- \( h_i([x \cap \lambda_i]^{<\omega}) \) is \( \omega \)-Jonsson for \( x \cap \lambda_i \), and
- \( e(x) \cap \lambda = x \).

By Fact 2.6, 2.11, and Proposition 3.1, we have \( X \in \text{NSh}^{\text{**}}_{\kappa, \lambda} \). We consider the set \( e``X = \{ e(x) : x \in X \} \). Note that this set is a WNS\( \kappa, \lambda^{<\kappa} \)-positive set, so it is stationary in \( \mathcal{P}_\kappa \lambda^{<\kappa} \). Fix a sufficiently large regular cardinal \( \theta \) and let \( C = \{ M \cap \lambda^{<\kappa} : M \ni \langle H_\theta, \in \rangle, |M| < \kappa, M \cap \lambda^{<\kappa} \in e``X, \{ \lambda_i : i < \text{cf}(\lambda) \}, \langle f_\alpha : \alpha < \lambda^+ \rangle, \pi, e \} \subseteq M \) and \( M \cap \lambda^{<\kappa} \) is \( \pi \)-closed}. Then \( C \) is stationary in \( \mathcal{P}_\kappa \lambda^{<\kappa} \). Note that if \( M \cap \lambda^{<\kappa} \in C \) then \( M \cap \lambda \in X \). Moreover by the definition of \( e \), we have that \( \{ M \cap \lambda \}^{<\text{cf}(\kappa)} \subseteq M \).

The following claim assures that \( \{ x \cap \lambda^+ : x \in C \} \) is an unbounded subset of \( \mathcal{P}_\kappa \lambda^+ \) with size \( \lambda^+ \), which completes the proof.
CLAI M 4.5. Let $M \cap \lambda^{<\kappa} \in C$ and $M' \cap \lambda^{<\kappa} \in C$. If $\sup(M \cap \lambda^+)$
$= \sup(M' \cap \lambda^+)$, then $M \cap \lambda^+ = M' \cap \lambda^+$.

PROOF OF THE CLAIM 4.5. Let $M \cap \lambda^{<\kappa}, M' \cap \lambda^{<\kappa} \in C$ be such that
$\sup(M \cap \lambda^+) = \sup(M' \cap \lambda^+)$. Let $N = M \cap M'$. Note that $\sup(N \cap \lambda^+)$
$= \sup(M \cap \lambda^+)$ and $N \cap \lambda_i$ is closed under $h_i$.

SUBCLAIM 4.6. If $M \cap \lambda = N \cap \lambda$, then $M \cap \lambda^+ = N \cap \lambda^+$.

PROOF OF THE SUBCLAIM 4.6. Choose any $\alpha \in (M \cap \lambda^+) \setminus \lambda$. We have
$\beta \in N \cap \lambda^+$ such that $\alpha < \beta$. Let $\tau \in N$ be a bijection from $\lambda$ to $\beta$. Since $\alpha < \beta$
and $\tau \in M$, there exists $\delta \in M \cap \lambda = N \cap \lambda$ such that $\pi(\delta) = \alpha$, hence $\alpha \in N \cap \lambda^+$.

We show $M \cap \lambda = N \cap \lambda$. To show this, we need the following claim.

SUBCLAIM 4.7. \{ $i < \text{cf}(\lambda) : \sup(N \cap \lambda_i) < \sup(M \cap \lambda_i)$ \} is bounded in $\text{cf}(\lambda)$.

PROOF OF THE SUBCLAIM 4.7. Assume otherwise. Then define $f \in \Pi\lambda_i$
by $f(i) \in (M \cap \lambda_i) \setminus \sup(N \cap \lambda_i)$ if $\sup(N \cap \lambda_i) < \sup(M \cap \lambda_i)$ and $f(i) = 0$
otherwise. Then $f \in M$ since $M \cap \lambda$ is closed under $< (M \cap \kappa)$-sequences. Because
$\langle f_\alpha : \alpha < \lambda \rangle$ is a scale for $\Pi\lambda_i$, there exists $\alpha \in M \cap \lambda^+$ such that $f <^* f_\alpha$, that is,
$\{ i < \text{cf}(\lambda) : f(i) \geq f_\alpha(i) \}$ is bounded in $\text{cf}(\lambda)$. Since $\sup(M \cap \lambda^+) = \sup(N \cap \lambda^+)$,
there exists $\beta \in N \cap \lambda^+$ such that $\alpha < \beta$. $f < f_\alpha <^* f_\beta$, so we can take $i < \text{cf}(\lambda)$
such that $f(i) \in (M \cap \lambda_i) \setminus \sup(N \cap \lambda_i)$ and $f(i) < f_\beta(i)$. However $f_\beta \in N$, hence
$f_\beta(i) \in N \cap \lambda_i$. This is a contradiction.

We return to the proof of the Claim. Let $i < \text{cf}(\lambda)$ be such that $\sup(M \cap \lambda_i)$
$= \sup(N \cap \lambda_i)$. Since $\text{ot}(M \cap \lambda_i)$ is regular, $\text{ot}(N \cap \lambda_i)$ is regular. Thus $|M \cap \lambda_i|
= |N \cap \lambda_i|$. Since $h_i[|M \cap \lambda_i|^{\omega} \text{ is } \omega$-Jonsson and $N \cap \lambda_i$ is closed under $h_i$, we have
$M \cap \lambda_i = N \cap \lambda_i$. There are unboundedly many such $i$, hence $M \cap \lambda = N \cap \lambda$. We
can show that $M' \cap \lambda = N \cap \lambda$ by the same argument. Thus $M \cap \lambda^+ = M' \cap \lambda^+$.

The following question is natural, but the author cannot answer:

QUESTION 1. Does $\lambda^{<\kappa} = \lambda^+$ follow from $\kappa$ is mildly $\lambda$-ineffable and $\text{cf}(\lambda)< \kappa$?

Of course $\lambda^{<\kappa} = \lambda^+$ follows from mild ineffability of $\mathcal{P}_\kappa \lambda^{<\kappa}$ when $\text{cf}(\lambda) < \kappa$.
Unfortunately, however, mild ineffability of $\mathcal{P}_\kappa \lambda$ does not always lift up to that of $\mathcal{P}_\kappa \lambda^{<\kappa}$. (See the next section.)
5. The equivalence of the Shelah property and ineffability.

Abe [3] showed that ineffability and almost ineffability of $\mathcal{P}_\kappa \lambda$ are equivalent if $2^\lambda = \lambda^{<\kappa}$. We will see that if $\lambda$ is strong limit and $\text{cf}(\lambda) < \kappa$ then ineffability and the Shelah property are equivalent. First we will check that such equivalence is impossible if $\text{cf}(\lambda) \geq \kappa$. Proposition 5.1 (3) was proved in Abe [3]. We present here a simple proof.

**Proposition 5.1.** Let $X$ be a subset of $\mathcal{P}_\kappa \lambda$.

1. If $X$ is Shelah, then $\{x \in X : X \cap \mathcal{P}_{x \cap \kappa} x \text{ is not Shelah in } \mathcal{P}_{x \cap \kappa} x\}$ has the Shelah property.
2. If $X$ is almost ineffable, then $\{x \in X : X \cap \mathcal{P}_{x \cap \kappa} x \text{ is not almost ineffable in } \mathcal{P}_{x \cap \kappa} x\}$ is almost ineffable.
3. If $X$ is ineffable, then $\{x \in X : X \cap \mathcal{P}_{x \cap \kappa} x \text{ is not ineffable in } \mathcal{P}_{x \cap \kappa} x\}$ is ineffable.

**Proof.** We will only show (3). (1) and (2) can be proved by a similar argument. Suppose $X \subseteq \mathcal{P}_\kappa \lambda$ is ineffable. We may assume that $x \cap \kappa$ is inaccessible for all $x \in X$. Let $Y = \{x \in X : X \cap \mathcal{P}_{x \cap \kappa} x \text{ is not ineffable in } \mathcal{P}_{x \cap \kappa} x\}$.

Let $D = \mathcal{P}_\kappa \lambda \cup \{\lambda\}$. Then the relation $<$ on $\mathcal{P}_\kappa \lambda$ can be extended to $D$ by identifying $\lambda$ as the maximal element of $D$ with respect to the relation $<$. We consider $\mathcal{P}_\kappa \lambda$ as $\mathcal{P}_{\lambda \cap \kappa} \lambda$. Note that the relation $<$ on $D$ is well-founded. To show that $Y$ is ineffable, we prove, by using induction on $<$, that, for any $x \in D \cap (X \cup \{\lambda\})$, $Y \cap \mathcal{P}_{x \cap \kappa} x$ is ineffable in $\mathcal{P}_{x \cap \kappa} x$ if $X \cap \mathcal{P}_{x \cap \kappa} x$ is ineffable. This is sufficient to show the proposition. Let $x \in X \cup \{\lambda\}$ and assume this claim is verified for all $y \in X$ with $y < x$. Suppose $X \cap \mathcal{P}_{x \cap \kappa} x$ is ineffable but $Y \cap \mathcal{P}_{x \cap \kappa} x$ is not ineffable. Let $\langle a_z : z \in Y \cap \mathcal{P}_{x \cap \kappa} x \rangle$ be a sequence which witnesses $Y \cap \mathcal{P}_{x \cap \kappa} x$ is not ineffable. Since $X \cap \mathcal{P}_{x \cap \kappa} x$ is ineffable but $Y \cap \mathcal{P}_{x \cap \kappa} x$ is not ineffable, $Z = (X \setminus Y) \cap \mathcal{P}_{x \cap \kappa} x$ is ineffable. For each $y \in Z$, $X \cap \mathcal{P}_{y \cap \kappa} y$ is ineffable in $\mathcal{P}_{y \cap \kappa} y$. Hence $Y \cap \mathcal{P}_{y \cap \kappa} y$ is ineffable by the induction hypothesis. Hence we can apply the ineffability of $Y \cap \mathcal{P}_{y \cap \kappa} y$ to $\langle a_z : z \in Y \cap \mathcal{P}_{y \cap \kappa} y \rangle$. So there exists $b_y \subseteq y$ such that $\{z \in Y \cap \mathcal{P}_{y \cap \kappa} y : b_y \cap z = a_z\}$ is stationary in $\mathcal{P}_{y \cap \kappa} y$. Since $Z$ is ineffable, there exists $B \subseteq x$ such that $\{y \in Z : B \cap y = b_y\}$ is stationary in $\mathcal{P}_{x \cap \kappa} x$. We check that $\{z \in Y \cap \mathcal{P}_{x \cap \kappa} x : a_z = B \cap z\}$ is stationary, which is a contradiction. Take $f : x \times x \rightarrow x$. We want to find $z \in Y \cap \mathcal{P}_{x \cap \kappa} x$ such that $B \cap z = a_z$, $z \cap \kappa \subseteq \kappa$, and $f^\kappa(z \times z) \subseteq z$. Since $\{y \in Z : B \cap y = b_y\}$ is stationary in $\mathcal{P}_{x \cap \kappa} x$, there exists $y \in Z$ such that $B \cap y = b_y$ and $f^\kappa(y \times y) \subseteq y$. Because $\{z \in Y \cap \mathcal{P}_{y \cap \kappa} y : b_y \cap z = a_z\}$ is stationary in $\mathcal{P}_{y \cap \kappa} y$, we can take $z \in Y \cap \mathcal{P}_{y \cap \kappa} y$ such that $a_z = b_y \cap z = B \cap z$, $z \cap \kappa \subseteq \kappa$, and $f^\kappa(z \times z) \subseteq z$. This completes the proof. □
Assume $\text{cf}(\lambda) \geq \kappa$.

(1) If $\kappa$ is $\lambda$-Shelah, then $\text{NSh}_{\kappa\lambda} \subseteq \text{NAIn}_{\kappa\lambda}$.

(2) If $\kappa$ is almost $\lambda$-ineffable, then $\text{NAIn}_{\kappa\lambda} \subseteq \text{NIn}_{\kappa\lambda}$.

**Proof.**

(1). By Abe [3], $\{x \in \mathcal{P}_{\kappa\lambda} : x \cap \kappa$ is $x$-Shelah} $\in \text{NAIn}_{\kappa\lambda}^{*\kappa\lambda}$. Hence by Proposition 5.1, $\{x \in \mathcal{P}_{\kappa\lambda} : x \cap \kappa$ is not $x$-Shelah} is Shelah but not almost ineffable.

(2). By Kamo [17], $\{x \in \mathcal{P}_{\kappa\lambda} : x \cap \kappa$ is almost $x$-ineffable} $\in \text{NIn}_{\kappa\lambda}^{*\kappa\lambda}$. So $\{x \in \mathcal{P}_{\kappa\lambda} : x \cap \kappa$ is not almost $x$-ineffable} is almost ineffable but not ineffable.

**PROPOSITION 5.3.** Assume $\lambda$ is a strong limit cardinal with $\text{cf}(\lambda) < \kappa$ (so $2^\lambda = \lambda^{<\kappa}$ holds). Let $\langle A_x : x \in \mathcal{P}_{\kappa\lambda} \rangle$ be an enumeration of $\mathcal{P}(\lambda)$ and $X = \{x \in \mathcal{P}_{\kappa\lambda} : \forall a \subseteq x \exists y < x (a = A_y \cap x)\}$. Then $\text{NSh}_{\kappa\lambda} = \text{NIn}_{\kappa\lambda} = \text{NAIn}_{\kappa\lambda} = \text{WNS}_{\kappa\lambda}|X$. In particular the following are equivalent:

(1) $\kappa$ is $\lambda$-Shelah.

(2) $\kappa$ is almost $\lambda$-ineffable.

(3) $\kappa$ is $\lambda$-ineffable.

(4) $X \in \text{WNS}_{\kappa\lambda}^{+\kappa\lambda}$.

**Proof.** Since $\text{WNS}_{\kappa\lambda} \subseteq \text{NSh}_{\kappa\lambda} \subseteq \text{NAIn}_{\kappa\lambda} \subseteq \text{NIn}_{\kappa\lambda}$, it is enough to show that $X \in \text{NSh}_{\kappa\lambda}^{*\kappa\lambda}$ and $\text{NIn}_{\kappa\lambda} \subseteq \text{WNS}_{\kappa\lambda}|X$. First we show that $X \in \text{NSh}_{\kappa\lambda}^{*\kappa\lambda}$. Let $\langle B_\xi : \xi < \lambda \rangle$ be an enumeration of all bounded subsets of $\lambda$. First we claim that $Z = \{x \in \mathcal{P}_{\kappa\lambda} : \forall a \subseteq x (a is bounded in $\lambda \rightarrow \exists \xi \in x (a = B_\xi \cap x))\} \in \text{NSh}_{\kappa\lambda}^{*\kappa\lambda}$. Assume otherwise, then by the normality of $\text{NSh}_{\kappa\lambda}$, there exists $\alpha < \lambda$ such that $Y = \{x \in \mathcal{P}_{\kappa\lambda} : \exists a \subseteq x \ni \exists \xi \in x (a \neq B_\xi \cap x)\} \in \text{NSh}_{\kappa\lambda}^{*\kappa\lambda}$. For each $x \in Y$, let $a_x \subseteq x \cap \alpha$ be a witness to $x \in Y$. Let $f_x : x \cap \alpha \rightarrow 2$ be the characteristic function of $a_x$ and $g_x : x \rightarrow x$ a function such that $g_x(\beta) = a_x \triangle (B_\beta \cap x)$ for each $\beta \in x$. By the Shelah property of $Z$, there exist $f : \alpha \rightarrow 2$ and $g : \lambda \rightarrow \lambda$ such that $\{x \in Y : f_x|y = f|y \text{ and } g_x|y = g|y\}$ is unbounded for all $y \in \mathcal{P}_{\kappa\lambda}$. Let $B = f^{-1}\{1\}$. $B \subseteq \alpha$, so $B = B_\xi$ for some $\xi < \lambda$. Take $y \in \mathcal{P}_{\kappa\lambda}$ such that $\xi \in y$ and $y$ is closed under $g$. Take $x \in Y$ such that $y \subseteq x$, $f_x|y = f|y$, and $g_x|y = g|y$. Then $B_\xi \cap x \neq a_x$ because $\xi \in y \subseteq X$. Since $f|x$ is the characteristic function of $B_\xi \cap x$, $f_x$ is the characteristic function of $B_\xi \cap x$, $f_x$ is that of $a_x$, and $g_x(\xi) \in a_x \triangle (B_\xi \cap x)$, we have $f_x(g_x(\xi)) \neq f(\xi)$. Since $\xi \in y$ and $y$ is closed under $g$, we have $g_x(\xi) = g(\xi) \in y$. But then $f_x(g(\xi)) = f(g(\xi))$, which is a contradiction.

Second we show that $X \in \text{NSh}_{\kappa\lambda}^{*\kappa\lambda}$. Fix an increasing sequence $\langle \lambda_i : i < \text{cf}(\lambda) \rangle$ which converges to $\lambda$. Assume $X \notin \text{NSh}_{\kappa\lambda}^{*\kappa\lambda}$. Then $Z' = \{x \in Z : \lambda_i \in x \land i < \text{cf}(\lambda) \subseteq x \land \exists a \subseteq x \forall y < x (a \neq A_y \cap x)\} \in \text{NSh}_{\kappa\lambda}^{+\kappa\lambda}$. For each $x \in Z'$,
let $a_x \subseteq x$ be a witness to $x \in Z'$. For $x \in Z'$ and $i < \text{cf}(\lambda)$, take $\xi^x_i \in x$ such that $a_x \cap \lambda_i = A_{\xi^x_i} \cap x$. Then, by the strong normality of $\text{NSh}_\kappa \lambda$, there exists $\langle \xi^x_i : i < \text{cf}(\lambda) \rangle$ such that $\{ x \in Z' : \forall i < \text{cf}(\lambda) (\xi^x_i = \xi_i) \} \in \text{NSh}_\kappa^+ \lambda$. Note that if $i < j < \text{cf}(\lambda)$, then $A_{\xi_i} = A_{\xi_j} \cap \lambda_i$. Thus we can define $A \subseteq \lambda$ by $A \cap \lambda_i = A_{\xi_i}$ for all $i < \text{cf}(\lambda)$. Take $y \in \mathcal{P}_\kappa \lambda$ such that $A = A_y$. It is easy to see that for $x \in Z'$, $a_x = A_y \cap x$ if $\xi^x_i = \xi_i$ for all $i < \text{cf}(\lambda)$, which is a contradiction. Thus we have $X \in \text{NSh}_\kappa^+ \lambda$.

Last we show that $\text{NIn}_{\kappa \lambda} \subseteq \text{WNS}_{\kappa \lambda}|X$. Let $W \in (\text{WNS}_{\kappa \lambda}|X)^+$. We may assume $W \subseteq X$. We claim that $W$ is ineffable. To see this, take an arbitrary sequence $\langle a_x : x \in W \rangle$ such that $a_x \subseteq x$ for all $x \in W$. By the definition of $X$, for each $x \in W$ there exists $y_x < x$ such that $a_x = A_{y_x} \cap x$. Since $W \in \text{WNS}_{\kappa \lambda}^+$, there exists $y \in \mathcal{P}_\kappa \lambda$ such that $W' = \{ x \in W : y_x = y \} \in \text{WNS}_{\kappa \lambda}^+$. Then $W'$ is stationary and it is clear that $a_x = A_y \cap x$ for all $x \in W'$.

**Remark.** If we replace “$\lambda$ is strong limit” by “$2^\lambda = \lambda^{<\kappa}$” in the assumption of the previous proposition, then we can obtain that $\text{NAIn}_{\kappa \lambda} = \text{NIn}_{\kappa \lambda} = \text{WNS}_{\kappa \lambda}|X$. The proof that $X \in \text{NAIn}_{\kappa \lambda}^+$ is easy, so we omit it.

**Corollary 5.4.** Assume $2^\lambda = \lambda^{<\kappa}$. For any $Y \in \text{NIn}_{\kappa \lambda}^+$ ($= \text{NAIn}_{\kappa \lambda}^+$) and $\langle a_x : x \in Y \rangle$ with $a_x \subseteq x$, there exists $A \subseteq \lambda$ such that $\{ x \in Y : A \cap x = a_x \} \in \text{NIn}_{\kappa \lambda}^+$.

**Proof.** By the above remark, $\text{NIn}_{\kappa \lambda} = \text{NAIn}_{\kappa \lambda} = \text{WNS}_{\kappa \lambda}|X$ holds, where $X$ is as in Proposition 5.3. We can argue as in the proof of $\text{NIn}_{\kappa \lambda} \subseteq \text{WNS}_{\kappa \lambda}|X$ in Proposition 5.3. □

Next we turn to completely ineffability of $\mathcal{P}_\kappa \lambda$.

**Definition 5.5.** Let $I$ be an ideal over $\mathcal{P}_\kappa \lambda$. $\mathcal{W}$ is called an $I$-partition if the following hold:

1. $\mathcal{W} \subseteq I^+$,
2. $\forall Y \in I^+ \exists Z \in \mathcal{W} (Y \cap Z \in I^+)$, and
3. $\forall Y, Z \in \mathcal{W} (Y \neq Z \Rightarrow Y \cap Z \in I)$.

Let $\mu$ and $\nu$ be cardinals. An ideal $I$ over $\mathcal{P}_\kappa \lambda$ is called $(\mu, \nu)$-distributive if, for every $X \in I^+$ and every $\langle \mathcal{W}_\alpha : \alpha < \mu \rangle$ where each $\mathcal{W}_\alpha$ is an $I$-partition with $|\mathcal{W}_\alpha| \leq \nu$, there exists $Y \in (I|X)^+$ such that $Y$ satisfies the following:

For every $\alpha < \mu$, there exists $Z \in \mathcal{W}_\alpha$ such that $Y \setminus Z \in I$.

**Fact 5.6** (Johnson [16]). Let $I$ be an ideal over $\mathcal{P}_\kappa \lambda$. Then the following are equivalent:

1. $I$ is normal and $(\lambda, \lambda)$-distributive.
(2) For all \(X \in I^+\) and \(\langle a_x : x \in X \rangle\) with \(a_x \subseteq x\), there exists \(A \subseteq \lambda\) such that \(\{x \in X : A \cap x = a_x\} \in I^+\).

We say that \(X \subseteq \mathcal{P}_\kappa \lambda\) is completely ineffable if there exists a proper \((\lambda, \lambda)\)-distributive normal ideal \(I\) such that \(X \in I^+\), and that \(\kappa\) is completely \(\lambda\)-ineffable if \(\mathcal{P}_\kappa \lambda\) is completely ineffable. Let \(\text{NCIn}_{\kappa \lambda} = \{X \subseteq \mathcal{P}_\kappa \lambda : X \text{ is not completely ineffable}\}\). Then \(\text{NCIn}_{\kappa \lambda}\) is the minimal normal \((\lambda, \lambda)\)-distributive ideal and, equivalently, is the minimal normal ideal which satisfies (2) of the above fact. Clearly \(\text{NIn}_{\kappa \lambda} \subseteq \text{NCIn}_{\kappa \lambda}\) holds.

**Proposition 5.7.** Assume \(\operatorname{cf}(\lambda) \geq \kappa\) and \(\kappa\) is \(\lambda\)-ineffable. Then \(\text{NIn}_{\kappa \lambda} \subseteq \text{NCIn}_{\kappa \lambda}\).

**Proof.** By Kamo [17], \(\{x \in \mathcal{P}_\kappa \lambda : x \cap \kappa \text{ is } x\text{-ineffable}\} \in \text{NCIn}_{\kappa \lambda}^*\). Hence the assertion follows from Proposition 5.1. \(\square\)

The next proposition can be easily verified by using Corollary 5.4 and Fact 5.6.

**Proposition 5.8.** Assume \(2^\lambda = \lambda^{<\kappa}\). Then \(\text{NIn}_{\kappa \lambda} = \text{NCIn}_{\kappa \lambda}\) holds. Thus \(\kappa\) is \(\lambda\)-ineffable if and only if \(\kappa\) is completely \(\lambda\)-ineffable under the assumption \(2^\lambda = \lambda^{<\kappa}\).

Now we show the preservation of ineffability under certain forcing methods. For a poset \(P\) and an ordinal \(\alpha\), \(\Gamma_\alpha(P)\) denotes the following 2-player game:

<table>
<thead>
<tr>
<th>Player I: (p_0)</th>
<th>(p_1)</th>
<th>(\cdots)</th>
<th>(p_{\omega+1})</th>
<th>(\cdots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player II: (q_0)</td>
<td>(q_1)</td>
<td>(\cdots)</td>
<td>(q_\omega)</td>
<td>(q_{\omega+1})</td>
</tr>
</tbody>
</table>

Player I and II choose elements of \(P\) alternately such that \(p_0 \geq q_0 \geq p_1 \geq q_1 \geq \cdots\). At limit stage \(\eta\), only Player II moves and Player II chooses a lower bound \(q_\eta\) of \(\{q_\xi : \xi < \eta\}\). Player II wins if this game can be continued to length \(\alpha\), that is, Player II can choose \(q_\beta\) for all \(\beta < \alpha\). A poset \(P\) is \(\alpha\)-strategically closed if Player II has a winning strategy in \(\Gamma_\alpha(P)\). It is well-known that \(\alpha\)-strategically closed posets add no new \(< \alpha\)-sequences.

**Proposition 5.9.** Assume \(\lambda\) is a strong limit cardinal with \(\operatorname{cf}(\lambda) < \kappa\). If \(\kappa\) is \(\lambda\)-ineffable (equivalently, \(\lambda\)-Shelah, almost \(\lambda\)-ineffable, or completely \(\lambda\)-ineffable), then \(\Vdash_P "\kappa\ is \lambda\text{-ineffable}"\) for every \(\lambda^+\)-strategically closed poset \(P\).

**Proof.** By Proposition 4.4, we have \(2^\lambda = \lambda^{<\kappa} = \lambda^+\). Let \(\langle A_x : x \in \mathcal{P}_\kappa \lambda \rangle\) be an enumeration of \(\mathcal{P}(\lambda)\) and define \(X\) as in Proposition 5.3. Then \(\text{NIn}_{\kappa \lambda} \subseteq \text{NCIn}_{\kappa \lambda}\).
= WNS_{λ^+}\lambda].

Since \( λ^+-\text{strategically closed} \) forcing adds no new subsets of \( λ \), \( \langle A_x : x ∈ ℙ_λ λ \rangle \) remains an enumeration of \( ℙ(λ) \) in \( V^P \). Thus it is enough to show that \( X ∈ WNS_{κλ}^+ \) in \( V^P \). Let \( p ∈ P \), and let \( ʃ \) be a \( P-\text{name} \) such that \( p ⊩ "ʃ : ℙ_λ λ → ℙ_λ \". \) Let \( \langle x_α : α < λ^+ \rangle \) be an enumeration of \( ℙ_λ \). Using the \( λ^+-\text{strategically closedness} \) of \( P \), we construct \( \langle y_α ∈ ℙ_λ : α < λ^+ \rangle \) and a descending sequence \( \langle p_α ∈ P : α < λ^+ \rangle \) such that \( p_0 ≤ p \) and \( p_α ⊩ "ʃ(x_α) = y_α" \) for all \( α < λ^+ \). Now define \( g : ℙ_λ λ → ℙ_λ \lambda \) by \( g(x_α) = y_α \). Since \( X ∈ WNS_{κλ}^+ \), there exists \( x ∈ X \) such that \( g^"ℙ_{x∩κ}x \subseteq ℙ_{x∩κ}x \". \) Take a sufficiently large \( β < λ^+ \) such that \( ℙ_{x∩κ}x \subseteq \{ x_α : α < β \} \). Then \( p_β \models "ʃ|ℙ_{x∩κ}x = g|ℙ_{x∩κ}x". \) Hence we conclude that \( p_β ∣ " x ∈ X \cap C_{ʃ}. \)

By Proposition 5.9, we have the following corollary:

**Corollary 5.10.** Assume \( λ \) is a strong limit cardinal with \( cf(λ) < κ \) and \( κ \) is \( λ \)-ineffable. Then there exists a poset which preserves all cofinalities and forces that \( κ \) remains completely \( λ \)-ineffable and \( \{ α < λ^+ : cf(α) < κ \} \) has a non-reflecting stationary subset.

*Proof.* Let \( P \) be the standard forcing notion which adds a non-reflecting stationary subset of \( \{ α < λ^+ : cf(α) < κ \} \) (see Burgess [5]). This poset is \( λ^+-\text{strategically closed} \), hence, by Lemma 5.9, \( κ \) is completely \( λ \)-ineffable in \( V^P \). \( □ \)

Abe [3] proved that \( λ \)-ineffability does not imply \( λ^{<κ} \)-ineffability if \( cf(λ) < κ \). We can improve Abe’s result to the following:

**Corollary 5.11.** Relative to a certain large cardinal assumption, it is consistent that \( κ \) is completely \( λ \)-ineffable with \( cf(λ) < κ \), but not mildly \( λ^{<κ} \)-ineffable.

*Proof.* We suppose that \( κ \) is completely \( λ \)-ineffable with \( cf(λ) < κ \) and that \( \{ α < λ^+ : cf(α) < κ \} \) has a non-reflecting stationary subset. This is consistent by Corollary 5.10. By Proposition 4.4, \( λ^{<κ} = λ^+ \) holds. By Proposition 4.3, \( κ \) is not mildly \( λ^+ \)-ineffable. Hence, in this model, \( κ \) is completely \( λ \)-ineffable but not mildly \( λ^{<κ} \)-ineffable. \( □ \)

Now we investigate the precipitousness of \( N_{nλ} \).

**Definition 5.12.** For an ideal \( I \) over \( ℙ_λ \), \( I \) is said to be precipitous if, for every \( X ∈ I^+ \) and for every \( I \)-partitions \( \langle ℨ_n : n < ω \rangle \) such that \( \forall n ∈ ω ∃ Y ∈ ℨ_{n+1} ∃ Z ∈ ℨ_n (Y ⊆ Z) \), there exists a sequence \( \langle X_n : n < ω \rangle \) such that \( X_n ∈ ℨ_n \) for all \( n < ω \), \( X ≥ X_0 ⊇ X_1 ⊇ ⋯ ⊇ X_n ⊇ ⋯ \), and \( ∩_{n<ω} X_n ≠ ∅ \).

For an information about precipitousness, see section 22 in Jech [14].
Furthermore it is easy to see that we can easily verify the following lemma:

**Lemma 5.14.** Let $\kappa$ be a Mahlo cardinal and let $e : P_\kappa \rightarrow P_\kappa^{<\kappa}$ be a canonical map. Then, for $X \in WNS^*_\kappa\lambda$, $e^"X \in WNS^*_\kappa^{<\kappa}$. For each $Y \subseteq X$, $Y \in WNS^*_\kappa\lambda$ if and only if $e^"Y \in WNS^*_\kappa^{<\kappa}$. Furthermore it is easy to see that $e|X$ is a bijection from $X$ to $e^"X$. Using this, we can easily verify the following lemma:

**Fact 5.13 (Abe [3]).** If $\text{cf}(\lambda) \geq \kappa$, then $\text{NCIn}_{\kappa\lambda}$ is not precipitous.

Now assume $\kappa$ is a Mahlo cardinal. Let $e : P_\kappa \rightarrow P_\kappa^{<\kappa}$ be a canonical map and $X = \{x \in P_\kappa : x \cap \kappa$ is inaccessible, $e(x) \cap \lambda = x\}$. Then $X \in WNS^*_\kappa\lambda$ and $e^"X \in WNS^*_\kappa^{<\kappa}$. For each $Y \subseteq X$, $Y \in WNS^*_\kappa\lambda$ if and only if $e^"Y \in WNS^*_\kappa^{<\kappa}$. Furthermore it is easy to see that $e|X$ is a bijection from $X$ to $e^"X$. Using this, we can easily verify the following lemma:

**Lemma 5.14.** Let $\kappa$ be a Mahlo cardinal and let $e : P_\kappa \rightarrow P_\kappa^{<\kappa}$ be a canonical map. Then, for $X \in WNS^*_\kappa\lambda$, $WNS^*_\kappa\lambda|X$ is precipitous if and only if $WNS^*_\kappa^{<\kappa}|e^"X$ is precipitous. In particular $WNS^*_\kappa\lambda$ is precipitous if and only if $WNS^*_\kappa^{<\kappa}$ is precipitous.

**Proposition 5.15.** Assume $\lambda$ is a strong limit cardinal with $\text{cf}(\lambda) < \kappa$ and $\kappa$ is $\lambda$-Shelah (and so is $\lambda$-ineffable, etc.). Let $\mu$ be a Woodin cardinal greater than $\lambda$. Then $\models \text{Col}(\lambda^+, < \mu)$ “$\text{NSh}_{\kappa\lambda} = \text{NAIn}_{\kappa\lambda} = \text{NIn}_{\kappa\lambda} = \text{NCIn}_{\kappa\lambda}$ is precipitous”, where $\text{Col}(\lambda^+, < \mu)$ is the standard $\lambda^+$-closed poset which collapses $\mu$ to $\lambda^{++}$.

**Proof.** Let $G$ be a $(V, \text{Col}(\lambda^+, < \mu))$-generic filter and work in $V[G]$. $\text{Col}(\lambda^+, < \mu)$ is $\lambda^+$-strategically closed. Hence $\kappa$ is $\lambda$-Shelah, and $\text{NSh}_{\kappa\lambda} = \text{NAIn}_{\kappa\lambda} = \text{NIn}_{\kappa\lambda} = \text{NCIn}_{\kappa\lambda}$ for some $X$ in $V[G]$. It is well-known that $\text{NSh}_{\kappa\lambda}$ is precipitous in $V[G]$ (see Goldring [12]). Since $(\lambda^+)^{<\kappa} = \lambda^+$, $\text{WNS}_{\kappa\lambda} = \text{NS}_{\kappa\lambda}^{++}|Y$ for some $Y$. Thus $\text{WNS}_{\kappa\lambda}^{++}$ is also precipitous. By the previous lemma, we have that $\text{WNS}_{\kappa\lambda}$ is precipitous. Hence $\text{WNS}_{\kappa\lambda}|X = \text{NSh}_{\kappa\lambda} = \text{NAIn}_{\kappa\lambda} = \text{NIn}_{\kappa\lambda} = \text{NCIn}_{\kappa\lambda}$ is precipitous.

**Question 2.** Can $\text{NSh}_{\kappa\lambda}$, $\text{NAIn}_{\kappa\lambda}$, and $\text{NIn}_{\kappa\lambda}$ be precipitous even if $\text{cf}(\lambda) \geq \kappa$? Furthermore can these ideals be $\lambda^+$-saturated?

### 6. Relationship between $\Pi^1_1$-indescribability and ineffability.

The indescribability of $P_\kappa\lambda$ was introduced by Baumgartner and Carr [8] as a generalization of the indescribability of a cardinal. First we explain some basic notation. A sentence $\varphi$ is a $\Pi^1_1$-sentence if $\varphi$ is of the form $\forall X_0 \forall X_1 \cdots \forall X_n \psi(X_0, X_1, \ldots, X_n)$, where $X_0, X_1, \ldots, X_n$ are type 2 variables, and $\psi(X_0, X_1, \ldots, X_n)$ is a first order sentence with language $\{\varepsilon, =, X_0, \ldots, X_n\}$ where $X_i$ is a unary predicate symbol. In the intended semantics, if $D$ is the domain of a structure, type 2 variables will range over $\mathcal{P}(D)$.

**Definition 6.1.** An uncountable cardinal $\kappa$ is $\Pi^1_1$-indecomposable if, for any $R \subseteq V_\kappa$ and $\Pi^1_1$-sentence $\varphi$ over the structure $(V_\kappa, \varepsilon, R)$ (that is, $\varphi$ is a $\Pi^1_1$-sentence with language $\{\varepsilon, =, R\}$),
Ineffability of $P_{\kappa\lambda}$ for $\lambda$ with small cofinality

$\langle V_\alpha, \in, R \rangle \models \phi \Rightarrow \exists \alpha < \kappa (\langle V_\alpha, \in, R \cap V_\alpha \rangle \models \phi)$,

where $V_\alpha$ is the set of all sets with rank less than $\alpha$.

**FACT 6.2.** An uncountable cardinal $\kappa$ is weakly compact if and only if $\kappa$ is $\Pi^1_1$-indescribable.

Baumgartner defined the following:

**Definition 6.3.** Let $S$ be a set with $\kappa \subseteq S$. Define $V_\alpha(\kappa, S)$ by induction on $\alpha \leq \kappa$ in the following way:

- $V_0(\kappa, S) = S$,
- $V_{\alpha+1}(\kappa, S) = V_\alpha(\kappa, S) \cup P_\kappa(V_\alpha(\kappa, S))$, and
- $V_\alpha(\kappa, S) = \bigcup_{\beta < \alpha} V_\beta(\kappa, S)$ if $\alpha$ is a limit ordinal.

For $X \subseteq P_\kappa S$, we say that $X$ is $\Pi^1_1$-indescribable if, for every $R \subseteq V_\kappa(\kappa, S)$ and $\Pi^1_1$-sentence $\phi$ over the structure $\langle V_\kappa(\kappa, S), \in, R \rangle$, the following holds:

If $\langle V_\kappa(\kappa, S), \in, R \rangle \models \phi$, then there exists $x \in X$ such that $|x \cap \kappa| = x \cap \kappa$ and $\phi$ reflects to $x$, that is,

$\langle V_{x \cap \kappa}(x \cap \kappa), \in, R \cap V_{x \cap \kappa}(x \cap \kappa, x) \rangle \models \phi$.

Let $\Pi_{\kappa\lambda}$ be the set of all $X \subseteq P_\kappa \lambda$ such that $X$ is not $\Pi^1_1$-indescribable.

**Fact 6.4** (Abe [2], Carr [8]).

1. $\Pi_{\kappa\lambda}$ is a strongly normal ideal over $P_\kappa \lambda$.
2. $\text{NSh}_{\kappa\lambda} \subseteq \Pi_{\kappa\lambda}$.
3. If $\text{cf}(\lambda) \geq \kappa$, then $\text{NSh}_{\kappa\lambda} = \Pi_{\kappa\lambda}$.

For further general background about indescribability of $P_\kappa \lambda$, see Abe [2] and Carr [8].

We will use the following combinatorial characterization of $\Pi^1_1$-indescribability.

**Fact 6.5** (Abe [2]). For $X \subseteq P_\kappa \lambda$, the following are equivalent:

1. $X$ is $\Pi^1_1$-indescribable.
2. $e^X$ is Shelah in $P_\kappa \lambda^{<\kappa}$, where $e$ is a canonical map from $P_\kappa \lambda$ to $P_\kappa \lambda^{<\kappa}$.
3. For all $\langle f_x : x \in X \rangle$ with $f_x : P_{x \cap \kappa} \lambda \rightarrow P_{x \cap \kappa} \lambda$, there exists $f : P_\kappa \lambda \rightarrow P_\kappa \lambda$ such that $\{ x \in X : f|P_{y \cap \kappa} \lambda = f_x|P_{y \cap \kappa} \lambda \}$ is unbounded for all $y \in P_\kappa \lambda$.

First we show that $\Pi^1_1$-indescribability implies a reflection principle for
WNS\(\kappa,\lambda\)-positive sets.

**Lemma 6.6.** Assume \(\mathcal{P}_\kappa\lambda\) is \(\Pi^1_1\)-indescribable. Then, for each \(X \in \text{WNS}^+\), \(\{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \text{ is regular, } X \cap \mathcal{P}_{x \cap \kappa}x \in \text{WNS}^+\} \in \Pi^*_\kappa\lambda\).

**Proof.** Assume otherwise. Then \(Y = \{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \text{ is regular and } X \cap \mathcal{P}_{x \cap \kappa}x \in \text{WNS}^+\} \in \Pi^*_\kappa\lambda\). For each \(x \in Y\), let \(f_x : \mathcal{P}_{x \cap \kappa}x \rightarrow \mathcal{P}_{x \cap \kappa}x\) be a function which witnesses \(X \cap \mathcal{P}_{x \cap \kappa}x \in \text{WNS}^+\). By Fact 6.5, we can take \(f : \mathcal{P}_\kappa\lambda \rightarrow \mathcal{P}_\kappa\lambda\) such that, for all \(y \in \mathcal{P}_\kappa\lambda\), \(\{x \in Y : f|\mathcal{P}_{y \cap \kappa}y = f_x|\mathcal{P}_{y \cap \kappa}y\}\) is unbounded. Since \(X \in \text{WNS}^+\), there exists \(y \in X\) such that \(f^y|\mathcal{P}_{y \cap \kappa}y \subseteq \mathcal{P}_{y \cap \kappa}y\). Take \(x \in Y\) such that \(y < x\) and \(f|\mathcal{P}_{y \cap \kappa}y = f_x|\mathcal{P}_{y \cap \kappa}y\). Then \(y \in X \cap \mathcal{P}_{x \cap \kappa}x\) and \(f^y|\mathcal{P}_{y \cap \kappa}y = f^x|\mathcal{P}_{y \cap \kappa}y \subseteq \mathcal{P}_{y \cap \kappa}y\), thus \(y \in (X \cap \mathcal{P}_{x \cap \kappa}x) \cap C_f\). This is a contradiction. \(\Box\)

We have another proof since \(\text{"}X \in \text{WNS}^+\text{"}\) can be stated in a \(\Pi^1_1\)-sentence over \(\langle V_{\kappa}(\kappa)\rangle\). Also note that, for every \(X \in \text{NS}^+\), \(\{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \text{ is regular, } X \cap \mathcal{P}_{x \cap \kappa}x \in \text{NS}^+\} \in \text{NS}^*_\kappa\lambda\).

The next proposition shows that \(\Pi^1_1\)-indescribability of \(\mathcal{P}_\kappa\lambda\) can be much stronger than ineffability if \(\text{cf}(\lambda) < \kappa\).

**Proposition 6.7.** Assume \(2^\lambda = \lambda^{<\kappa}\). Then the following hold:

1. \(\text{NIn}_{\kappa,\lambda} \subseteq \Pi^*_\kappa\lambda\). Hence \(\kappa\) is \(\lambda\)-ineffable if \(\mathcal{P}_\kappa\lambda\) is \(\Pi^1_1\)-indescribable.
2. If \(Y \subseteq \mathcal{P}_\kappa\lambda\) is ineffable, then \(\{x \in \mathcal{P}_\kappa\lambda : Y \cap \mathcal{P}_{x \cap \kappa}x \text{ is ineffable}\} \in \Pi^*_\kappa\lambda\).
3. If \(\kappa\) is \(\lambda\)-ineffable, then \(\text{NIn}_{\kappa,\lambda} \nsubseteq \Pi^*_\kappa\lambda\).

**Proof.** Take \(X\) and \(\langle A_x : x \in \mathcal{P}_\kappa\lambda\rangle\) as in Proposition 5.3.

1. By the remark after Proposition 5.3, \(\text{NIn}_{\kappa,\lambda} = \text{WNS}_{\kappa,\lambda}|X\) holds. Since \(\text{WNS}_{\kappa,\lambda} \subseteq \text{NSH}_{\kappa,\lambda} \subseteq \Pi^*_\kappa\lambda\), it is enough to show that \(X \in \Pi^*_\kappa\lambda\). Assume otherwise. Then \(Y = \{x \in \mathcal{P}_\kappa\lambda : \exists a_x \subseteq x \forall y < x (a_x \neq A_y \cap x)\} \in \Pi^*_\kappa\lambda\). For each \(x \in Y\), let \(a_x \subseteq x\) be a witness to \(x \in Y\). Now define \(f_x : x \rightarrow 2\) and \(g_x : \mathcal{P}_{x \cap \kappa}x \rightarrow x\) as follows: \(f_x\) is the characteristic function of \(a_x\) and \(g_x(y) \in a_x \triangle (A_y \cap x)\). Then there exist \(f : \lambda \rightarrow 2\) and \(g : \mathcal{P}_\kappa\lambda \rightarrow \lambda\) such that \(\{x \in Y : f_x|y = f|y, g_x|\mathcal{P}_{y \cap \kappa}y = g|\mathcal{P}_{y \cap \kappa}y\}\) is unbounded for all \(y \in \mathcal{P}_\kappa\lambda\). Let \(A = f^{-1}_x(\{1\})\). Then \(A = A_x\) for some \(z \in \mathcal{P}_\kappa\lambda\). Take \(y \in \mathcal{P}_\kappa\lambda\) such that \(z < y\) and \(g^y|\mathcal{P}_{y \cap \kappa}y \subseteq y\). Then we can find \(x \in Y\) such that \(y < x, f|y = f_x|y, g_x|\mathcal{P}_{y \cap \kappa}y = g|\mathcal{P}_{y \cap \kappa}y\). Since \(z < y\) \(< x, a_x \neq A_z \cap x\). Since \(g_x(z) = g(z)\), we have that \(g(z) \in a_x \triangle (A_z \cap x)\). However, \(g(z) \in y\), thus \(f(g(z)) = f_x(g(z))\), which contradicts to \(g(z) \in a_x \triangle (A_x \cap x)\).

2. Let \(Z \subseteq \mathcal{P}_\kappa\lambda\) be ineffable. Since \(\text{NIn}_{\kappa,\lambda} = \text{WNS}_{\kappa,\lambda}|X\), we may assume that \(Z \subseteq X\). Let \(x \in X\) such that \(x \cap \kappa\) is regular. By the definition of \(X\), \(\langle A_y \cap x : y < x\rangle\) can be seen as an enumeration of \(\mathcal{P}(x)\) which is indexed by elements of \(\mathcal{P}_{x \cap \kappa}x\). Let \(X' = \{y \in \mathcal{P}_{x \cap \kappa}x : \forall a \subseteq y \exists z < y (a = A_z \cap y)\}\).

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Then $X' = X \cap \mathcal{P}_{x \cap \kappa} x$. By the proof of Proposition 5.3, we see that, for $x \in X$ such that $x \cap \kappa$ is regular, $Z \cap \mathcal{P}_{x \cap \kappa} x$ is ineffable if $Z \cap \mathcal{P}_{x \cap \kappa} x \in \text{WNS}_{x \cap \kappa}$. It is clear that $\{x \in X \mid x \cap \kappa$ is regular, $Z \cap \mathcal{P}_{x \cap \kappa} x \in \text{WNS}_{x \cap \kappa}\} \in \Pi^*_{\kappa \lambda}$ by Lemma 6.6.

(3). By (2), it is enough to show that $\{x \in \mathcal{P}_{\kappa \lambda} \mid x \cap \kappa$ is not $x$-ineffable$\} \in \text{NIn}^+_{\kappa \lambda}$. This follows from Proposition 5.1. □

Assume $\lambda = \kappa^+\omega$, $2^\lambda = \lambda^{<\kappa}$, and $\mathcal{P}_{\kappa \lambda}$ is $\Pi^1_2$-indescribable. Then $\{x \in \mathcal{P}_{\kappa \lambda} \mid \text{ot}(x) = (x \cap \kappa)^{+\omega}\} \in \Pi^*_{\kappa \lambda}$. By the above proposition, we have $\{x \in \mathcal{P}_{\kappa \lambda} \mid \text{ot}(x) = x \cap \kappa^{+\omega}$ and $x \cap \kappa$ is $x$-ineffable$\} \in \Pi^*_{\kappa \lambda}$, thus we can show that $\{\alpha < \kappa : \alpha$ is $\alpha^{+\omega}$-ineffable$\}$ is stationary in $\kappa$. In particular, under GCH, if $\kappa = \min\{\alpha : \alpha$ is $\alpha^{+\omega}$-ineffable$\}$, then $\mathcal{P}_{\kappa \kappa^{+\omega}}$ is not $\Pi^1_2$-indescribable. Hence, the assumption that $\text{cf}(\lambda) \geq \kappa$ in (3) of Fact 6.4 cannot be dropped.

**Lemma 6.8.** Let $X \subseteq \mathcal{P}_{\kappa \lambda}$ be $\Pi^1_2$-indescribable. Then $\{x \in X \mid x \cap \mathcal{P}_{x \cap \kappa} x$ is not $\Pi^1_2$-indescribable$\}$.

**Proof.** Let $Y = \{x \in X \mid x \cap \mathcal{P}_{x \cap \kappa} x$ is not $\Pi^1_2$-indescribable$, R \subseteq V_\kappa(\kappa, \lambda)$, and $\varphi$ be a $\Pi^1_1$-sentence such that $(V_\kappa(\kappa, \lambda), \in, R) \models \varphi$. We show that there exists $x \in Y$ such that $\varphi$ reflects to $x$. Take $x \in X$ such that $x$ is a $<$-minimal element of $\{y \in X : \varphi$ reflects to $y\}$. Then $\varphi$ holds in $(V_{x \cap \kappa}(x \cap \kappa, x), \in, R \cap V_{x \cap \kappa}(x \cap \kappa, x))$ but there is no $y \in X \cap \mathcal{P}_{x \cap \kappa} x$ such that $\varphi$ reflects to $y$ by the minimality of $x$. Hence $x$ is an element of $Y$. □

As an immediate corollary, we have the following:

**Corollary 6.9.** Assume $2^\lambda = \lambda^{<\kappa}$ and $\mathcal{P}_{\kappa \lambda}$ is $\Pi^1_2$-indescribable. Then $\{x \in \mathcal{P}_{\kappa \lambda} \mid x \cap \kappa$ is $x$-ineffable but $\mathcal{P}_{x \cap \kappa} x$ is not $\Pi^1_2$-indescribable$\} \in \Pi^*_{\kappa \lambda}$.

Thus, for instance, $\{\alpha < \kappa : \alpha$ is $\alpha^{+\omega}$-ineffable but $\mathcal{P}_{\alpha \alpha^{+\omega}}$ is not $\Pi^1_2$-indescribable$\}$ is stationary in $\kappa$ if $\mathcal{P}_{\kappa \kappa^{+\omega}}$ is $\Pi^1_2$-indescribable.

**Question 3.** In this paper, we frequently used the assumptions that “$\lambda$ is a strong limit cardinal” or “$2^\lambda = \lambda^{<\kappa}$”. Can we eliminate these assumptions?

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