Feynman-Kac penalization problem for additive functionals with jumping functions

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Abstract. Takeda ([30]) solved the Feynman-Kac penalization problem for positive continuous additive functionals. We extend his result to additive functionals with jumps. We further give concrete examples of jumping functions.

1. Introduction.

Let \( X := (\Omega, \mathcal{F}, \mathcal{F}_t, \{X_t\}_{t \geq 0}, x \in \mathbb{R}^n) \) be a symmetric \( \alpha \)-stable process \((0 < \alpha < 2)\) and let \( A_t \) be an additive functional of \( X \). We call the next problems the Feynman-Kac penalization problem.

(i) Does there exist a probability measure \( \tilde{\mathbb{P}}_x \) such that

\[
\lim_{t \to \infty} \frac{\mathbb{E}_x[e^{A_t}S]}{\mathbb{E}_x[e^{A_t}]} = \int Sd\tilde{\mathbb{P}}_x
\]

for every \( x \in \mathbb{R}^n \), every \( s \geq 0 \), and every bounded \( S \in \mathcal{M}_s \)?

(ii) Does there exist a martingale \( M \) by which the limit distribution \( \tilde{\mathbb{P}}_x \) is determined:

\[
d\tilde{\mathbb{P}}_x = M_sd\mathbb{P}_x?
\]

Roynette, Vallois, and Yor considered the Feynman-Kac penalization problem of one or two dimensional Brownian motions ([22], [23], and [25]). K. Yano, Y. Yano, and Yor solved that of one dimensional recurrent symmetric \( \alpha \)-stable processes ([35]) \((1 < \alpha \leq 2)\). Though the previous results treated the case that Feynman-Kac functionals are killing, we deal with Feynman-Kac functionals with creation.

Takeda solved the Feynman-Kac penalization problem for \( e^{A_{\mu}^t} \) ([30]), where \( A_{\mu}^t \) is a positive continuous additive functional (PCAF, as an abbreviation) with Revuz measure \( \mu \) which is Green-tight. We consider this problem in the case that symmetric jumps are added to \( A_{\mu}^t \):

\[
A_{\mu,F}^t := A_{\mu}^t + \sum_{0 < u \leq t} F(X_{u-}, X_u),
\]
where $F$ is a bounded measurable positive symmetric function.

The Feynman-Kac multiplicative functional (MF, as abbreviation) $e^{A_t\mu,F}$ is non-local. We decompose this non-local Feynman-Kac MF as the product of an exponential type martingale $L_t$ and the local Feynman-Kac MF $e^{-A_t\mu,F_1}$:

$$e^{A_t\mu,F} = L_te^{A_t\mu,F_1}. \quad (1)$$

Here,

$$L_t := \exp\left(\sum_{0<s\leq t} F(X_{s-},X_s) - c_{\alpha,n} \int_0^t \int F_1(X_s,y)|X_s-y|^{-(\alpha+n)}dy ds\right),$$

$$\mu_{F_1}(dx) := c_{\alpha,n} \left\{ \int_{\mathbb{R}^n} F_1(x,y)|x-y|^{-(n+\alpha)}dy \right\} dx,$$

$F_1 := e^F - 1$ and $c_{\alpha,n}$ is a positive constant. We assume that $\mu_{F_1}$ is a Green-tight Kato measure. We then transform the symmetric $\alpha$-stable process $X$ by the martingale MF $L_t$ and denote by $Y$ the transformed process. The Dirichlet form of the transformed process $Y$ is given by

$$\mathcal{E}^Y(u,u) = \frac{c_{\alpha,n}}{2} \int d\nu \int d\nu u(x) - u(y))^2 e^{F(x,y)}|x-y|^{-(\alpha+n)}dxdy,$$

where $d$ is the diagonal set, that is, $d := \{(x,x); x \in \mathbb{R}^n\}$. We then see that the Lévy kernel of the transformed process is equivalent to that of the symmetric stable process: it holds

$$c^{-1}|x-y|^{-(\alpha+n)} \leq e^{F(x,y)}|x-y|^{-(\alpha+n)} \leq c|x-y|^{-(\alpha+n)}$$

for some $c > 1$. This implies the equivalence of transition probabilities (Bass and Levin (3)). Thus Kato classes are invariant under the transform by $L_t$.

We define the function $\lambda(\theta)$ for $\theta \geq 0$ by

$$\lambda(\theta) := \inf \left\{ \mathcal{E}^Y_\theta(u,u); \int u(x)^2 d(\mu + \mu_{F_1}) = 1 \right\}.$$

We see by the definition that $\lambda(\theta)$ is increasing and concave and satisfies $\lim_{\theta \to \infty} \lambda(\theta) = \infty$. We denote the generator of the transformed process $Y$ by $\mathcal{L}^Y$: let

$$\mathcal{L}^Y u(x) := \lim_{t \to 0} \frac{e^{L_t}[u(Y_t)] - u(x)}{t}, \quad (2)$$

where $d\mathbb{P}_x := L_t d\mathbb{P}_x$. Note that $\mathcal{E}^Y(u,u) = (-\mathcal{L}^Y u, u)$. We divide cases in terms of the value of $\lambda(0)$: if $\lambda(0) > 1$, $\lambda(0) < 1$, and $\lambda(0) = 1$, then $\mathcal{L}^Y + \mu + \mu_{F_1}$ is said to be
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subcritical, supercritical, and critical respectively.

(a) $\lambda(0) > 1$: We define

$$h(x) := \mathbb{E}_x^L\left[e^{A^\mu + \mu F_1 x}\right].$$

Z.-Q. Chen proved that the boundedness of $h(x)$ is equivalent to $\lambda(0) > 1$. We change his proof by using the equivalence of $\beta$-resolvent kernel of $X$ and $Y$. The weight process $M_s$ is identified with

$$M_s := \frac{h(X_s)}{h(x)} e^{A^\mu + \mu F_1 s}.$$ 

We treat this case in Section 4.

(b) $\lambda(0) < 1$: Since there exists $\theta_0 > 0$ such that $\lambda(\theta_0) = 1$ and $\mu$ and $\mu F_1$ are in the Green-tight Kato class, the embedding of $\mathcal{D}[\mathcal{E}^Y](= \mathcal{D}[\mathcal{E}])$ into $L^2(\mu + \mu F_1)$ is compact so that we can take a positive function $h$ in $\mathcal{D}[\mathcal{E}]$ such that $\mathcal{E}^Y_{\theta_0}(h, h) = 1$. We use the limit theorem of Feynman-Kac MFs like [30, Theorem 4.1] in the supercritical case. The weight process is then given by

$$M_s := e^{-\theta_0 s} \frac{h(X_s)}{h(x)} e^{A^\mu + \mu F_1 s}.$$ 

We treat this case in Section 5.

(c) $\lambda(0) = 1$: We use the compact embedding theorem of the extended Dirichlet form $\mathcal{D}_e[\mathcal{E}^Y]$ into $L^2(\mu + \mu F_1)$ by Takeda and Tsuchida ([33, Theorem 10]). This implies the existence of a positive function $h$ in $\mathcal{D}_e[\mathcal{E}^Y](= \mathcal{D}_e[\mathcal{E}])$ such that $\mathcal{E}^Y(h, h) = 1$. We then obtain a $h$-transformed process $(\mathbb{P}_x^L, Y_t, h^2 dx)_{x \in \mathbb{R}^n}$ and see that the semigroup of this process becomes recurrent. The function

$$\psi(t) := \mathbb{E}_x^{L, h}\left[\int_0^t k(Y_u)du\right], \quad k \in C_0^+ (\mathbb{R}^n)$$

diverges to infinity as $t \to \infty$ and $\mathbb{E}_x^{L, h}[e^{A^\mu + \mu F_1 x} S]/\psi(t)$ and $\mathbb{E}_x^{L, h}[e^{A^\mu + \mu F_1}]/\psi(t)$ converge only if $\mu$ and $\mu F_1$ are in the special Kato class. Then the problem is solved for a restricted class of Feynman-Kac MFs. The weight process of the critical case is given by

$$M_s := \frac{h(X_s)}{h(x)} e^{A^\mu + \mu F_1 S}.$$ 

We treat this case in Section 6.

Let $\mathcal{A}$ (resp. $\mathcal{A}_s$) be the set of jumping functions such that $\mu_F$ is in the Green-tight Kato class (resp. the special Kato class). The conditions that $F \in \mathcal{A}$ or $F \in \mathcal{A}_s$ are then
analytically characterized. We have used the equivalence of transition probabilities of $X$ and $Y$ instead of the conditional gaugeability to solve the Feynman-Kac penalization problem. Since the condition $F \in A_2$ (see [7, Definition 2.3] for the definition of $A_2$) is needed for the conditional gaugeability, we then see $A \supset A_2$. Furthermore, there exists a jumping function with a full support in $A$. For example, $F \in A$ and $F$ has a full support if the jumping function is

$$F(x, y) := (1 \wedge |x - y|^p) \langle x \rangle^{-q} \langle y \rangle^{-q} \quad \text{for } p > \alpha \text{ and } q > n,$$

where $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$. If we further assume $q > 2n - \alpha$ in this example, then $F \in A_s$.

In Section 2, we prepare for fundamental notations related to Green functions and Kato classes to describe our main results. In Section 3, we show the decomposition (1) of $e^{A_t^{\mu,F}}$, the equivalence of transition probabilities of $X$ and $Y$, and the invariance of Kato classes under the transform by $L_t$. We solve our problem in Sections 4, 5, and 6 in the subcritical, supercritical, and critical case respectively. In Section 7, we check that $F \in A$ or $F \in A_s$ for the functions $F$ described by (3).

2. Preliminaries.

Let $X$ be a symmetric $\alpha$-stable process ($0 < \alpha < 2$). The Dirichlet form of $X$ is given by

$$\mathcal{E}(u, u) := \frac{c_{\alpha,n}}{2} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^\alpha + n} dxdy,$$

$$\mathcal{D}[\mathcal{E}] := \left\{ u \in L^2(\mathbb{R}^n, dx); \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^\alpha + n} dxdy < \infty \right\},$$

where $c_{\alpha,n}$ is given by

$$c_{\alpha,n} := \frac{\alpha 2^{n-1} \Gamma((\alpha + n)/2)}{\pi^{n/2} \Gamma(1 - (\alpha/2))}.$$  \hspace{1cm} (4)

It is well known that Lévy system of the symmetric stable process $X_t$ is $(c_{\alpha,n}|x - y|^{-(n+\alpha)}, t)$. Note that the Revuz measure of $t$ is the Lebesgue measure (see [7, Example 2.1] for further details). Let $A_{t}^{\mu}$ be a PCAF with the corresponding Revuz measure $\mu$ and let $F$ be a bounded measurable positive symmetric function vanishing on diagonal set throughout this paper. We consider following additive functionals (AFs, as an abbreviation) with symmetric jumps

$$A_{t}^{\mu,F} := A_{t}^{\mu} + \sum_{0 < s \leq t} F(X_{s^-}, X_s).$$

We define ($\beta$-)resolvent kernels, ($\beta$-)potentials, and Kato classes.
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Definition 2.1. Let $p(t, x, y)$ be the transition probability of the symmetric $\alpha$-stable process $X$. The following function $G_\beta(x, y)$ is said to be $\beta$-resolvent kernel.

$$G_\beta(x, y) := \int_0^\infty e^{-\beta t} p(t, x, y) dt \quad \text{for } \beta \geq 0.$$ 

We write 

$$G(x, y) := G_0(x, y)$$

if the process $X$ is transient. Let $\mu$ be a positive Radon measure. We denote the $\beta$-potential of $\mu$ by $G_\beta \mu$, that is,

$$G_\beta \mu(x) := \int_{\mathbb{R}^n} G_\beta(x, y) \mu(dy) \quad \text{for } \beta \geq 0.$$ 

Definition 2.2. A positive Radon measure $\mu$ on $\mathbb{R}^n$ is said to be in the Kato class $K$ if it satisfies

$$\lim_{\beta \to \infty} \|G_\beta \mu\|_\infty = 0.$$ 

Given $\beta \geq 0$, a measure $\mu \in K$ is said to be in $\beta$-Green-tight Kato class if

$$\lim_{R \to \infty, r \to 0} \|G_\beta(1_{B(0, R) \cup B(x, r)}) \mu\|_\infty = 0.$$ 

We denote by $K_{\infty, \beta}$ the set of $\beta$-Green-tight measures. We write $K_\infty$ for $K_{\infty, 0}$ simply and call this Green-tight Kato class.

It follows from the definition of $K_{\infty, \beta}$ that $K_\infty \subset K_{\infty, 1}$ and $K_{\infty, \beta} = K_{\infty, 1}$ for all $\beta > 0$.

We define a measure

$$\mu_F(dx) := c_{\alpha, n} \left\{ \int_{\mathbb{R}^n} F(x, y) |x - y|^{-(\alpha + n)} dy \right\} dx.$$ 

We define the class $A$ of jumping functions. This class plays an important role in our results.

Definition 2.3. The function $F$ is said to be in the class $A$ if $\mu_F \in K_\infty$ (resp. $\mu_F \in K_{\infty, 1}$) for $n > \alpha$ (resp. $n \leq \alpha$).

Our goal is to obtain the next theorem.

Theorem 2.4. Assume that the Revuz measure $\mu$ is in Kato class $K_\infty$ (resp. $K_{\infty, 1}$) for $n > \alpha$ (resp. $n \leq \alpha$) and that the function $F_1 := e^F - 1$ belongs to the class $A$. Then
there exists a probability measure \( \mathbb{P}_x^M \) such that it holds

\[
\lim_{t \to \infty} \frac{\mathbb{E}_x[e^{A_{\mu,F}^t} S]}{\mathbb{E}_x[e^{A_{\mu,F}^t}]} = \mathbb{P}_x^M[S]
\]

for every \( s \geq 0 \), every bounded \( \mathcal{M}_s \)-measurable random variable \( S \), and every \( x \in \mathbb{R}^n \).

Moreover, the limit distribution \( \mathbb{P}_x^M \) is characterized as

\[
\mathbb{P}_x^M[A] := \int_A M_s d\mathbb{P}_x \quad \text{for } A \in \mathcal{M}_s,
\]

where \( M_s \) is a martingale MF defined in (9), (12), and (15) below.

Here and in what follows, we let \( F_1 = e^F - 1 \) without mentioning.

3. Decomposition of non-local Feynman-Kac MF.

In this section, to employ the result for local Feynman-Kac functionals we decompose a non-local Feynman-Kac MF as the product of a local Feynman-Kac MF and an exponential martingale.

We define an exponential martingale \( L_t \) by

\[
L_t = \exp \left( \sum_{0 < u \leq t} F(X_{u-}, X_u) - c_{\alpha,n} \int_0^t \int_{\mathbb{R}^n} F_1(X_u, y) |X_u - y|^{-(n+\alpha)} dy \, du \right).
\]

This is the unique solution of Doléans-Dade equation

\[
Z_t = 1 + \int_0^t Z_{u-} dK_u,
\]

where \( K_t \) is a purely discontinuous martingale defined by

\[
K_t := \sum_{0 < u \leq t} F_1(X_{u-}, X_u) - c_{\alpha,n} \int_0^t \int_{\mathbb{R}^n} F_1(X_u, y) |X_u - y|^{-(n+\alpha)} dy \, du
\]

(see [8, Remark 3.4]). We thus obtain

\[
e^{A_{\mu,F}^t} = e^{A_{\mu}^t} \prod_{0 < u \leq t} (1 + F_1(X_{u-}, X_u))
\]

\[
= L_t \exp \left( A_{\mu}^t + c_{\alpha,n} \int_0^t \int_{\mathbb{R}^n} F_1(X_u, y) |X_u - y|^{-(n+\alpha)} dy \, du \right)
\]

\[
= L_t e^{A_{\mu,F_1}^t}.
\]
Note that $t \mapsto c_{\alpha,n}(\int_0^t \int_{\mathbb{R}^n} F_1(X_u, y)|X_u - y|^{-(n+\alpha)} dy du)$ is a PCAF and its Revuz measure is $\mu_{F_1}$ as in the formula (5). We transform the symmetric stable process $X$ by the martingale MF $L_t$ and denote its law by $\mathbb{P}^L_x$, that is, $d\mathbb{P}^L_x := L_t d\mathbb{P}_x$. We further denote the associated symmetric strong Markov process by $(\mathbb{P}^L_x, Y_t)_{t \in \mathbb{R}^n}$. The Dirichlet form $\mathcal{E}^Y$ of the process $Y$ is identified as follows (see also [7]):

$$\mathcal{E}^Y(u, u) = \mathcal{E}(u, u) + \frac{c_{\alpha,n}}{2} \int_{\mathbb{R}^n} (u(x) - u(y))^2 F_1(x, y)|x - y|^{-(n+\alpha)} dy dx$$

$$= \frac{c_{\alpha,n}}{2} \int_{\mathbb{R}^n} (u(x) - u(y))^2 e^{F(x,y)}|x - y|^{-(n+\alpha)} dy dx.$$ 

Recall that $F$ is bounded. We then find that the Lévy kernel of $Y$ is equivalent to that of $X$, that is,

$$c^{-1}|x - y|^{-(n+\alpha)} \leq e^{F(x,y)}|x - y|^{-(n+\alpha)} \leq c|x - y|^{-(n+\alpha)}$$

for some $c > 1$. We then see from Bass and Levin ([3]) that the transition probability of $Y$ is equivalent to that of $X$.

**Theorem 3.1 ([3]).** If the Lévy kernel of $Y$ satisfies (6), then the transition probability $p^Y$ is also equivalent to $p$: it holds

$$c^{-1}p(t, x, y) \leq p^Y(t, x, y) \leq cp(t, x, y)$$

for some $c > 1$, every $t \geq 0$, and every $x, y \in \mathbb{R}^n$.

In the sequel, we denote positive constants by $c$ or $C$. They may be different at each occurrence.

Theorem 3.1 implies the equivalence of the $\beta$-resolvent kernel of $X$ and $Y$.

**Corollary 3.2.** Let $G^Y$ (resp. $G^Y_{\beta}$) be the Green function (resp. the $\beta$-resolvent kernel) of $Y$. It then holds

$$c^{-1}G_\beta(x, y) \leq G^Y_\beta(x, y) \leq cG_\beta(x, y)$$

for every $x, y \in \mathbb{R}^n$ and some $c > 1$.

For the rest part of this section, if the process $X$ is transient (resp. recurrent), then we assume that $\mu \in K_\infty$ (resp. $\mu \in K_{\infty,1}$) and $F_1 \in \mathcal{A}$.

We define the spectral function $\lambda(\theta)$ by the transformed Dirichlet form $\mathcal{E}^Y$.

$$\lambda(\theta) := \inf \left\{ \mathcal{E}^Y_\theta(u, u); \int_{\mathbb{R}^n} u(x)^2 d(\mu + \mu_{F_1}) = 1 \right\} \quad \text{for } \theta \geq 0,$$

where $\mathcal{E}^Y_\theta(\cdot, \cdot) := \mathcal{E}^Y(\cdot, \cdot) + \theta(\cdot, \cdot)L^2(dx)$. We here summarize some properties of the spectral function.
**Theorem 3.3.**  
(i) $\lambda(\theta)$ is concave (and hence continuous) and increasing.  
(ii) $\lim_{\theta \to \infty} \lambda(\theta) = \infty$.

**Proof.**  
(i) follows just as [30, Lemma 3.1]. Note that it holds

$$\int_{\mathbb{R}^n} u(x)^2 d(\mu + \mu_{F_1}) \leq \| G^Y_\theta (\mu + \mu_{F_1}) \|_\infty \mathcal{E}^Y_\theta (u, u)$$

for all $u \in D[E]$ (see [29, Proposition 2.3]). It then follows that

$$\lambda(\theta) \geq \frac{1}{\| G^Y_\theta (\mu + \mu_{F_1}) \|_\infty}.$$  

Since $G^Y_\theta (\mu + \mu_{F_1})$ is equivalent to $G_\theta(\mu + \mu_{F_1})$ and so $\| G^Y_\theta (\mu + \mu_{F_1}) \|_\infty \to 0$ as $\theta \to \infty$, we complete the proof of (ii). □

Since the transformed Dirichlet form of $Y$ is equivalent to that of $X$, we obtain the compact embedding of the domain of Dirichlet forms into $L^2(\mu + \mu_{F_1})$ (Takeda and Tsuchida [33]).

**Theorem 3.4.**  
(i) If $\mu \in K_\infty, F_1 \in A$ and $\mathcal{E}^Y$ is transient, then the embedding of $D_e[\mathcal{E}^Y]$ into $L^2(\mu + \mu_{F_1})$ is compact, where $(D_e[\mathcal{E}^Y], \mathcal{E}^Y)$ is the extended Dirichlet form.

(ii) If $\mu \in K_{\infty,1}$ and $F_1 \in A$, then the embedding of $D[\mathcal{E}^Y]$ into $L^2(\mu + \mu_{F_1})$ is compact.

**Proof.** Note that if $\mathcal{E}^Y$ is transient then $(D_e[\mathcal{E}^Y], \mathcal{E}^Y)$ is a Hilbert space whose norm is $\sqrt{\mathcal{E}^Y(\cdot, \cdot)}$. One can prove this by imitating the proofs of [33, Theorem 10] and [31, Theorem 2.7]. □

We will divide the following three cases in terms of the value of $\lambda(0)$. If $\lambda(0) > 1$, $\lambda(0) < 1$, and $\lambda(0) = 1$, then we call $\mathcal{L}^Y + \mu + \mu_{F_1}$ subcritical, supercritical, and critical respectively. $\mathcal{L}^Y$ is the generator defined by the formula (2) of the process $Y$.

**Remark 3.5.** The formula (8) implies

$$\lambda(0) \int_{\mathbb{R}^n} u(x)^2 d(\mu + \mu_{F_1}) \leq \mathcal{E}^Y(u, u)$$

for all $u \in D[E]$. The recurrence of the semigroup associated with $Y$ implies the existence of $\{u_n\}_{n \geq 1} \subset D[E]$ such that $\lim_{n \to \infty} \mathcal{E}(u_n, u_n) = 0$ and $\lim_{n \to \infty} u_n = 1$ a.e. by [13, Theorem 1.6.3 (i)(ii)], that is, $1 \in D_e[E]$ from [13, Theorem 1.6.3 (iii)]. [13, Theorem 2.1.7] then yields the last “a.e.” can be replaced by “q.e.” If $\lambda(0) > 0$ then the last inequality causes contradiction as $n \to \infty$. Therefore, we find that if the process $Y$ is recurrent then $\lambda(0) = 0$. 


4. Subcritical cases.

We use the gaugeability of \((Y, A^{\mu+F_1})\) in the subcritical case. One can modify the proof of ([7, Theorem 3.4]) by using the equivalence of the \(\beta\)-resolvent kernel of \(X\) and \(Y\). This modification is needed for the extension of the class of jumping functions.

**Theorem 4.1.** Assume that \(\mu \in K_\infty\) and \(F_1 \in \mathcal{A}\). The following three conditions are equivalent.

(i) \(\lambda(0) > 1\)
(ii) \((X, A^{\mu}_t)\) is gaugeable, that is, the function \(x \mapsto \mathbb{E}_x[e^{A^{\mu}_s \cdot (e^{A^{\mu}_t}_s \circ \theta_s)}]\) is bounded.
(iii) \((Y, A^{\mu + F_1})\) is gaugeable, that is, the function \(x \mapsto \mathbb{E}_x[e^{A^{\mu + F_1}_s}]\) is bounded.

In the subcritical case, we define the function \(h\) by

\[
h(x) := \mathbb{E}_x[e^{A^{\mu + F_1}_s}].
\]

We now solve the Feynman-Kac penalization problem in the subcritical case. We have only to consider the following ratio.

\[
\frac{\mathbb{E}_x[e^{A^{\mu}_s} | \mathcal{M}_s]}{\mathbb{E}_x[e^{A^{\mu}_t}]} = \frac{\mathbb{E}_x[e^{A^{\mu}_s} \cdot (e^{A^{\mu}_t}_s \circ \theta_s) | \mathcal{M}_s]}{\mathbb{E}_x[e^{A^{\mu}_t}]} = \frac{e^{A^{\mu}_s} \mathbb{E}_x[e^{A^{\mu}_t}_s \circ \theta_s | \mathcal{M}_s]}{\mathbb{E}_x[e^{A^{\mu}_t}]} = \frac{e^{A^{\mu}_s} \mathbb{E}_x[e^{A^{\mu}_t}_s]}{\mathbb{E}_x[e^{A^{\mu}_t}]} = L_s \mathbb{E}_x[e^{A^{\mu + F_1}_s} | \mathcal{M}_s] / \mathbb{E}_x[e^{A^{\mu + F_1}_t}].
\]

Letting \(t \to \infty\),

\[
\lim_{t \to \infty} \frac{\mathbb{E}_x[e^{A^{\mu}_s} | \mathcal{M}_s]}{\mathbb{E}_x[e^{A^{\mu}_t}]} = L_s \frac{e^{A^{\mu + F_1}_s} h(X_s)}{h(x)}.
\]

The problem is solved in this case by setting \(M_s\) as follows:

\[
M_s := \frac{e^{A^{\mu}_s} h(X_s)}{h(x)}.
\]

**Remark 4.2.** Let \(P_t^{\mu + F_1}\) be the Feynman-Kac semigroup.
\[ P^{\mu+\mu F_1}_t f(x) = \mathbb{E}_x^L [ e^{A_t^{\mu+\mu F_1}} f(Y_t) ] . \] (10)

Let \( \mathcal{L}^{\mu+\mu F_1} \) be the generator of \( P^{\mu+\mu F_1}_t \). We may regard the above \( h \) as a harmonic function such that \( \mathcal{L}^{\mu+\mu F_1} h(x) = 0 \). Indeed, the Markov property implies

\[
\begin{align*}
  h(x) &= \mathbb{E}^L_x [ e^{A_t^{\mu+\mu F_1}} \circ \theta_t ] \\
  &= \mathbb{E}^L_x [ e^{A_t^{\mu+\mu F_1}} E_{Y_t}^{\mu+\mu F_1} [ e^{A_{\infty}^{\mu+\mu F_1}} ] ] \\
  &= \mathbb{E}^L_x [ e^{A_t^{\mu+\mu F_1}} h(Y_t) ] \\
  &= P^{\mu+\mu F_1}_t h(x)
\end{align*}
\]

so that we obtain

\[
\mathcal{L}^{\mu+\mu F_1} h(x) = \lim_{t \to 0} \frac{P^{\mu+\mu F_1}_t h(x) - h(x)}{t} = 0.
\]

5. Supercritical cases.

If the process \( X \) is transient (resp. recurrent) then we assume \( \mu \in \mathcal{K}_\infty \) (resp. \( \mu \in \mathcal{K}_{\infty,1} \)) and \( F_1 \in \mathcal{A} \). Since \( \lambda(0) < 1 \), there exists \( \theta_0 > 0 \) such that \( \lambda(\theta_0) = 1 \). We then see the asymptotic behavior of \( \mathbb{E}_x [ e^{A_t^{\mu}} ] \) by using the next theorem.

**Theorem 5.1.** Suppose that the process \( X \) is transient (resp. recurrent). If \( \mu \in \mathcal{K}_\infty \) (resp. \( \mu \in \mathcal{K}_{\infty,1} \)) and \( F_1 \in \mathcal{A} \), then there exists a positive function \( h \in L^2(\mu + \mu F_1) \) and \( \theta_0 > 0 \) such that \( \lambda(\theta_0) = 1 \), \( \mathbb{E}^{Y_{\theta_0}_{\theta_0}}_{\theta_0}(h; h) = 1 \), and

\[
\lim_{t \to \infty} e^{-\theta_0 t} \mathbb{E}_x [ e^{A_t^{\mu}} ] = h(x) \int_{\mathbb{R}^n} h(x) dx.
\]

**Proof.** The existence of \( \theta_0 > 0 \) immediately follows from Theorem 3.3. It is trivial that if both \( \mu \) and \( \mu F_1 \) are in \( \mathcal{K}_\infty \) (resp. \( \mathcal{K}_{\infty,1} \)) then \( \mu + \mu F_1 \) is also a member of \( \mathcal{K}_\infty \) (resp. \( \mathcal{K}_{\infty,1} \)).

The compactness of the embedding from \( \mathcal{D}[\mathcal{E}] \) into \( L^2(\mu + \mu F_1) \) (see [31, Theorem 2.7]) implies the existence of the function \( h \in L^2(\mu + \mu F_1) \). In particular, if \( X \) is transient then the uniform boundedness principle implies the existence of \( h \in L^2(\mu + \mu F_1) \) since \( (\mathcal{E}_{\theta_0}^{Y}, \mathcal{D}[\mathcal{E}]) \) is a Hilbert space. Applying Lemma 5.2 stated below with \( g = 1/h \) completes the proof. Note that \( h \) is in \( L^1(dx) \).

**Lemma 5.2.** (i) Let \( \mu \in \mathcal{K}_{\infty,1} \) and let \( h \) be the function given in the proof of Theorem 5.1. Then it holds

\[
\int_{\mathbb{R}^n} p^h(t, x, y)^2 h(y)^2 dy < \infty
\]
for every $t \geq 0$, every $x \in \mathbb{R}^n$, and all $q > 1$. Here, $p^h$ is the heat kernel given by

$$p^h(t, x, y) = e^{-\theta_0 t} \frac{p^{\mu + \mu F_1}(t, x, y)}{h(x)h(y)}$$

(11)

and $p^{\mu + \mu F_1}$ is the heat kernel of the Feynman-Kac semigroup defined in (10).

(ii) Let $\mu \in K_{\infty,1}$, let $h$ be the function as in (i) and let $P^h_t$ be a semigroup whose heat kernel is $p^h$. Then it holds

$$\lim_{t \to \infty} P^h_t g(x) = \int_{\mathbb{R}^n} g(x)h(x)^2 dx$$

for all $x \in \mathbb{R}^n$, all $g \in L^p(h^2 dx)$, and all $p > 1$.

Proof. (i) Note that $h$ is a harmonic function of the equation.

$$(L^Y + \mu + \mu F_1)h(x) = \theta_0 h(x).$$

Since $\mu + \mu F_1 \in K_{\infty,1}$, [31, Lemma 4.1] implies that it holds

$$c|x|^{-(n+\alpha)} \leq h(x) \leq C|x|^{-(n+\alpha)/q_1}$$

for all $1 < q_1 < 2$ and all $|x| > 1$. We also find the upper bound of the heat kernel $p^{\mu + \mu F_1}$ off the diagonal set:

$$p^{\mu + \mu F_1}(t, x, y) \leq c|x - y|^{-(n+\alpha)/q_2}$$

for all $q_2 > 1$ and every $t > 0$ from [31, Lemma 4.3]. Combining the last two results, we find $L^p(h^2 dx)$-integrability of the heat kernel $p^h$ if we take $q_1$ and $q_2$ close to 1. (ii) Take $g \in L^p(h^2 dx)$ arbitrarily. The maximal ergodic theorem follows from [31, Lemma 4.5] since $\mu + \mu F_1 \in K_{\infty,1}$.

$$\left\| \sup_{t>0} P^h_t g \right\|_{L^p(h^2 dx)} \leq C_p \| g \|_{L^p(h^2 dx)}.$$

This implies $\sup_{t>0} P^h_t g(x) < \infty$. Thus we have the desired result

$$\lim_{t \to \infty} P^h_t g(x) = \int_{\mathbb{R}^n} g(x)h(x)^2 dx$$

for every $x \in \mathbb{R}^n$. \qed

We now solve the Feynman-Kac penalization problem in the supercritical case. Using Theorem 5.1, we find
\[
\begin{align*}
\mathbb{E}_x[e^{A^{\mu,F}_t} | M_s] &= \frac{\mathbb{E}_x[e^{-\theta_0 t} e^{A^{\mu,F}_t} | M_s]}{\mathbb{E}_x[e^{-\theta_0 t} e^{A^{\mu,F}_t}]} \\
&= e^{-\theta_0 s} e^{A^{\mu,F}_s} \mathbb{E}_x[e^{-\theta_0 (t-s) e^{A^{\mu,F}_{t-s}}} | M_s] \\
&= e^{-\theta_0 s} e^{A^{\mu,F}_s} \{ e^{-\theta_0 (t-s)} \mathbb{E}_x[e^{A^{\mu,F}_{t-s}}] \} \\
&= e^{-\theta_0 t} \mathbb{E}_x[e^{A^{\mu,F}_t}].
\end{align*}
\]

Letting \( t \to \infty \), we obtain

\[
\lim_{t \to \infty} \mathbb{E}_x[e^{A^{\mu,F}_t} | M_s] = \frac{e^{-\theta_0 s} e^{A^{\mu,F}_s} h(X_s) \int_{\mathbb{R}^n} h(x) dx}{h(x) \int_{\mathbb{R}^n} h(x) dx} = \frac{e^{-\theta_0 s} e^{A^{\mu,F}_s} h(X_s)}{h(x)}.
\]

Scheffé’s lemma implies that the above convergence is in \( L^1(\mathbb{P}_x) \). The weight process \( M_s \) is given by

\[
M_s := \frac{e^{-\theta_0 s} e^{A^{\mu,F}_s} h(X_s)}{h(x)}.
\]

\section*{6. Critical cases.}

We use Chacon-Ornstein type ergodic theorem to solve the Feynman-Kac penalization problem in this case. We have to treat a subclass of the Green-tight Kato class to use this theorem.

Note that the extended Dirichlet space \( D_e[\mathcal{E}] \) is a Hilbert space (see \cite[Lemma 1.5.5]{13}). We further see that the embedding of \( D_e[\mathcal{E}] \) into \( L^2(\mu + \mu_F) \) is also compact by \cite[Theorem 10]{33}. Then we obtain a harmonic function in \( D_e[\mathcal{E}] \). We weight the probability measure by an exponential martingale \( L_t \) and the harmonic function \( h \):

\[
d_{\mathbb{P}_x^{L,h}} := N_t d_{\mathbb{P}_x}^{L}, \quad N_t := \frac{h(Y_t)}{h(x)} e^{A^{\mu+F}_t}. \]

Hereafter, we consider the Markov process \((\Omega, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_x^{L,h}, Y_t)\), or \( \mathbb{M}^{L,h} \) for short.

We define the special Kato class.

\textbf{Definition 6.1. } (i) Let \( \mu \) be a measure of Kato class \( \mathcal{K} \). \( \mu \) is said to be in the special Kato class if it holds

\[
\sup_{x \in \mathbb{R}^n} |x|^\alpha \int_{\mathbb{R}^n} |x-y|^\alpha \mu(dy) < \infty.
\]

We denote this class by \( \mathcal{K}_s \).
(ii) A PCAF $A$ is said to be special with respect to $\mathbb{M}^{L,h}$, if it holds
\[ \sup_{x \in \mathbb{R}^n} \mathbb{E}_x^{L,h} \left[ \int_0^\infty \exp \left( - \int_0^t g(X_u)du \right) dA_t \right] < \infty \]
for any positive Borel function $g$ with $\int_{\mathbb{R}^n} g(x)dx > 0$.

**Definition 6.2.** A bounded measurable symmetric positive function $F$ vanishing on the diagonal set is said to be in the class $A_s$ if $\mu_F$ is in $K_s$.

For the rest part of this section, we assume $\mu \in K_s$ and $F_1 \in A_s$. One can easily check the following properties.

**Lemma 6.3.** (i) $K_s$ is the subset of $K_\infty$.
(ii) $\mu + \mu_{F_1}$ is also a member of $K_s$.

**Proof.** See [30, Section 4] about (i). Noting $G(x, y) = c_{\alpha,n}|x - y|^{\alpha-n}$, (ii) also immediately follows the equivalence of Green function $G$ and the transformed Green function $G_Y$. □

One can prove the following lemmas just as in [30, Section 4].

**Lemma 6.4.** For all PCAFs $B$, it holds
\[ \mathbb{E}_x^{L}\left[ \int_0^t e^{A_{\mu+\mu F_1}^u - Bu}dA_u \right] = h(x)\mathbb{E}_x^{L,h}\left[ \int_0^t e^{-Bu} \frac{dA_{\mu+\mu F_1}^u}{h(Y_u)} \right] \]
for every $x \in \mathbb{R}^n$ and $t \geq 0$.

**Lemma 6.5.** $\int_0^t (1/h(Y_u))dA_{\mu+\mu F_1}^u$ is special with respect to $\mathbb{M}^{L,h}$ if $F_1 \in A_s$.

**Proof.** See the proof of [30, Lemma 4.3] and [30, Lemma 4.4]. □

By using Lemma 6.4,
\[ \mathbb{E}_x^{L}\left[ e^{A_{\mu+\mu F_1}^u} \right] = 1 + \mathbb{E}_x^{L}\left[ \int_0^t e^{A_{\mu+\mu F_1}^u} dA_{\mu+\mu F_1}^u \right] = 1 + h(x)\mathbb{E}_x^{L,h}\left[ \int_0^t \frac{dA_{\mu+\mu F_1}^u}{h(Y_u)} \right]. \]

Integrating by an arbitrary finite positive measure $\nu$, we see
\[ \mathbb{E}_x^{L}\left[ e^{A_{\mu+\mu F_1}^u} \right] = \nu(\mathbb{R}^n) + \langle \nu, h \rangle \mathbb{E}_x^{L,h}\left[ \int_0^t \frac{dA_{\mu+\mu F_1}^u}{h(Y_u)} \right], \] (13)
where $\nu^h := (h \cdot \nu)/\langle \nu, h \rangle$ and $\langle \nu, h \rangle := \int_{\mathbb{R}^n} h(x) d\nu$.

We define a function $\psi$ as follows.
\[ \psi(t) := \mathbb{E}_{x}^{L,h} \left[ \int_{0}^{t} k(Y_{u})du \right], \quad (14) \]

where \( k \) is an arbitrary continuous and positive function with a compact support. We here give some properties of the function \( \psi \).

**Lemma 6.6.** Let \( \psi \) be as in (14).

(i) \( \lim_{t \to \infty} \psi(t) = \infty \).

(ii) For every \( s > 0 \) it holds

\[ \lim_{t \to \infty} \frac{\psi(t+s)}{\psi(t)} = 1. \]

**Proof.** Let \( \psi(y) := \int_{\mathbb{R}^n} G^{h}(y, z)k(z)h^2(z)dz \) and \( G^{h}(y, z) := \int_{0}^{\infty} p^{h}(t, y, z)dt \) for every \( y, z \in \mathbb{R}^n \), where \( p^{h} \) is the heat kernel given by the formula (11). The recurrence of \( M_{L,h} \) implies \( G^{h}(y) = \infty \) \( h^2dy \)-a.e. We then obtain (i): The Markov property implies

\[ \psi(t) \geq \mathbb{E}_{x}^{L,h} \left[ \int_{1}^{t} k(Y_{u})du \right] \]

\[ = \mathbb{E}_{x}^{L,h} \left[ \mathbb{E}_{Y_{1}}^{L,h} \left[ \int_{0}^{t-1} k(Y_{u})du \right] \theta_{1} \left| \mathcal{M}_{1} \right. \right] \]

\[ = \mathbb{E}_{x}^{L,h} \left[ \mathbb{E}_{Y_{1}}^{L,h} \left[ \int_{0}^{t-1} k(Y_{u})du \right] \right] \]

\[ = \int_{\mathbb{R}^n} p^{h}(1, x, y) \int_{\mathbb{R}^n} \int_{0}^{t-1} p^{h}(u, y, z)duk(z)h^2(z)dzh^2(y)dy \]

\[ \to \int_{\mathbb{R}^n} p^{h}(1, x, y)G^{h}(y)h^2(y)dy \]

\[ = \infty \]

as \( t \to \infty \). Combining (i) and the boundedness of \( k \), we see that (ii) follows. \( \square \)

We quote Chacon-Ornstein type ergodic theorem.

**Theorem 6.7 ([4]).** Let \( \nu_{1} \) and \( \nu_{2} \) be arbitrary probability measures and let \( B_{t} \) and \( C_{t} \) be special PCAFs with respect to \( M_{L,h} \). Suppose \( \int_{0}^{t} f(Y_{u})dB_{u} \) and \( \int_{0}^{t} g(Y_{u})dC_{u} \) are special PCAFs with respect to \( M_{L,h} \). It then holds

\[ \lim_{t \to \infty} \frac{\mathbb{E}_{\nu_{1}}^{L,h} \left[ \int_{0}^{t} f(Y_{u})dB_{u} \right]}{\mathbb{E}_{\nu_{2}}^{L,h} \left[ \int_{0}^{t} g(Y_{u})dC_{u} \right]} = \frac{\langle h^2\mu_{B}, f \rangle}{\langle h^2\mu_{C}, g \rangle} \]

for arbitrary bounded positive Borel-measurable functions \( f \) and \( g \). Here, \( \mu_{B} \) and \( \mu_{C} \) are Revuz measures corresponding to \( B_{t} \) and \( C_{t} \) respectively.
Now, we solve the Feynman-Kac penalization problem in the critical case. Using the formula (13), Lemma 6.6, and Theorem 6.7,

\[
\lim_{t \to \infty} \mathbb{E}_L \left[ \frac{e^{A \mu + \mu F_1}}{\psi(t)} \right] = \lim_{t \to \infty} \frac{\nu(\mathbb{R}^n)}{\psi(t)} + \langle \nu, h \rangle \mathbb{E}_L^h \left[ \int_0^t \frac{1}{h(Y_u)} dA_u^{\mu + \mu F_1} \right] \mathbb{E}_L^h \left[ \int_0^t k(Y_u) du \right]
\]

\[
= \lim_{t \to \infty} \langle \nu, h \rangle \mathbb{E}_L^h \left[ \int_0^t \frac{1}{h(Y_u)} dA_u^{\mu + \mu F_1} \right] \mathbb{E}_L^h \left[ \int_0^t k(Y_u) du \right]
\]

\[
= \langle \nu, h \rangle \frac{\langle \mu + \mu F_1, h \rangle}{\langle h^2 dx, k \rangle}.
\]

We set a finite positive measure \( \nu \) for every \( B \in \mathcal{B} \) as follows:

\[
\nu(B) := \mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} S; Y_s \in B \right].
\]

Note that the Markov property implies

\[
\mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} S \right] = \mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} S \mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} \circ \theta_s | \mathcal{M}_s \right] \right]
\]

\[
= \mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} \mathbb{E}_Y^L \left[ e^{A \mu + \mu F_1} \right] \right]
\]

\[
= \mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} \right].
\]

Lemma 6.6 and Theorem 6.7 yield

\[
\lim_{t \to \infty} \mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} S \right] = \lim_{t \to \infty} \mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} \right] / \psi(t)
\]

\[
= \lim_{t \to \infty} \frac{\mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} \right] / \psi(t)}{\mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} \right] / \psi(t)}
\]

\[
= \lim_{t \to \infty} \frac{\mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} \right] / \psi(t)}{\mathbb{E}_x^L \left[ e^{A \mu + \mu F_1} \right] / \psi(t)}
\]

\[
= \frac{\langle \nu, h \rangle \langle \mu + \mu F_1, h \rangle}{\langle h^2 dx, k \rangle}
\]

\[
= \frac{\langle \nu, h \rangle}{h(x)}.
\]

Rewriting the last limit, we completely solve the problem.
\[
\frac{\langle \nu, h \rangle}{h(x)} = \frac{\mathbb{E}_x^L [e^{A_{x}^{\nu}F_1} h(Y_s)S]}{h(x)} = \mathbb{E}_x \left[ L_s e^{A_{x}^{\nu}F_1} h(X_s) \frac{S}{h(x)} \right] = \mathbb{E}_x [M_s S] = \mathbb{E}_x^M [S].
\]

Here, the weight process \(M_s\) is as follows:

\[
M_s := e^{A_{X}^{\mu,F}} h(X_s) \frac{S}{h(x)}. \tag{15}
\]

**Remark 6.8.** Since \(M^{L,h}\) is an irreducible recurrent \(h^2 dx\)-symmetric right process, the ergodic theorem yields

\[
\lim_{t \to \infty} \frac{\psi(t)}{t} = \begin{cases} 
\langle h^2 dx, k \rangle & \text{if } h \in L^2(\mathbb{R}^n, dx) \\
0 & \text{if } h \not\in L^2(\mathbb{R}^n, dx).
\end{cases}
\]

We see that \(h \in L^2(\mathbb{R}^n, dx)\) (positive critical) if and only if \(n > 2\alpha\) and \(h \not\in L^2(\mathbb{R}^n, dx)\) (null critical) if and only if \(\alpha < n \leq 2\alpha\), since \(c^{-1} |x|^{\alpha-n} \leq h(x) \leq c|x|^{\alpha-n}\) for all \(|x| > 1\). Consequently, we see the asymptotic behavior of the non-local Feynman-Kac semigroup \(P_t f(x) := \mathbb{E}_x [e^{A_{X}^{\mu,F}} f(X_t)]:\)

\[
P_t^{\mu,F} f(x) \begin{cases} 
\sim \mathbb{E}_x [e^{A_{X}^{\mu,F}}] & \text{if } \lambda(0) > 1 \\
\sim \left( h(x) \int_{\mathbb{R}^n} h(x) dx \right) e^{\theta_0 t} & \text{if } \lambda(0) < 1 \\
\sim \left( h(x) \int_{\mathbb{R}^n} h(x) d(\mu + \mu_{F_1}) \right) t & \text{if } \lambda(0) = 1 \text{ and } n \geq 2\alpha \\
= o(t) & \text{if } \lambda(0) = 1 \text{ and } \alpha < n \leq 2\alpha
\end{cases}
\]

as \(t \to \infty\). We further see the growth of \(L^p\)-spectral bounds for all \(1 \leq p \leq \infty\) (see [32, Theorem 5.6]). Let \(l_p := -\lim_{t \to \infty} (1/t) \log \| P_t^{\mu,F} \|_{p,p}\). Then

\[
l_p = \begin{cases} 
0 & \text{if } \lambda(0) \geq 1 \\
-\theta_0 & \text{if } \lambda(0) < 1.
\end{cases}
\]

This implies that our definition of (sub-, super-)criticality corresponds to Simon’s definition (see p. 218 of [26]).
7. Examples of jumping functions.

We give some concrete examples of jumping functions which belong to the class $\mathcal{A}$ and the class $\mathcal{A}_s$ (see Definition 2.3 and Definition 6.2 for the definitions of them). We assume $n > \alpha$ in this section. Since the Green function of $X$ is $c_{\alpha,n}|x - y|^{\alpha - n}$ and the Lévy kernel of $X$ is $c_{\alpha,n}|x - y|^{-\alpha - n}$, $F \in \mathcal{A}$ is equivalent to

$$\lim_{R \to \infty, r \to 0} \sup_{x \in \mathbb{R}^n} \int_{B(0,R) \cup B(x,r)} dy \, |x - y|^{\alpha - n} \int_{\mathbb{R}^n} dz \, |y - z|^{-\alpha - n} F(y, z) = 0$$

(16)

and $F \in \mathcal{A}_s$ is equivalent to

$$\sup_{x \in \mathbb{R}^n} |x|^{n - \alpha} \int_{\mathbb{R}^n} dy \, |x - y|^{\alpha - n} \int_{\mathbb{R}^n} dz \, |y - z|^{-\alpha - n} F(y, z) < \infty.$$  

(17)

We first give a well-known example.

**Example 7.1.** Let $K_1$ and $K_2$ be two disjoint compact subsets on $\mathbb{R}^n$ and let $F(x, y)$ be as follows:

$$F(x, y) := \mathbf{1}_{K_1}(x) \mathbf{1}_{K_2}(y) + \mathbf{1}_{K_2}(x) \mathbf{1}_{K_1}(y).$$

We here check that this satisfies the conditions (16) and (17). It suffices to estimate the integral

$$I(x) := \int_{B(0,R) \cup B(x,r)} dy \, |x - y|^{\alpha - n} \int_{\mathbb{R}^n} dz \, |y - z|^{-\alpha - n} (\mathbf{1}_{K_1}(y) \mathbf{1}_{K_2}(z) + \mathbf{1}_{K_2}(y) \mathbf{1}_{K_1}(z)).$$

sup\{|y - z|^{-\alpha - n}; y \in K_1 \text{ and } z \in K_2\} is bounded so that the integral $I(x)$ can be estimated:

$$I(x) \leq c \int_{B(0,R) \cup B(x,r)} dy \, |x - y|^{\alpha - n} \int_{\mathbb{R}^n} dz \, (\mathbf{1}_{K_1}(y) \mathbf{1}_{K_2}(z) + \mathbf{1}_{K_2}(y) \mathbf{1}_{K_1}(z))$$

$$\leq c \int_{B(0,R) \cup B(x,r)} dy \, |x - y|^{\alpha - n} (|K_2| \mathbf{1}_{K_1}(y) + |K_1| \mathbf{1}_{K_2}(y))$$

$$\leq c \int_{(B(0,R) \cup B(x,r)) \cap (K_1 \cup K_2)} dy \, |x - y|^{\alpha - n}$$

$$\leq c \int_{B(x,r)} dy \, |y - x|^{\alpha - n},$$

where $|K_j|$ is the Lebesgue measure of $K_j$ ($j = 1, 2$). We take $R > 0$ such that $B(0, R) \supset K_1 \cup K_2$ in the fourth line. We then see that
\[ \sup_{x \in \mathbb{R}^n} I(x) \leq c \int_{B(0,r)} dy |y|^{\alpha-n} \]
\[ \leq c \int_0^r \rho^{\alpha-1} d\rho \]
\[ \leq cr^\alpha. \]

We have used the polar coordinates transform in the second line. Since the constant \( c \) is independent of \( x \), letting \( r \to 0 \) completes the check. One can also check the condition (17).

We give another example. Thus far, it has been unknown whether there exists a jumping function of the class \( \mathcal{A} \) that has a full support. We provide such a function in the following example.

**Example 7.2.**

\[ F(x, y) = (1 \wedge |x - y|^p)\langle x \rangle^{-q} \langle y \rangle^{-q} \quad \text{for } p > \alpha \text{ and } q > n. \]

Here, \( \langle x \rangle := \sqrt{1 + |x|^2} \). We check the condition (16). Note that \( \langle x + y \rangle \leq \sqrt{2}\langle x \rangle \langle y \rangle \) for all \( x, y \in \mathbb{R}^n \). We take \( x \) arbitrarily.

\[
\int_{B(0,R)^c \cup B(x,r)} dy \int_{\mathbb{R}^n} dz |x - y|^{\alpha-n} |y - z|^{-\alpha-n} (1 \wedge |y - z|^p) \langle y \rangle^{-q} \langle z \rangle^{-q} \\
\leq \int_{B(x,R)^c \cup -B(x,r) + x} dy \int_{\mathbb{R}^n} dz |y|^{\alpha-n} |z|^{-\alpha-n} (1 \wedge |z|^p) \langle x - y \rangle^{-q} \langle x - y - z \rangle^{-q} \\
\leq c \int_{B(x,R)^c \cup -B(x,r) + x} dy \int_{\mathbb{R}^n} dz |y|^{\alpha-n} |z|^{-\alpha-n} (1 \wedge |z|^p) \langle x - y \rangle^{-q} \langle x \rangle^{\delta q} \langle y \rangle^{\delta q} \langle z \rangle^{-\delta q} \\
\leq c \langle x \rangle^{\delta q} \int_{B(x,R)^c \cup -B(x,r) + x} dy |y|^{\alpha-n} \langle x - y \rangle^{-q} \langle y \rangle^{\delta q} \\
\times \left( \int_{\mathbb{R}^n} dz |z|^{-\alpha-n} (1 \wedge |z|^p) \langle z \rangle^{-\delta q} \right).
\]

Here, \(-B(x,r) + x := \{ x - y; y \in B(x,r) \} \) and \( \delta > 0 \) is so close to 0. We have replaced \( x - y \) and \( y - z \) by \( y \) and \( z \) in the second line respectively. We have used two estimates: \( \langle y - z \rangle^{-q} \leq c \langle y \rangle^{-\delta q} \) and \( \langle y - z \rangle^{-1} \leq \sqrt{2}\langle y \rangle \langle z \rangle^{-1} \) in the third line. It is easy to see that the second factor of the fourth line is dominated by a constant independent of \( x \). We take \( x \) arbitrarily. Note that the condition \( p > \alpha \) is needed for this estimate.

The first factor of the fourth line is estimated as follows.

\[
\langle x \rangle^{\delta q} \int_{-B(x,r) + x \cup B(x,R)^c} |y|^{\alpha-n} \langle y - x \rangle^{-q} \langle y \rangle^{\delta q} dy \\
\leq c \langle x \rangle^{\delta q} \int_{B(x,r) \cup B(0,R)^c} |x - w|^{\alpha-n} \langle x - w \rangle^{\delta q} \langle x \rangle^{-(1-\delta)q} \langle x - w \rangle^{\delta q} dw
\]
\[ = c \int_{B(x,r) \cup B(0,R)^c} |x - w|^{\alpha-n} \langle x - w \rangle^{2 \delta q} \langle w \rangle^{-(1-\delta)q} dw \quad (=: I(x)). \]

Here, \( w := x - y \). We consider two cases: one is \( x \in B(0, R) \) and the other is \( x \in B(0, R)^c \).

\[
\sup_{x \in B(0, R)} I(x) \leq \sup_{x \in B(0, R)} \left( \int_{B(x,r)} + \int_{B(0,R)^c} \right) |x - w|^{\alpha-n} \langle x - w \rangle^{2 \delta q} \langle w \rangle^{-(1-\delta)q} dw \\
\leq c \sup_{x \in B(0, R)} \int_0^r \rho^{\alpha-n} \cdot \rho^{n-1} d\rho + r^{\alpha-n} \int_R^{\infty} \rho^{-(1-\delta)q+n-1} d\rho \\
\leq c (r^\alpha + r^{\alpha-n} R^{n-q+\delta q}).
\]

\[
\sup_{x \in B(0, R)^c} I(x) \leq \sup_{x \in B(0, R)^c} \left( \int_{B(x,r) \cap B(0,R)^c} + \int_{B(x,r)^c \cap B(0,R)^c} \right) |x - w|^{\alpha-n} \\
\times \langle x - w \rangle^{2 \delta q} \langle w \rangle^{-(1-\delta)q} dw \\
\leq cr^\alpha + cr^{\alpha-n} R^{n-q+\delta q} \\
\leq c (r^\alpha + r^{\alpha-n} R^{n-q+\delta q}).
\]

Here, \(|B(0, r)|\) is the volume of \( B(0, r) \). If for an arbitrary \( \varepsilon > 0 \) we take \( r, R > 0 \) such that \( r^\alpha < \varepsilon \) and \( r^{\alpha-n} R^{n-q+\delta q} < \varepsilon \) then \( F \in A \).

We see that some jumping functions of \( A \) have full support and are in \( L^1(\mathbb{R}^n \times \mathbb{R}^n) \). We also give a concrete example of jumping functions in the class \( A_s \).

**Example 7.3.**

\[ F(x, y) = (1 \wedge |x - y|^p) \langle x \rangle^{\gamma-q} \langle y \rangle^{-q} \quad \text{for } p > \alpha \text{ and } q > 2n - \alpha. \]

We check that this function satisfies the condition (17).

\[
|x|^{\alpha-n} \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} dz |x - y|^{\alpha-n} |y - z|^{\alpha-n} (1 \wedge |y - z|^p) \langle y \rangle^{-q} \langle z \rangle^{-q} \\
\leq |x|^{\alpha-n} \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} dz |y|^{\alpha-n} |z|^{\alpha-n} (1 \wedge |z|^p) \langle x - y \rangle^{-q} \langle x - y - z \rangle^{-q} \\
\leq c |x|^{\alpha-n} \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} dz |y|^{\alpha-n} |z|^{\alpha-n} (1 \wedge |z|^p) \langle x - y \rangle^{-q} \langle x \rangle^{\delta q} \langle y \rangle^{\delta q} \langle z \rangle^{-\delta q} \\
\leq c (|x|^{\alpha-n} \langle x \rangle^{\delta q} \int_{\mathbb{R}^n} dy |y|^{\alpha-n} \langle x - y \rangle^{-q} \langle y \rangle^{\delta q}) \cdot \left( \int_{\mathbb{R}^n} dz |z|^{\alpha-n} (1 \wedge |z|^p) \langle z \rangle^{-\delta q} \right).
\]

Here, \( \delta > 0 \) is so close to 0. It is easy to see that the second factor of the fourth line is dominated by a constant independent of \( x \). The condition \( p > \alpha \) is then needed.

Consequently, we have only to prove that
$|x|^{n-\alpha} \langle x \rangle^{\delta q} \int_{\mathbb{R}^n} |y|^{\alpha-n} \langle y-x \rangle^{-q} \langle y \rangle^{\delta q} dy$

is uniformly dominated by a constant independent of $x$. We divide the last integral into three parts:

$$\int_{\mathbb{R}^n} = \int_{\{y\leq 1\}} + \int_{\{y>1 \text{ and } |y-x|\leq |x|/2\}} + \int_{\{y>1 \text{ and } |y-x|>|x|/2\}} (=: I + II + III).$$

Noting $\langle y-x \rangle^{-q} \leq 2^{q/2} \langle x \rangle^{-q} \langle y \rangle^q, \langle x \rangle^{(\delta-1)q} \leq 1$, and $\langle y \rangle^{(1+\delta)q} \leq 2^q$ for all $x \in \mathbb{R}^n$ and $|y| \leq 1$,

$$I \leq c|x|^{n-\alpha} \langle x \rangle^{\delta q} \int_{\{y\leq 1\}} |y|^{\alpha-n} \langle y-x \rangle^{-q} \langle y \rangle^{(1+\delta)q} dy$$

$$\leq c|x|^{n-\alpha} \langle x \rangle^{(\delta-1)q} \int_0^1 \rho^{\alpha-n} \rho^{n-1} d\rho$$

$$\leq c(|x|^{n-\alpha} \wedge |x|^{n-\alpha+(\delta-1)q}).$$

We use the polar coordinates transform in the second line. Thus I is dominated by a constant independent of $x$. The estimate of II is tricky. Note that if $|x| < 2/3$ then II = 0 since the subset $\{y \geq 1 \text{ and } |y-x| \leq |x|/2\}$ is empty, that $|x-y| \leq |x|/2$ implies $|y|^{\alpha-n} \leq 2^{n-\alpha}|x|^{\alpha-n}$ and that $\langle y \rangle^{\delta q} \leq 2^{\delta q/2} \langle x \rangle^{\delta q} \langle y-x \rangle^{\delta q}$ holds. We then see

$$II \leq c\langle x \rangle^{\delta q} \int_{\{y\geq 1 \text{ and } |y-x|\leq |x|/2\}} \langle y-x \rangle^{-q} \langle y \rangle^{\delta q} dy$$

$$\leq c\langle x \rangle^{2\delta q} \int_{\{y\geq 1 \text{ and } |y-x|\leq |x|/2\}} \langle y-x \rangle^{(\delta-1)q} dy$$

$$\leq c|x|^{2\delta q} \int_{|x|/2}^{|x|/2} \rho^{n-1}(1+\rho^2)^{(\delta-1)q/2} d\rho$$

$$\leq c|x|^{2\delta q} \cdot |x|^{n+(\delta-1)q}$$

$$\leq c|x|^{n+(3\delta-1)q}$$

for $|x| \geq 2/3$. Since $n+(3\delta-1)q < 0$, II is also dominated by a constant independent of $x$. The estimate of III is also tricky.

$$III \leq c|x|^{n-\alpha} \langle x \rangle^{\delta q} \int_{\{y\geq 1 \text{ and } |y-x|\geq |x|/2\}} |y|^{\alpha-n} \langle y-x \rangle^{(\delta-1)q} \langle y \rangle^{\delta q} dy$$

$$\leq c|x|^{n-\alpha} \langle x \rangle^{\delta q} \int_{\{y\geq 1 \text{ and } |y-x|\geq |x|/2\}} \langle y \rangle^{\alpha-n+\delta q} \langle y-x \rangle^{(\delta-1)q} dy$$
\[ \leq c|x|^{n-\alpha} \langle x \rangle^{\delta q} \int_{\{ |y| \geq 1 \text{ and } |y-x| \geq |x|/2 \}} \langle x-y \rangle^{\alpha-n-\delta q} \langle y-x \rangle^{(\delta-1)q} dy \]
\[ \leq c|x|^{n-\alpha} \langle x \rangle^{n-\alpha} \int_{\{ |y-x| \geq |x|/2 \}} \langle x-y \rangle^{\alpha-n+(\delta-1)q} dy \]
\[ \leq c|x|^{n-\alpha} \langle x \rangle^{n-\alpha} \int_{|x|/2}^{\infty} (1 + \rho^2) (\alpha-n+(\delta-1)q)/2 \rho^{n-1} d\rho \]
\[ \leq c|x|^{n-\alpha} \langle x \rangle^{n-\alpha} (|x|^n \land |x|^{\alpha+(\delta-1)q}) \]
\[ \leq c(|x|^{2n-\alpha} \land |x|^{2n-\alpha+(\delta-1)q}). \]

We have used the estimate \(|y|^{n-n} \leq 2^{(n-\alpha)/2} \langle y \rangle^{\alpha-n}\) for all \(|y| \geq 1\) in the first line. Since \(\alpha-n+\delta q < 0\), \(\langle y \rangle^{\alpha-n+\delta q} \leq 2^{(n-\alpha-\delta q)/2} \langle x-y \rangle^{\alpha-n+\delta q} \langle x \rangle^{\alpha-n-\delta q}\). We use this in the third line. Since \(2n-\alpha+(\delta-1)q < 0\), III is also uniformly dominated by a constant independent of \(x\).

**Remark 7.4.** We further see that the jumping function of Example 7.3 does not belong to \(A_2\) (see also [7, Definition 2.3]), that is, it does not hold

\[ \lim_{R \to \infty, r \to 0} \sup_{(x,w) \in d^c} |x-w|^{n-\alpha} \int_{B(x,r) \cup B(0,R)^c \times B(x,r) \cup B(0,R)^c} |x-y|^{\alpha-n} \times (1 \land |y-z|^{-(\alpha+n)}) \langle y \rangle^{-q} \langle z \rangle^{-q} |z-w|^{\alpha-n} |y-z|^{-(n+\alpha)} dydz = 0. \]

Indeed, we may take a closed ball \(B_{x,w}\) with radius 1 in \(\{(y, z); |y-x| \leq 1, |z-w| \leq 1, 1 \leq |y-z| \leq 5, \text{ and } |y|, |z| \geq R\}\) for an arbitrary \(R > 0\). It then follows

\[ |x-w|^{n-\alpha} \int_{B(x,r) \cup B(0,R)^c \times B(x,r) \cup B(0,R)^c} |x-y|^{\alpha-n} (1 \land |y-z|^{-(\alpha+n)}) \times \langle z \rangle^{-q} |z-w|^{\alpha-n} |y-z|^{-(n+\alpha)} dydz \]
\[ \geq c|B(0,1)||x-w|^{n-\alpha} \langle x \rangle^{-q} \langle w \rangle^{-q}. \]

Here, \(|B(0,1)|\) is the volume of \(B(0,1)\). Therefore we find that \(A_2 \not\supset A_s\) and \(A_2 \not\subset A_s\).

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**References**

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