On the limit of a monotonous sequence of Cousin's domains

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§ 0. Introduction.

In the previous paper [8] it is remarked that the limit of a monotonously decreasing sequence of Cousin-I domains in \( C^n \) is not necessarily a Cousin-I domain for \( n \geq 3 \). In the present paper we shall prove that the limit of a monotonously increasing sequence of Cousin-I domains over a Stein manifold is a Cousin-I domain. Concerning the Cousin-II problem, however, we can prove that the limit of the monotonously increasing sequence of Cousin-II domains over a Stein manifold is a Cousin-II domain, only in case that it is simply connected. The proof is based on the theory of domains of holomorphy due to Docquier-Grauert [5] and the approximation theory due to Behnke [1].

§ 1. Increasing sequence of domains.

Let \( \mathbb{M} \) be a complex manifold. We say that \( (D, \Phi) \) is a domain over \( \mathbb{M} \) if \( \Phi \) is a holomorphic mapping of a complex manifold \( D \) into \( \mathbb{M} \) such that \( \Phi \) is locally biholomorphic. A domain \( (D, \Phi) \) over \( \mathbb{M} \) is called a covering manifold over \( \mathbb{M} \) if there exists a neighbourhood \( U \) of any point \( x \) of \( \mathbb{M} \) such that \( \Phi \) maps each connected component of \( \Phi^{-1}(U) \) biholomorphically onto \( U \). Let \( (D_1, \Phi_1) \) and \( (D_2, \Phi_2) \) be domains over \( \mathbb{M} \). We say that \( (D_1, \Phi_1) \) is a domain over \( (D_2, \Phi_2) \) and write \( (D_1, \Phi_1) < (D_2, \Phi_2) \) if there exists a holomorphic mapping \( \tau \) of \( D_1 \) in \( D_2 \) such that \( \Phi_1 = \Phi_2 \circ \tau \). By this relation \( < \) the set of all domains over \( \mathbb{M} \) forms a partially ordered set. We consider a sequence \( \{(D_n, \Phi_n); n = 1, 2, 3, \ldots\} \) of domains over \( \mathbb{M} \) such that \( (D_n, \Phi_n) < (D_{n+1}, \Phi_{n+1}) \) for \( n \geq 1 \) and call it a monotonously increasing sequence of domains over \( \mathbb{M} \). Then there exists a holomorphic mapping \( \tau_m^n \) of \( D_n \) in \( D_m \) such that \( \Phi_n = \Phi_m \circ \tau_m^n \) for any \( m \geq n \).

Let \( E \) be the subset of the product set \( \prod_{n=1}^{\infty} D_n \) consisting of all \( (x_n) \) which satisfies \( x_n = \tau_n^N(x_N) \) (\( n \geq N \)) for some \( N \). We say that \( (x_n) \) and \( (y_n) \in E \) are equivalent modulo \( R \) if \( x_n = y_n \) (\( n \geq N \)) for some \( N \). The factor set \( E/R \) is denoted by \( D \). Let \( x_n \) be a point of \( D_n \). We put \( x_m = \tau_m^n(x_n) \) for \( m \geq n \) and
take $x_m$ arbitrarily for $m < n$. If we define that $\tau_n(x_n)$ is the class which has $(x_n)$ as a representative, then we have a mapping $\tau_n$ of $D_n$ in $D$. Let $x$ be an element of $D$, which is represented by $(x_n) \in E$ such that $x_n = \tau_n^x(x_n)$ $(n \geq N)$. If we put $\Phi(x) = \Phi_n(x_n)$, we have a mapping $\Phi$ of $D$ in $\mathbb{M}$ such that $\Phi_n = \Phi \circ \tau_n$ for $n \geq 1$. Let $U_N$ be an open neighbourhood of $x_N$ such that the restriction $\Phi_N|U_N$ of $\Phi_N$ to $U_N$ is a biholomorphic mapping of $U_N$ onto a local coordinate neighbourhood of $\Phi_N(x_n)$ in $\mathbb{M}$. If we put $U_n = \tau_n^x(U_N)$ for $n \geq N$, $\tau_n^x$ maps $U_N$ biholomorphically onto $U_n$. Hence $\tau_n|U_N$ is an injective mapping of $U_N$ onto the subset $V(x)$ of $D$ consisting of all elements of $D$ which have a representative $(x_n) \in E$ such that $x_n = \tau_n^x(x_n)$ $(n \geq N)$ for $x_n \in U_N$. A subset of $D$ containing such $V(x)$ is called a neighbourhood of $x$. If we define neighbourhoods of $D$ in this way, $D$ is a Hausdorff space. Let $\mu$ be a biholomorphic mapping of $\Phi_n(U_N)$ onto an open set $Z$ of a complex Euclidean space. Then $\mu \circ (\Phi|V) = \mu \circ (\tau_n|U_N)^{-1}$ is a homeomorphism of $V(x)$ onto $Z$. Let $x'$ be another point of $D$. Then there exist, respectively, neighbourhoods $V(x')$ and $U_N$, of $x'$ and $x_N$, $D_N$, such that $\Phi_N$, maps $U_N$ biholomorphically onto a local coordinate neighbourhood of $\Phi_N(x_N)$ in $\mathbb{M}$ and that $\Phi_N$, maps $U_N$, homeomorphically onto $V(x')$. Let $\mu'$ be a biholomorphic mapping of $\Phi_N(U'_N)$ onto an open set $Z'$ of a complex Euclidean space. Suppose that $V(x) \cap V'(x') \neq \emptyset$. Then $$(\mu \circ (\Phi|V)) \circ (\mu' \circ (\Phi'|V'))^{-1} = \mu \circ (\Phi_N|U_N) \circ (\tau_N^x|U_N)^{-1} \circ (\tau_N^{x'}|U'_N)^{-1} \circ (\Phi_N'|U'_N)^{-1} \circ \mu'^{-1}$$ is a biholomorphic mapping of $\mu'(\Phi'(V \cap V'))$ onto $\mu(\Phi(V \cap V'))$ where $N'' = \max (N, N')$. Hence we can induce a complex structure in $D$ such that $\tau_n$ is a holomorphic mapping of $D_n$ in $D$ $(n \geq 1)$ and that $\Phi$ is a holomorphic mapping of $D$ in $\mathbb{M}$ which is locally biholomorphic. Therefore $(D, \Phi)$ is a domain over $\mathbb{M}$.

Let $(D', \Phi')$ be a domain over $\mathbb{M}$ such that $(D_n, \Phi_n) < (D', \Phi')$ $(n \geq 1)$ with a holomorphic mapping $\tau_n'$ of $D_n$ in $D'$ satisfying $\tau_n' = \tau_n' \circ \Phi_n'$ for $m \geq n$. In this case $\{(D_n, \Phi_n); n = 1, 2, 3, \ldots \}$ is called a monotonously increasing sequence over $(D', \Phi')$. Let $(x_n) \in E$ be a representative of $x \in D$ such that $x_n = \tau_n^x(x_n)$ $(n \geq N)$ for $x_n \in D_N$. If we put $\tau'(x) = \tau_n^x(x_n)$, $\tau'$ is well-defined and a holomorphic mapping of $D$ in $D'$ such that $\Phi = \Phi' \circ \tau'$. Hence $(D, \Phi)$ is a domain over $\mathbb{M}$ such that $\{(D_n, \Phi_n); n = 1, 2, 3, \ldots \}$ is a monotonously increasing sequence over $(D, \Phi)$ and that $(D, \Phi) < (D', \Phi')$ for all $(D', \Phi')$ over which $\{(D_n, \Phi_n); n = 1, 2, 3, \ldots \}$ is a monotonously increasing sequence. We call this domain $(D, \Phi)$ over $\mathbb{M}$ the limit of a monotonously increasing sequence $\{(D_n, \Phi_n); n = 1, 2, 3, \ldots \}$ of domains over $\mathbb{M}$ and denote it by $\lim (D_n, \Phi_n)$. If $D_n$ is a domain in $\mathbb{M}$ for each $n$, then $D$ coincides with the usual $\lim D_n = \bigcup_{n=1}^{\infty} D_n$.

**Lemma 1.** Let $G$ be a relatively compact subdomain of the limit $(D, \Phi)$ of a monotonously increasing sequence $\{(D_n, \Phi_n); n = 1, 2, 3, \ldots \}$ of domains over
a complex manifold $\mathcal{M}$. Then there exist an integer $m$ and a relatively compact subdomain $G_m$ of $D_m$ such that $\tau_m$ maps $G_m$ biholomorphically onto $G$.

**Proof.** Let $E_n \subset D_n$ be a relatively compact subdomain of $D_n$ such that $\tau_n^m(E_n) \subset E_n$ and $\overline{C} \subset \bigcup_{n=1}^{\infty} \tau_n(E_n)$ for $m \geq n \geq 1$. Since $\{\tau_n(E_n); n = 1, 2, 3, \ldots\}$ is an open covering of a compact set $\overline{C}$, there exists an integer $n$ such that $\overline{C} \subset \tau_n(\overline{E_n})$. We shall prove that $\tau_m$ maps $K_m = \tau_m(E_n)$ injectively onto $D$ for sufficiently large $m$. If this is not true, there exist sequences $\{y_v; v = n, n+1, n+2, \ldots\}$ and $\{x_v'; v = n, n+1, n+2, \ldots\}$ of points $y_v$ in $D$ and $x_v'$ in $\overline{E_n}$ such that $\tau_v'(x_v') \neq \tau_v'(x_v'')$ and $y_v = \tau_n(x_v') = \tau_n(x_v'')$. Since $\overline{E_n}$ and $\tau_n(\overline{E_n})$ are compact, there exists a subsequence $\{v_v\}$ of $\{n, n+1, n+2, \ldots\}$ such that $x_v' \to x' \in \overline{E_n}$ and $x_v'' \to x'' \in \overline{E_n}$ as $v \to \infty$. Since $y_v = \tau_n(x_v') = \tau_n(x_v'')$, we have $y_v = \tau_n(x_v') = \tau_n(x_v'')$. Hence there exists an integer $l$ such that $\tau_l(x') = \tau_l(x'')$. Therefore there exist neighbourhoods $U'$ and $U''$ of $x'$ and $x''$ such that $\tau_l$ maps $U'$ and $U''$ biholomorphically onto $U_l = \tau_l(U') = \tau_l(U'')$ and that $\tau_l$ maps $U_l$ biholomorphically onto $\tau_l(U)$. Since $x_v' \to x'$ and $x_v'' \to x''$ as $v \to \infty$, there exists an integer $\mu$ such that $x_v' \in U'$, $x_v'' \in U''$ and $\mu > l$. Therefore we have $\tau_v'(x_v') = \tau_v'(x_v'')$. Hence we have $\tau_v''(x_v') = \tau_v''(x_v'')$. But this is a contradiction. If we put $G_m = \tau_m(G) \cap K_m$, we have our lemma.

§ 2. Domain of holomorphy.

Let $\{(D_i, \Phi_i); i \in I\}$ be a set of domains over $\mathcal{M}$. We denote by $D$ the set of all $(x_i)$ such that a neighbourhood $U$ of a point $x$ in $\mathcal{M}$ and a neighbourhood $U_i$ of $x_i$ in $D_i$ for each $i$ satisfy $x = \Phi_i(x_i)$ and $U = \Phi_i(U_i)$. We can naturally induce a complex structure in $D$ such that the canonical mapping $\lambda_i$ of $D$ in $D_i$ is holomorphic for each $i$ and the mapping $\Phi$ defined by $\Phi = \Phi_i \circ \lambda_i$ is a mapping of $D$ in $\mathcal{M}$ which is locally biholomorphic. Hence $(D, \Phi)$ is a domain over $\mathcal{M}$. $(D, \Phi)$ is called the intersection of domains $(D_i, \Phi_i)$ $(i \in I)$ and denoted by $\bigcap_{i \in I} (D_i, \Phi_i)$. If each $D_i$ is a subdomain of $\mathcal{M}$, then $D$ coincides with the open kernel of the usual intersection $\bigcap_{i \in I} D_i$.

Let $(X, \Phi)$ be a domain over $\mathcal{M}$ and $f$ be a holomorphic function in $X$. A domain $(X', \Phi')$ over $\mathcal{M}$ is called a domain of holomorphic prolongation of $f$ if there exist a holomorphic function $f'$ in $X'$ and a holomorphic mapping $\tau$ of $X$ in $X'$ such that $\Phi = \Phi' \circ \tau$ and $f = f' \circ \tau$. In this case it holds that $(X, \Phi) \prec (X', \Phi')$. $f'$ is called a holomorphic prolongation of $f$ over $(X', \Phi')$. Consider a fixed domain $(X, \Phi)$ over $\mathcal{M}$ and a holomorphic function $f$ in $X$. A domain $(\tilde{X}_f, \tilde{\Phi}_f)$ is called the domain of maximal holomorphic prolongation of $f$ if the following conditions are satisfied:

(i) There exists a holomorphic function $\tilde{f}$ in $\tilde{X}_f$ which is a holomorphic
prolongation of \( f \) over \((\tilde{X}_f, \tilde{\Phi}_f)\).

(ii) If \( f' \) is a holomorphic prolongation of \( f \) over a domain \((X', \Phi')\) over \(\mathcal{M}\), then \( \tilde{f} \) is a holomorphic prolongation of \( f' \) over \((\tilde{X}_f, \tilde{\Phi}_f)\).

A domain over \(\mathcal{M}\) is called a domain of holomorphy if it is the domain of maximal holomorphic prolongation of a holomorphic function in a domain over \(\mathcal{M}\). Due to Cartan [3] (exposé 7) there exists such domain \((\tilde{X}_f, \tilde{\Phi}_f)\) for any holomorphic function \( f \) in a domain \((X, \Phi)\) over \(\mathcal{M}\). If \(\mathcal{M}\) is a Stein manifold, a domain of holomorphy over \(\mathcal{M}\) is holomorphically convex from Docquier-Grauert [5].

Conversely, suppose that \((X, \Phi)\) is a holomorphically convex domain over a Stein manifold \(\mathcal{M}\). We can construct a holomorphic function \( f \) in \( X \) by using Bochner-Martin's method [2] such that \( f \) is unbounded at each boundary point of \((X, \Phi)\). Since the holomorphically convex domain \((X, \Phi)\) over \(\mathcal{M}\) is a Stein manifold (see Grauert [7]), \( X \) is holomorphically separable, that is, there exists a holomorphic function in \( X \) which takes different values at two given different points in \( X \). Let \( A = \{x_i; i = 1, 2, 3, \ldots\} \) be a dense subset of \( X \) such that \( \Phi^{-1}\{x_i\} \subset A \) for any \( i \). There exists a holomorphic function \( f_{ij} = f_{ij} \) in \( X \) such that \( f_{ij}(x_i) \neq f_{ij}(x_j) \) for \( i \neq j \). If we take a suitable double sequence \( \{a_{ij}; i, j = 1, 2, 3, \ldots\} \) of complex numbers, \( g = \sum_{i<j} a_{ij} f_{ij} \) converges absolutely and uniformly in any compact subset of \( X \) and \( g(x_i) \neq g(x_j) \) for any \( i \neq j \). Then, for suitable complex numbers \( a \) and \( b \), \( h = af + bg \) is a holomorphic function in \( X \) which is unbounded at each boundary point of \((X, \Phi)\) and satisfies \( h(x_i) \neq h(x_j) \) for any \( i \neq j \). \((X, \Phi)\) is the domain of maximal holomorphic prolongation of \( h \) and is a domain of holomorphy.

Hence we obtained

**Lemma 2.** A domain \((X, \Phi)\) over a Stein manifold \(\mathcal{M}\) is holomorphically convex, if and only if \((X, \Phi)\) is a domain of holomorphy.

Hereafter we shall denote a Stein manifold by \(\mathcal{M}\). Let \((X, \Phi)\) be a domain over \(\mathcal{M}\) and \(O_X\) be the set of all holomorphic functions in \( X \). For any \( f \in O_X \) we denote by \((\tilde{X}_f, \tilde{\Phi}_f)\) the domain of maximal holomorphic prolongation of \( f \). We denote by \((\tilde{X}, \tilde{\Phi})\) the intersection of \((\tilde{X}_f, \tilde{\Phi}_f)\) for all \( f \in O_X \). \((\tilde{X}, \tilde{\Phi})\) can be characterized by the following properties:

(i) For any \( f \in O_X \), there exists a holomorphic prolongation of \( f \) over \((\tilde{X}, \tilde{\Phi})\).

(ii) If a domain \((X', \Phi')\) has the above property, then \((X', \Phi') < (\tilde{X}, \tilde{\Phi})\).

\((\tilde{X}, \tilde{\Phi})\) is called the envelope of holomorphy of a domain \((X, \Phi)\).

**Lemma 3.** The envelope of holomorphy \((\tilde{X}, \tilde{\Phi})\) of a domain \((X, \Phi)\) over a Stein manifold \(\mathcal{M}\) is a domain of holomorphy.

**Proof.** Let \(O_X\) be the set of all holomorphic functions in \( X \) and \((\tilde{X}_f, \tilde{\Phi}_f)\) be the domain of maximal holomorphic prolongation of \( f \in O_X \). We consider
an open covering \( \{ V_i ; i \in I \} \) of \( \mathcal{M} \) with the following properties:

There exists a holomorphic mapping \( \mu_i \) of \( V_i \) onto a domain of holomorphy \( W_i \) of a complex Euclidean space. We put \( U_i = \phi^{-1}(V_i) \) for any \( i \in I \). Then \( (U_i, \mu_i \circ \phi_i | U_i) \) is the intersection of holomorphically convex open sets \( (\phi_f^{-1}(V_i), \mu_i \circ \phi_f | \phi_f^{-1}(V_i)) \) over the complex Euclidean space for all \( f \in \mathcal{O}_X \). Therefore \( (U_i, \mu_i \circ \phi_i | U_i) \) is holomorphically convex by Cartan-Thullen's results \([4]\). Hence \( (\tilde{X}, \tilde{\phi}) \) is in the sense of Docquier-Grauert \([5]\). Therefore \( (\tilde{X}, \tilde{\phi}) \) is holomorphically convex and is a domain of holomorphy from Lemma 2.

**Lemma 4.** Let \( (D_1, \Phi_1) < (D_2, \Phi_2) \) be domains over a Stein manifold \( \mathcal{M} \) and \( \tau \) be a holomorphic mapping of \( D_1 \) in \( D_2 \) with \( \Phi_1 = \Phi_2 \circ \tau \). Let \( (D_1, \Phi_1) \) and \( (D_2, \Phi_2) \) be, respectively, the envelopes of holomorphy of \( (D_1, \Phi_1) \) and \( (D_2, \Phi_2) \). Let \( \lambda_1 \) and \( \lambda_2 \) be, respectively, the canonical mapping of \( D_1 \) in \( \tilde{D}_1 \) and that of \( D_2 \) in \( \tilde{D}_2 \). Then there exists a holomorphic mapping \( \tilde{\tau} \) of \( \tilde{D}_1 \) in \( \tilde{D}_2 \) such that \( \lambda_2 \circ \tau = \tilde{\tau} \circ \lambda_1 \).

**Proof.** By Remmert's result \([9]\) there exists a biholomorphic mapping \( \mu \) of \( \tilde{D}_2 \) onto a regular analytic set \( A \) in \( C^a \) as \( \tilde{D}_2 \) is a Stein manifold. Then \( \mu \circ \lambda_a \circ \tau \) is a holomorphic mapping of \( D_1 \) in \( C^a \). Since \( (\tilde{D}_1, \tilde{\Phi}_1) \) is the envelope of holomorphy of \( (D_1, \Phi_1) \), there exists a holomorphic mapping \( \phi \) of \( \tilde{D}_1 \) in \( C^a \) such that \( \mu \circ \lambda_a \circ \tau = \phi \circ \lambda_1 \). We shall prove that \( \phi(\tilde{D}_1) \subseteq A \). Suppose that \( \phi(x) \in A \) for \( x \in D_1 \). Let \( \lambda_1 = \phi \circ \lambda_a \circ \tau \). We shall prove that \( \phi(\tilde{D}_1) \subseteq A \). Suppose that \( \phi(x(t)) \in A \) for some \( 0 \leq t \leq 1 \). Since \( \phi \) is holomorphic in \( \tilde{D}_1 \), there exist \( f_1, f_2, \ldots \) and \( f_s \) in \( C^a \) such that \( \forall x \in D_1 \), \( f_i(x(t)) = 0 \) for some \( 1 \leq i \leq s \). Then \( \phi(\tilde{D}_1) \subseteq A \). Hence we have \( \phi(\tilde{D}_1) \subseteq A \). Therefore \( \phi = \mu \circ \lambda_a \circ \tau \) is a holomorphic mapping of \( \tilde{D}_1 \) in \( \tilde{D}_2 \) such that \( \lambda_a \circ \tau = \tilde{\tau} \circ \lambda_1 \). Since \( \tilde{\Phi}_1 \circ \lambda_1 = \tilde{\Phi}_2 \circ \tilde{\tau} \circ \lambda_1 \), we have \( \tilde{\Phi}_1 = \tilde{\Phi}_2 \circ \tilde{\tau} \).

**Lemma 5.** Let \( \{(D_n, \Phi_n) ; n = 1, 2, 3, \ldots \} \) be a monotonously increasing sequence of domains of holomorphy over a Stein manifold \( \mathcal{M} \). Then its limit \( (D, \Phi) \) is also a domain of holomorphy.

**Proof.** It suffices to prove that \( D \) is \( p_\alpha \)-convex in the sense of Docquier-Grauert \([5]\). We put \( B(a) = \{ z ; |z_1| \leq a, \ldots, |z_n| < a \} \) and \( \delta B(a) = \{ z ; |z_1| = 1, |z_2| < a, \ldots, |z_n| < a \} \) where \( \alpha \) is the dimension of \( \mathcal{M} \). Let \( \varphi \) be a biholomorphic mapping of \( B = B(1) \) in \( D \) such that \( \varphi(\delta B) \subseteq D \). Let \( W \) be a relatively compact open neighbourhood of \( \varphi(\delta B \cup \overline{B(1/2)}) \). From Lemma 1
there exist an integer \( m_0 \) and a relatively compact open set \( W_0 \) in \( D_{m_0} \) such that \( \tau_{m_0} \) maps \( W_0 \) biholomorphically onto \( W \). For any \( 1/2 < a < 1 \) there exists \( \varepsilon > 0 \) such that \( \varphi(G(a)) \subseteq W_0 \) for \( G(a) = \{ z; 1-\varepsilon < |z_1| < 1+\varepsilon, |z_2| < a, \ldots, |z_n| < a \} \cup \{ z; |z_1| < 1+\varepsilon, |z_2| < 1/2, \ldots, |z_n| < 1/2 \} \). \( \eta = (\tau_{m_0}W)^{-1} \circ \varphi \) maps \( G(a) \) biholomorphically in \( D_{m_0} \). As in the proof of Lemma 4 there exists a holomorphic mapping \( \tilde{\eta}_a \) of \( \overline{G(a)} = \{ z; |z_1| < 1+\varepsilon, |z_2| < a, \ldots, |z_n| < a \} \cup \{ z; |z_1| < 1+s, |z_2| < 1/2, \ldots, |z_n| < 1/2 \} \) in \( D_{m_0} \) such that \( \tilde{\eta}_a = \eta_a \) in \( G(a) \). From the theorem of identity we have \( \varphi = \tau_{m_0} \circ \tilde{\eta}_a \) in \( \overline{G(a)} \). Since \( \varphi \) is biholomorphic, \( \tilde{\eta}_a \) is also biholomorphic. Thus we have proved that there exists a biholomorphic mapping \( \tilde{\eta} \) of \( B \) in \( D_{m_0} \) such that \( \varphi = \tau_{m_0} \circ \tilde{\eta} \) and \( \tilde{\eta}(B) \subseteq D_{m_0} \). Since \( D_{m_0} \) is \( p_\theta \)-convex, we have \( \tilde{\eta}(B) \subseteq D_{m_0} \). Therefore we have \( \varphi(B) = \tau_{m_0}(\tilde{\eta}(B)) \subseteq D \). Hence \( D \) is \( p_\theta \)-convex.

§ 3. Cohomology of an increasing sequence of domains.

Let \( \{ (D_n, \Phi_n); n = 1, 2, 3, \ldots \} \) be a monotonously increasing sequence of domains over \( \mathbb{R} \), \( (D, \Phi) \) be its limit and \( \tau_n \) and \( \tau_m \) be, respectively, the canonical mapping of \( D_n \) in \( D_m \) (\( m \geq n \)) and that of \( D_n \) in \( D \) (\( n \geq 1 \)). Then there exists a canonical homomorphism \( \pi^m_n \) of \( H^1(D_m, \mathcal{O}) \) in \( H^1(D_n, \mathcal{O}) \) for \( m \geq n \) such that \( \pi^l_n = \pi^m_n \circ \pi^m_n \) for \( l \geq m \geq n \) where \( \mathcal{O} \) is the sheaf of all germs of holomorphic functions. Hence \( \{ H^1(D_n, \mathcal{O}); \pi^m_n \} \) is an inverse system of \( \mathbb{C} \)-module over a directed set \( \{ 1, 2, 3, \ldots \} \). We consider its inverse limit and denote it by \( \lim H^1(D_n, \mathcal{O}) \). The canonical homomorphisms of \( H^1(D, \mathcal{O}) \) in \( H^1(D_n, \mathcal{O}) \) induce the canonical homomorphism of \( H^1(D, \mathcal{O}) \) in \( \lim H^1(D_n, \mathcal{O}) \). Under these assumptions we have

**Lemma 6.** The canonical homomorphism of \( H^1(D, \mathcal{O}) \) in \( \lim H^1(D_n, \mathcal{O}) \) is injective.

**Proof.** Since the canonical homomorphism of \( H^1(V, \mathcal{F}) \) in \( H^1(X, \mathcal{F}) \) is injective for any sheaf \( \mathcal{F} \) of abelian groups in a topological space \( X \) and for any open covering \( \mathcal{U} \) of \( X \), it suffices to prove the following statement:

If \( \{ f_{ij} \} \) is an element of \( Z^1(\mathcal{U}, \mathcal{F}) \) (cocycle) for any open covering \( \mathcal{V} = \{ V_i; i \in I \} \) of \( D \) such that \( \{ f_{ij} \circ \tau_n \} \subseteq B^1(\tau_n^{-1}(\mathcal{V}); \mathcal{F}) \) (coboundary) for any \( n \geq 1 \) where \( \tau_n^{-1}(\mathcal{V}) = \{ \tau_n^{-1}(V_i); i \in I \} \), then \( \{ f_{ij} \} \subseteq B^1(\mathcal{V}, \mathcal{F}) \).

Let \( (\tilde{D}_n, \tilde{\Phi}_n) \) and \( (\tilde{D}, \tilde{\Phi}) \) be, respectively, the envelope of holomorphy of \( (D_n, \Phi_n) \) (\( n = 1, 2, 3, \ldots \)) and \( (D, \Phi) \). From Lemma 4 \( \{ (\tilde{D}_n, \tilde{\Phi}_n); n = 1, 2, 3, \ldots \} \) is a monotonously increasing sequence of domains over \( (D, \Phi) \). Hence from Lemma 5 \( \lim (\tilde{D}_n, \tilde{\Phi}_n) \) is a domain of holomorphy satisfying \( (D, \Phi) < \lim (\tilde{D}_n, \tilde{\Phi}_n) \) < \( (\tilde{D}, \tilde{\Phi}) \). Since \( (\tilde{D}, \tilde{\Phi}) \) is the envelope of holomorphy of \( (D, \Phi) \), we have \( (\tilde{D}, \tilde{\Phi}) = \lim (\tilde{D}_n, \tilde{\Phi}_n) \). We denote by \( \tau_n, \tau_n^m, \tau_n^m, \lambda_n, \lambda \) and \( \lambda \) the canonical mapping of \( D_n \) in \( D \), that of \( D_n \) in \( D_m \), that of \( D_n \) in \( \tilde{D} \), that of \( \tilde{D}_n \) in \( \tilde{D}_m \).
that of $D_n$ in $\bar{D}_n$ and that of $D$ in $\bar{D}$, respectively. Then the commutativity holds in the following diagram:

\[ \begin{array}{ccc}
D_n & \xrightarrow{\tau_m^n} & D_m \\
\downarrow{\lambda_n} & & \downarrow{\lambda_m} \\
\bar{D}_n & \xrightarrow{\bar{\tau}_m^n} & \bar{D}_m
\end{array} \]

Let \( \{Q_n; n = 1, 2, 3, \ldots\} \) be a sequence of relatively compact open subsets of $D$ such that $Q_n \subseteq Q_{n+1}$ (\( n \geq 1 \)) and $D = \bigcup_{n=1}^{\infty} Q_n$. From Lemmas 2 and 3 $\bar{D}$ is holomorphically convex. There exists a sequence of analytic polyhedra $P_n$ defined by holomorphic functions in $\bar{D}$ such that $P_n \subseteq P_{n+1}$, $\lambda(Q_n) \subseteq P_n$ for $n \geq 1$ and $\bar{D} = \bigcup_{n=1}^{\infty} P_n$. Since $(D, \Phi) = \lim (D_n, \Phi_n)$ and $(\bar{D}, \bar{\Phi}) = \lim (\bar{D}_n, \bar{\Phi}_n)$, there exists a monotonously increasing sequence \( \{\nu_n; n = 1, 2, 3, \ldots\} \) of integers such that $\tau_{\nu_n}$ and $\bar{\tau}_{\nu_n}$ map, respectively, relatively compact open subsets of $D_n$ and $\bar{D}_n$, biholomorphically onto $Q_{\nu_n}$ of $D$ and $P_{\nu_n}$ of $\bar{D}$ for $n \geq 1$ and further that

\[ \tau_{\nu_{n+1}}(Q_{\nu_n}) \subseteq Q_{\nu_{n+1}}, \quad \tau_{\nu_{n+1}}(P_{\nu_n}) \subseteq P_{\nu_{n+1}}, \quad \lambda_{\nu_n}(Q_{\nu_n}) \subseteq \bar{P}_{\nu_n} \quad (n \geq 1). \]

Without losing generality, we may suppose that $\nu_n = n$.

Since \( \{f_{ij} \circ \tau_n\} \in B(\tau_n^{-1}(\mathbb{V}), \mathcal{C}) \), there exists $\{f^n\} \in C(\tau_n^{-1}(\mathbb{V}), \mathcal{C})$ such that $f_{ij} \circ \tau_n = f^n \circ \tau_{n+1}^{-1}$ in $\tau_n^{-1}(V_j) \cap \tau_{n+1}^{-1}(V_j)$. If we put $f^n = f^n \circ \tau_{n+1}^{-1}$ in $\tau_n^{-1}(V_j)$, then, $f^n$ is well-defined and holomorphic in $D_n$. Since $(\bar{D}_n, \bar{\Phi}_n)$ is the envelope of holomorphy of $(D_n, \Phi_n)$, there exists a holomorphic prolongation $f^n$ of $f^n$ over $(\bar{D}_n, \bar{\Phi}_n)$. There holds $f^n = f^n \circ \lambda_n$ for $n \geq 1$. Since $f^n \circ \lambda_n^{-1}(P_n)$ is a holomorphic function in $P_n$, there exists a holomorphic function $h^n$ in $\bar{D}$ (\( n \geq 1 \)) which satisfies $|f^n \circ \lambda_n^{-1}(P_n) - h^n| < 2^{-n}$ in $P_{n-1}$ for $n \geq 2$ from Behnke’s approximation theory [1]. If we put $h^n = h^n \circ \lambda$, the holomorphic function $h^n$ in $D$ satisfies $|f^n \circ \lambda_n^{-1}(Q_n) - h^n| < 2^{-n}$ in $Q_{n-1}$ for $n \geq 2$. We consider holomorphic functions in $D$ defined by $g^1 = 0$, $g^n = h^1 + h^2 + \cdots + h^{n-1}$ for $n \geq 2$. Then the coboundary of $\{f^n \circ \lambda_n^{-1}(Q_n) - g^n\} \in C(\mathbb{V} \cap Q_n, \mathcal{C})$, where $\mathbb{V} \cap Q_n = \{V_i \cap Q_n; i \in I\}$, is $\{f_{ij} \circ Q_n\} \in Z(\mathbb{V} \cap Q_n, \mathcal{C})$. There holds

\[ (f^n \circ \lambda_n^{-1}(Q_n) - g^n) - (f^{n+1} \circ \lambda_{n+1}^{-1}(Q_{n+1}) - g^{n+1}) = f^n \circ \lambda_n^{-1}(Q_n) - h^n \]

in any $V_i \cap Q_n$. Hence $f^n \circ \lambda_n^{-1}(Q_n) - g^n$ converges uniformly in any compact subset of $V_i$ to a holomorphic function $f_i$ in $V_i$. Since

\[ f_{ij} = (f^n \circ \lambda_n^{-1}(Q_n) - g^n) - (f^n \circ \lambda_n^{-1}(Q_n) - g^n) \]

A collection \( \mathcal{E} = \{(m_i, V_i) ; i \in I \} \) of pairs of an open subset \( V_i \) of a complex manifold \( X \) and a meromorphic function \( m_i \) in \( V_i \) is called a Cousin-I (or Cousin-II) distribution in \( X \) if \( m_i - m_j \in H^0(V_i \cap V_j, \mathcal{O}) \) (or \( m_i/m_j \in H^0(V_i \cap V_j, \mathcal{O}^*) \)) for any \( V_i \cap V_j \neq \emptyset \) and \( \{ V_i ; i \in I \} \) is an open covering of \( X \) where \( \mathcal{O}^* \) is the sheaf of all germs of holomorphic mapping in \( \hat{C} = GL(1, C) \). A meromorphic function \( m \) in \( X \) is called a solution of the Cousin-I (or Cousin-II) distribution \( \mathcal{E} \) if \( m - m_i \in H^0(V_i, \mathcal{O}) \) (or \( m/m_i \in H^0(V_i, \mathcal{O}^*) \)) for any \( i \in I \). A meromorphic function \( M \) in the universal covering manifold \((X, \pi)\) of \( X \) is called a multiform solution of \( \mathcal{E} \) if \( M \) is the solution of the Cousin distribution \( \{(m_1 \circ \tau^n, \pi^{-1}(V_i)) ; i \in I \} \). If any Cousin-I (or Cousin-II) distribution in \( X \) has a solution, \( X \) is called a Cousin-I (or Cousin-II) manifold. If any Cousin-I (or Cousin-II) distribution in \( X \) has a multiform solution, \( X \) is called a multiform Cousin-I (or Cousin-II) manifold. A complex manifold \( X \) with the vanishing fundamental group \( \pi_1(X) \) is called simply connected.

PROPOSITION 1. The limit \( (D, \Phi) \) of a monotonously increasing sequence \( \{(D_n, \Phi_n) ; n = 1, 2, 3, \ldots \} \) of Cousin-I domains over a Stein manifold \( \mathcal{M} \) is a Cousin-I domain. However, for any \( \alpha \geq 3 \) there exists an example of the limit of a monotonously decreasing sequence of Cousin-I domains in \( C^\alpha \) which is not even a multiform Cousin-I domain.

PROOF. Let \( \mathcal{E} = \{(m_i, V_i) ; i \in I \} \) be a Cousin-I distribution in \( D \). Then \( \{(m_i \circ \tau^n, \tau^{-1}_n(V_i)) ; i \in I \} \) is a Cousin-I distribution in \( D_n \). If we put \( f_{ij} = m_i - m_j \in H^0(V_i \cap V_j, \mathcal{O}) \) and \( f_{ij}^n = m_i \circ \tau^n - m_j \circ \tau^n \in H^0(\tau^{-1}_n(V_i \cap V_j), \mathcal{O}) \), then \( \{f_{ij}^n\} \in Z^1(\tau^{-1}_n(\mathcal{O}_\mathcal{M})), \mathcal{O}) \) is the canonical image of \( \{f_{ij}\} \in Z^1(\mathcal{O}_\mathcal{M}), \mathcal{O}) \) where \( \mathcal{O}_\mathcal{M} = \{V_i; i \in I\} \) and \( \tau^{-1}_n(\mathcal{O}_\mathcal{M}) = \{\tau^{-1}_n(V_i); i \in I\} \). Since \( D_n \) is a Cousin-I domain for any \( n \), \( \{f_{ij}^n\} \in B^1(\tau^{-1}_n(\mathcal{O}_\mathcal{M}), \mathcal{O}) \) for any \( n \). Hence there exists a holomorphic function \( f_i \) in \( V_i \) for any \( i \in I \) such that \( f_{ij} = f_i - f_j \) in \( V_i \cap V_j \) from Lemma 6. If we put \( m = m_i - f_i \) in \( V_i \), \( m \) is well-defined and a solution of \( \mathcal{E} \).

As for the latter half, we put

\[ D = \{z; |z_1| < 1, |z_2| < 1, \ldots, |z_n| < 1\} - \{z; z_1 = z_2 = 0\} \]

and

\[ D_p = \{z; |z_1| < (p+1)/p, |z_2| < (p+1)/p, \ldots, |z_n| < (p+1)/p\} \]

\[ - \mathcal{D} \cap \{z; z_1 = z_2 = 0\} \]

for \( p = 1, 2, 3, \ldots \). As shown in the previous paper [8], \( D \) is the limit (precisely the open kernel of \( \bigcap_{p=1}^\infty D_p \) of the monotonously decreasing sequence of Cousin-I domains in \( C^\alpha \)).
domains $D_p$ in $C^n$ but is not a Cousin-I domain. Since $D$ is simply connected, $D$ is not even a multiform Cousin-I domain.

Let $(D, \Phi)$ be a domain over a Stein manifold $\mathfrak{M}$ and $(D^*, \lambda^*)$ be the universal covering manifold of $D$. If we put $\Phi^* = \Phi \circ \lambda^*$, $(D^*, \Phi^*)$ is a domain over $\mathfrak{M}$. If $(D, \Phi)$ is a domain of holomorphy, it is $p_r$-convex in the sense of Docquier-Grauert [5]. Hence $(D^*, \Phi^*)$ is $p_r$-convex and is a domain of holomorphy. This follows also from the result of Stein [11] that a covering manifold over a Stein manifold is a Stein manifold.

Let $\{(D_n, \Phi_n) ; n = 1, 2, 3, \cdots \}$ be a monotonously increasing sequence of domains over $\mathfrak{M}$, $(D, \Phi)$ be its limit, $(\check{D}_n, \check{\Phi}_n)$ and $(\check{D}, \check{\Phi})$ be, respectively, the envelope of holomorphy of $(D_n, \Phi_n)$ and $(D, \Phi)$. Let $(D^*_n, \lambda^*_n)$, $(D^*, \lambda^*)$, $(\check{D}^*_n, \check{\lambda}^*_n)$ and $(\check{D}^*, \check{\lambda}^*)$ be, respectively, the universal covering manifolds of $D_n$, $D$, $\check{D}_n$ and $\check{D}$ for any $n$. If we put

$$\Phi^*_n = \Phi_n \circ \lambda^*_n, \quad \Phi^* = \Phi \circ \lambda^*, \quad \check{\Phi}^*_n = \check{\Phi}_n \circ \check{\lambda}^*_n, \quad \check{\Phi}^* = \check{\Phi} \circ \check{\lambda}^*,$$

then the commutativity holds in the three-dimensional diagram obtained by adding holomorphic mappings

$$D_n \xrightarrow{\tau_{mn}} D_m \xrightarrow{\tau_{nm}} D^*$$

to the following diagram and identifying the same symbols in it where each holomorphic mapping is the canonical one $(m \geq n)$:

From Lemma 1 $(D^*, \Phi^*)$ and $(\check{D}^*, \check{\Phi}^*)$ are, respectively, the limits of monotonously increasing sequences $\{(D^*_n, \Phi^*_n) ; n = 1, 2, 3, \cdots \}$ and $\{ (\check{D}^*_n, \check{\Phi}^*_n) ; n = 1, 2, 3, \cdots \}$. Let $\{P_n ; n = 1, 2, 3, \cdots \}, \{Q_n ; n = 1, 2, 3, \cdots \}, \{R_n ; n = 1, 2, 3, \cdots \}$ and $\{S_n ; n = 1, 2, 3, \cdots \}$ be, respectively, sequences of relatively compact subdomains of $\check{D}, D, \check{D}^*$ and $D^*$ with the following properties:

$$P_n \subset P_{n+1}, \quad Q_n \subset Q_{n+1}, \quad R_n \subset R_{n+1}, \quad S_n \subset S_{n+1},$$

$$\bigcup_{n=1}^{\infty} P_n = \check{D}, \quad \bigcup_{n=1}^{\infty} Q_n = D, \quad \bigcup_{n=1}^{\infty} R_n = \check{D}^*, \quad \bigcup_{n=1}^{\infty} S_n = D^*,$$

$$\lambda(Q_n) \subset P_n, \quad \check{\lambda}(R_n) \subset P_n, \quad \lambda^*(S_n) \subset Q_n, \quad j(S_n) \subset R_n.$$
$P_n$ and $Q_n$ are, respectively, analytic polycylinders defined by holomorphic functions in $\bar{D}$ and $\bar{D}^*$. There exists a subsequence $\{\nu_n \; ; \; n = 1, 2, 3, \ldots\}$ of $\{1, 2, 3, \ldots\}$ with the following properties:

There exist, respectively, subdomains $P'_n$, $Q'_n$, $R'_n$ and $S'_n$ of $\bar{D}_{\nu_n}$, $D_{\nu_n}$, $\bar{D}_{\nu_n}$ and $D^*_{\nu_n}$ such that $\tau_{\nu_n}$, $\tau_{\nu_n}$, $\tau^*_{\nu_n}$ and $\tau^*_{\nu_n}$ map biholomorphically $P'_n$, $Q'_n$, $R'_n$ and $S'_n$ onto $P_{\nu_n}$, $Q_{\nu_n}$, $R_{\nu_n}$ and $S_{\nu_n}$ and that

\[
\lambda_{\nu_n}(Q'_n) \subset P'_n, \quad \lambda^*_{\nu_n}(R'_n) \subset P'_n, \quad \lambda^*_{\nu_n}(S'_n) \subset Q'_n, \quad \lambda_{\nu_n}(S'_n) \subset R'_n,
\]

\[
\tau_{\nu_n+1}(P'_n) \subset P'_{n+1}, \quad \tau^*_{\nu_n+1}(Q'_n) \subset Q'_{n+1}, \quad \tau^*_{\nu_n+1}(R'_n) \subset R'_{n+1}, \quad \tau^*_{\nu_n+1}(S'_n) \subset S'_{n+1}.
\]

We may suppose that $\nu_n = n$. Let $\mathcal{C} = \{(m_i, V_i) \ ; \ i \in I\}$ be a Cousin-II distribution in $(D, \Phi)$. We shall suppose that the Cousin-II distribution $\{(m_i \circ \tau_n, \tau_n(V_i)) \ ; \ i \in I\}$ has a solution in $D_n$ for any $n$. There exists a meromorphic function $m^n$ in $D_n$ such that $m^n/m_i \circ \tau_n \in H^\infty(\tau_n(V_i), \mathbb{C}^*)$ for any $i \in I$. If we put $f^n = m^n/m^{n+1} \circ \tau_n$, then $f^n \in H^\infty(D_n, \mathbb{C}^*)$. Since $(\bar{D}_n, \bar{\Phi}_n)$ is the envelope of holomorphy of $(D_n, \Phi_n)$, there exists a holomorphic prolongation $\tilde{f}^n$ of $f^n$ which satisfies $\tilde{f}^n \in H^\infty(\bar{D}_n, \mathbb{C}^*)$ and $f^n = \tilde{f}^n \circ \tilde{\tau}_n$ for any $n$. Then $\log(\tilde{f}^n \circ \tilde{\tau}_n) \in H^\infty(\bar{D}_n, \mathbb{C})$ for $n \geq 1$. There holds $\log(\tilde{f}^n \circ \tilde{\tau}_n(\bar{\tau}_n(V_{n+1}))) \in H^\infty(\bar{D}_n, \mathbb{C})$ for $n \geq 1$. Since $R_n$ is an analytic polycylinder defined by holomorphic functions in $\bar{D}_n$, from Behnke's approximation theory [1] there exists a holomorphic function $\tilde{h}_n$ in $\bar{D}_n$ such that

\[
| \log(\tilde{f}^n \circ \tilde{\tau}_n(\bar{\tau}_n(V_{n+1}))) - \tilde{h}_n | < 2^{-n-2} \text{ in } R_{n-1} \text{ for } n \geq 2.
\]

We put $H^n = \exp(\tilde{h}_n \circ \tilde{\tau}_n) \in H^\infty(D_n, \mathbb{C}^*)$. There holds $|f^n \circ \tau_n(\bar{\tau}_n(V_{n+1})) - \lambda^n/(H^n - 1)| < 2^{-n} \text{ in } S_{n-1}$ for $n \geq 2$. We put $G^1 = 1$, $G^n = H^2 \circ H^2 \cdots H^2 \circ H^2 \in H^\infty(D_n, \mathbb{C}^*)$ for $n \geq 2$. Then $M^n = (m^n \circ \tau_n)(G^n)$ is a meromorphic function in $S_n$. There holds $|\overline{M^n}/M^n - \overline{\lambda^n}/\lambda^n| < 2^{-n} \text{ in } S_{n-1}$. Therefore $\{M^n \; ; \; n = 1, 2, 3, \ldots\}$ converges uniformly to a meromorphic function $M$ in any compact subset of $D^*$. There holds $M/m_i \circ \lambda^n \in H^\infty(\tau_n(V_i), \mathbb{C}^*)$ for any $i \in I$. Hence $M$ is the solution of the Cousin-II distribution $\{m_i \circ \lambda^n, \lambda^n(\tau_n(V_i)) \ ; \ i \in I\}$ in $D^*$ and is a multiform solution of $\mathcal{C}$. Therefore we have

\[\text{PROPOSITION 2. The limit of a monotonously increasing sequence of Cousin-II domains over a Stein manifold is a multiform Cousin-II domain.}\]

\[\text{COROLLARY. If the limit of a monotonously increasing sequence of Cousin-II domains over a Stein manifold is simply connected, it is a Cousin-II domain.}\]
References


