Holomorphic imbeddings of symmetric domains

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(Received Feb. 23, 1967)

The purpose of this paper is to determine all (equivariant) holomorphic imbeddings of a symmetric domain \( D \) into another symmetric domain \( D' \); a part of results has been announced in [3] without proofs.

In the case \( D' \) is of type (III)\(_p\) or of type (I)\(_{p,q}\), this problem was solved completely (and partially in the case \( D' \) is of type (II)\(_p\)) by Satake in his paper [4]. Our methods are similar to those adopted in [4], but depend further on general properties of Lie algebras. Our results are essentially applicable to any cases.

Let \( g \) and \( g' \) be the Lie algebras of the groups of all analytic automorphisms of \( D \) and \( D' \) respectively. Then the problem is equivalent to that of finding all monomorphisms of \( g \) into \( g' \) satisfying a condition called \( (H_1) \) in [4]. Therefore we shall consider a slightly generalized problem as was done in [4], that is to determine all homomorphisms of a Lie algebra of hermitian type into another satisfying \( (H_1) \). A more precise exposition of our problem will be given in §1.

We shall make reductions of the problem in §2. Namely, if we find all regular subalgebras (see 2.3) of \( g' \), and if we determine all pairs \( (g, \rho) \) of a Lie algebra of hermitian type and a homomorphism of \( g \) into a regular subalgebra \( g'_i \) of \( g' \) satisfying a certain condition \( (H_2) \) stronger than \( (H_1) \), we shall get all solutions; moreover, we shall be able to assume that both \( g \) and \( g' \) are simple. The determination of regular subalgebras of each non-compact simple Lie algebra of hermitian type will be done in §4. For a simple Lie algebra \( g' \) of type (I)\(_{p,q}\), (II)\(_p\), or (III)\(_p\), all pairs \( (g, \rho) \) of a non-compact simple Lie algebra and a homomorphism into \( g' \) satisfying \( (H_2) \) can be determined by combining the results tabulated in [4] and our results in §4; for the remaining cases \( g' = (IV)\(_p\), (EIII), or (EVII)\), they are determined in §5. Our §3 is devoted to the preparations to §5. In the Appendix, we shall refer to some results, supplementary to those given by Satake in [5], about the correspondence of boundary components by holomorphic imbeddings of symmetric domains.

Throughout this paper, the usual symbols (I)\(_p\), (II)\(_p\), (III)\(_p\), (IV)\(_p\), (EIII), and (EVII) for the irreducible symmetric domains will be also used to denote...
the corresponding simple Lie algebras, and the symbols \((A_l), (B_l), \ldots\) for the complex simple Lie algebras to denote their compact real forms.

On account of the well known isomorphisms between simple Lie algebras, one may restrict the parameters as follows:

\[
\begin{align*}
(I)_{p,q} & : 1 \leq p \leq q, \\
(II)_p & : p \geq 5, \\
(III)_p & : p \geq 2, \\
(IV)_p & : p \geq 5.
\end{align*}
\]

§ 1. Exposition of the problem.

1.1. Let \(D\) and \(D'\) be symmetric domains. A holomorphic imbedding of \(D\) into \(D'\) is a holomorphic isometry \(\varphi\) of \(D\) into \(D'\) such that \(\varphi(D)\) is totally geodesic in \(D'\). Let \(G\) (resp. \(G'\)) be the connected component of the identity of the group of all analytic automorphisms. Two holomorphic imbeddings \(\varphi_1\) and \(\varphi_2\) of \(D\) into \(D'\) are said to be equivalent if there is an element \(g\) of \(G'\) such that \(\varphi_2 = g \cdot \varphi_1\). (For convenience, we shall not say equivalent when \(g\) is an analytic automorphism not contained in \(G'\).) The problem is precisely to find out, for a given symmetric domain \(D'\), all equivalence classes (defined naturally) of pairs \((D, \varphi)\) of a symmetric domain \(D\) and a holomorphic imbedding \(\varphi\) of \(D\) into \(D'\). Hence, if \(o\) and \(o'\) are origins fixed in \(D\) and \(D'\) respectively, we may assume that

\[
\varphi(o) = o'.
\]

If \(g\) is the Lie algebra of \(G\), one has the Cartan decomposition

\[
g = \mathfrak{f} + \mathfrak{p}
\]

corresponding to \(o\), where \(\mathfrak{f}\) is the (maximal compact) subalgebra of \(g\), corresponding to the isotropy subgroup at \(o\). The vector space \(\mathfrak{p}\) is identified with the tangent space to \(D\) at \(o\), and has a complex structure \(J\) which agrees with that of \(D\). Similarly we have \(g' = \mathfrak{f}' + \mathfrak{p}'\) and \(J'\). For a holomorphic imbedding \(\varphi\) of \(D\) into \(D'\) satisfying (1), we have a monomorphism \(\rho\) of \(g\) into \(g'\) such that

\[
\rho(\mathfrak{f}) \subset \mathfrak{f}', \quad \rho(\mathfrak{p}) \subset \mathfrak{p}'
\]

and

\[
\rho \circ J = J' \circ \rho \text{ on } \mathfrak{p}.
\]

Conversely, for a monomorphism \(\rho\) satisfying (2) and (3), there corresponds a holomorphic imbedding satisfying (1).

On the other hand, there is the uniquely determined element \(H_0\) (resp. \(H_0'\)) in the center of \(\mathfrak{f}\) (resp. \(\mathfrak{f}'\)) such that

\[
J = \text{ad}(H_0) \text{ on } \mathfrak{p}, \quad J' = \text{ad}(H_0') \text{ on } \mathfrak{p}'.
\]
The inner automorphism \( \exp \pi \text{ad}(H_0) \) (resp. \( \exp \pi \text{ad}(H_0') \)) of \( \mathfrak{g} \) (resp. \( \mathfrak{g}' \)) gives rise to the involution corresponding to the Cartan decomposition \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) (resp. \( \mathfrak{g}' = \mathfrak{t'} + \mathfrak{p'} \)). Thus we have a condition

\[
(H_1) \quad \rho \circ \text{ad}(H_0) = \text{ad}(H_0) \circ \rho
\]

which is equivalent to (2) and (3) taken together.

Clearly the problem is equivalent to determine all equivalence classes under the group of all inner automorphisms of \( \mathfrak{g}' \) of monomorphisms \( \rho \) of \( \mathfrak{g} \) into \( \mathfrak{g}' \) satisfying (\( H_1 \)). We shall investigate the problem in this manner, but actually in a slightly generalized form similar to that treated in [4].

1.2. A semi-simple Lie algebra over \( \mathbb{R} \) is of hermitian type if all the non-compact simple components of it correspond to symmetric domains (i.e. the center of a maximal compact subalgebra of each non-compact simple component has the dimension 1). Let \( \mathfrak{g} \) be a Lie algebra of hermitian type, and \( \mathfrak{f} \) a maximal compact subalgebra of \( \mathfrak{g} \). Let further

\[
\mathfrak{g} = \mathfrak{g}_0 + \sum_{i=1}^d \mathfrak{g}_i
\]

be a decomposition of \( \mathfrak{g} \) into the direct sum of ideals, where \( \mathfrak{g}_0 \) is compact and \( \mathfrak{g}_i \) (\( 1 \leq i \leq d \)) are simple and non-compact, and put

\[
\mathfrak{f}_i = \mathfrak{g}_i \cap \mathfrak{f} \quad (1 \leq i \leq d).
\]

Hence \( \mathfrak{f} = \mathfrak{g}_0 + \sum_{i=1}^d \mathfrak{f}_i \). To each pair \( (\mathfrak{g}_i, \mathfrak{f}_i) \) (\( 1 \leq i \leq d \)), there corresponds a (simply connected) symmetric space \( D_i \) and an element \( H_{0i} \) of the center of \( \mathfrak{f}_i \) which determines a complex structure of \( D_i \) so as to make it a symmetric domain. Actually, \( H_{0i} \) is unique within the sign \( \pm \). For simplicity, we shall say that the element of \( \mathfrak{g} \) defined by

\[
H_0 = \sum_{i=1}^d H_{0i}
\]

is a complex structure of the pair \( (\mathfrak{g}, \mathfrak{f}) \). If \( \mathfrak{g} \) is compact, clearly \( H_0 = 0 \). It is easy to see that a complex structure \( H_0 \) of \( (\mathfrak{g}, \mathfrak{f}) \) is contained in the center of \( \mathfrak{f} \) and the inner automorphism \( \exp \pi \text{ad}(H_0) \) of \( \mathfrak{g} \) is the Cartan involution of \( (\mathfrak{g}, \mathfrak{f}) \). Let \( \mathfrak{g}' \) be another Lie algebra of hermitian type, \( \mathfrak{f}' \) its maximal compact subalgebra, and \( H_0' \) a complex structure of \( (\mathfrak{g}', \mathfrak{f}') \). The condition (\( H_1 \)) may be stated for any homomorphism \( \rho \) of \( \mathfrak{g} \) into \( \mathfrak{g}' \) and complex structures \( H_0 \) and \( H_0' \). Then our problem will be as follows:

For a given Lie algebra \( \mathfrak{g}' \) of hermitian type, determine all equivalence classes of pairs \( (\mathfrak{g}, \rho) \) of a Lie algebra \( \mathfrak{g} \) of hermitian type and a homomorphism \( \rho \) of \( \mathfrak{g} \) into \( \mathfrak{g}' \) satisfying (\( H_1 \)) (for given \( H_0 \) and \( H_0' \) w.r.t. fixed maximal compact subalgebras \( \mathfrak{f} \) and \( \mathfrak{f}' \)).

Since \( \exp \pi \text{ad}(H_0) \) (resp. \( \exp \pi \text{ad}(H_0') \)) is the Cartan involution attached to
g = f+p (resp. g' = f'+p'), a homomorphism satisfying (H₁) also satisfies (2).

1.3. For a semi-simple Lie algebra g and a maximal compact subalgebra f of g, Int (g) and K will denote the group of all inner automorphisms of g and its subgroup corresponding to f. We shall say briefly that two subalgebras g₁ and g₂ of g are (k)-conjugate if they are conjugate in g under an element of K (i.e. there is an element k ∈ K such that g₂ = k(g₁)). Let ρ₁ and ρ₂ be two homomorphisms of a semi-simple Lie algebra g into another g'. We shall say that they are (k)-equivalent if there is an element k of K' such that ρ₂ = k ∘ ρ₁.

LEMMA 1. Let g = f+p be a semi-simple Lie algebra over R with a fixed Cartan decomposition, and g₁ = t₁+p₁, g₂ = t₂+p₂ be Cartan decompositions of two semi-simple subalgebras of g such that tᵢ ⊂ f and pᵢ ⊂ p (i = 1, 2). Then g₁ and g₂ are conjugate in g if and only if they are (k)-conjugate.

PROOF. The only if part should be shown. Since g₁ and g₂ are conjugate in g, there is an element s₀ in Int (g) such that g₂ = s₀(g₁). Put t₂ = s₀(t₁) and p₂ = s₀(p₁). By the conjugacy of Cartan decompositions, we have an element t in Int (g₂) such that t(t₁) = t₂ and t(p₁) = p₂. Let s₁ be the element in the subgroup of Int (g) corresponding to g₂ such that its restriction on g₂ is t, and put s = s₁s₀. Then,

\[ s(g₁) = g₂, \quad s(t₁) = t₂, \quad s(p₁) = p₂. \]

On the other hand, one knows that

\[ s = kv, \]

where k is a uniquely determined element in K and \[ v = \exp ad(X), \quad X ∈ p. \]

By (7) and (8), we have

\[ v(t₁) ⊂ f, \quad v(p₁) ⊂ p. \]

Putting \( v₁ = \cosh ad(X) \) and \( v₂ = \sinh ad(X) \), we have

\[ v₁(t₁) ⊂ f, \quad v₁(p₁) ⊂ p, \]

\[ v₂(t₁) ⊂ p, \quad v₂(p₁) ⊂ f, \]

and \( v = v₁ + v₂ \). Since \( v₂(t₁) = v₂(p₁) = 0 \) by (9) and (10), the subalgebra \( g₁ = t₁ + p₁ \) is contained in the kernel of semi-simple linear transformation \( v₂ = \sinh ad(X) \) whose eigen-values are altogether real, hence also in the kernel of \( ad(X) \). Thus the inner automorphism \( v \) acts on \( g₁ \) as the identity, and we see that

\[ s(g₁) = k(g₁), \quad q.e.d. \]

PROPOSITION 1. Let \( g = f+p \) and \( g' = f'+p' \) be Lie algebras of hermitian types with fixed Cartan decompositions. Suppose that two homomorphisms \( ρ₁ \)
and \( \rho_2 \) satisfy \((H_1)\) for same complex structures \( H_0 \) of \((g, t)\) and \( H'_0 \) of \((g', t')\). Then \( \rho_1 \) and \( \rho_2 \) are equivalent (in \( g' \)) if and only if they are \((k)\)-equivalent.

**PROOF.** Let \( s = kv (k \in K', v \in \exp \Ad(p')) \) be an element of \( \text{Int}(g') \) such that \( \rho_2 = s \circ \rho_1 \). Clearly, \( s(\rho_1(t)) = \rho_2(t) \subset t' \). Then, we can prove by the same way as in the proof of Lemma 1 that \( v \) acts on \( \rho_1(g) \) as the identity, and we have \( \rho_2 = k \circ \rho_1 \).

It is easy to see that the condition \((H_1)\) is invariant under \((k)\)-equivalence, hence our problem is equivalent to determining all \((k)\)-equivalence classes of \((g, \rho)\).

### 1.4

Let \( g \) be a Lie algebra of hermitian type, and \( t \) a (fixed) maximal compact subalgebra. It is known that there is a Cartan subalgebra \( h \) of \( g \) in \( t \). All such Cartan subalgebras are mutually \((k)\)-conjugate, and can be regarded as those of \( t \). We denote by \( g_C \) and \( q_0 \) the complexifications of \( g \) and a vector subspace \( q \) of it. One knows that the root system \( \tau \) of \( g_0 \) relative to \( h_0 \) is the disjoint union of two subsets \( u \) and \( v \) consisting of so-called compact roots and non-compact roots respectively, i.e.

\[
\tau = \tau_u \cup \tau_v
\]

where \( g_\alpha = \{ X \in g_0 | \Ad(H)X = \alpha(H)X \text{ for every } H \in h_0 \} \) (see e.g. [2]). Moreover, one knows that \( \alpha(H_0) = 0 \) or \( \pm \sqrt{-1} \) according as \( \alpha \in u \) or \( \alpha \in v \).

Put

\[
v_\alpha = \{ \alpha \in \tau | \alpha(H_0) = \pm \sqrt{-1} \},
\]

and take an order of \( \tau \) so as to make all the roots of \( v_\alpha \) positive. Let further \( \Pi \) be the fundamental root system w.r.t. this order, and

\[
\Pi = \Pi_0 \cup \Pi_1 \cup \cdots \cup \Pi_d
\]

the decomposition of \( \Pi \) into a union of ideals; where \( \Pi_0 \) contains only compact roots, and \( \Pi_i \) (\( 1 \leq i \leq d \)) has a connected Dynkin diagram in which there is one and only one non-compact positive root. The decomposition (13) corresponds actually to (5). In fact, each \( \Pi_i \) (\( 0 \leq i \leq d \)) is a fundamental system of roots of the ideal \( g_0 \) of \( g \) relative to \( h_0 = h_0 \cap g_0 \).

If \( \Pi = \{ \alpha_1, \alpha_2, \ldots, \alpha_d \} \), we have a basis \( H_1, \ldots, H_d \) of the vector space \( \sqrt{-1} h \) over \( R \) such that

\[
\alpha_i(H_j) = 2\langle \alpha_i, \alpha_j \rangle.
\]

In other words by the usual identification, \( H_j = 2\alpha_j / \langle \alpha, \alpha \rangle \). In the following \( \S\S \), we shall also put \( H_\alpha = 2\alpha / \langle \alpha, \alpha \rangle \) for every root \( \alpha \in \tau \), and regard it as an element of \( \sqrt{-1} h \).

Since \( \sqrt{-1} h_0 \subset \sqrt{-1} h \), we can express it as a linear combination of \( H_1, \ldots, H_d \) with real coefficients. We shall give here such expressions of
$\sqrt{-1}^{-1}H_0$ for all the simple Lie algebras of hermitian types. The element $\alpha_*$ in $\Pi$ will always be the unique non-compact simple root, hence $\alpha_*(\sqrt{-1}H_0) = 1$ and $\alpha_*(\sqrt{-1}H_0) = 0$ ($i \geq 2$). On the other hand, putting

$$a_{ij} = \alpha_i(H_j), \quad \frac{1}{\sqrt{-1}}H_0 = \sum_{j=1}^i r_j H_j,$$

we have

$$\alpha_i\left(\frac{1}{\sqrt{-1}}H_0\right) = \sum_{j=1}^i a_{ij} r_j \quad (1 \leq i \leq l).$$

Since the Cartan matrix $(a_{ij})$ is non-singular, this equation in $r_j$ can be solved. Now we have the following list:

(I)$_{p,q}$ ($1 \leq p \leq q$): $[\mathfrak{t}, \mathfrak{f}] \cong (A_{p-1}) \times (A_{q-1})$.

\[ \alpha_1 \cdots \alpha_{q+1} \alpha_{q+2} \alpha_1 \alpha_2 \cdots \alpha_q \quad (l = p + q - 1). \]

$$\frac{1}{\sqrt{-1}}H_0 = \frac{1}{p+q} (qH_{q+1} + 2H_{q+2} + \cdots + (p-1)qH_1 + pqH_1 + p(q-1)H_2 + \cdots + pH_p).$$

(II)$_p$ ($p \geq 5$): $[\mathfrak{t}, \mathfrak{f}] \cong (A_{p-1})$.

\[ \alpha_1 \cdots \alpha_2 \cdots \alpha_p \]

$$\frac{1}{\sqrt{-1}}H_0 = \frac{1}{4} (pH_1 + 2(p-2)H_2 + 2(p-3)H_3 + \cdots + 2H_{p-1} + (p-2)H_p).$$

(III)$_p$ ($p \geq 2$): $[\mathfrak{t}, \mathfrak{f}] \cong (A_{p-1})$.

\[ \alpha_1 \cdots \alpha_2 \cdots \alpha_p \]

$$\frac{1}{\sqrt{-1}}H_0 = \frac{1}{2} (pH_1 + (p-1)H_2 + \cdots + 2H_{p-1} + H_p).$$

(IV)$_p$ ($p \geq 5$):

$p$ odd: $[\mathfrak{t}, \mathfrak{f}] \cong (B_{l-1})$, $l = \frac{p+1}{2}$.

\[ \alpha_1 \cdots \alpha_2 \cdots \alpha_{l-1} \cdots \alpha_l \]

$$\frac{1}{\sqrt{-1}}H_0 = \frac{1}{2} (2H_1 + 2H_2 + \cdots + 2H_{l-1} + H_l).$$

$p$ even: $[\mathfrak{t}, \mathfrak{f}] \cong (D_{l-1})$, $l = \frac{p}{2} + 1$. 
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\[ \alpha_1 \quad \alpha_2 \quad \alpha_{1-2} \quad \alpha_{1-1} \]

\[ \frac{1}{\sqrt{-1}} H_0 = \frac{1}{2} (2H_1 + \cdots + 2H_{1-a} + H_{1-1} + H_0). \]

(EIII): \([t, t] \cong (D_6).\)

\[ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \]

\[ \frac{1}{\sqrt{-1}} H_0 = \frac{1}{3} (4H_1 + 5H_2 + 6H_3 + 4H_4 + 2H_5 + 3H_6). \]

(EVII): \([t, t] \cong (E_6).\)

\[ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \]

\[ \frac{1}{\sqrt{-1}} H_0 = \frac{1}{2} (3H_1 + 4H_2 + 5H_3 + 6H_4 + 4H_5 + 2H_6 + 3H_7). \]

§ 2. Reduction of the problem.

2.1. Let the notations \( \mathfrak{g}, \mathfrak{g}', \cdots, H_0, H_0' \) be as before. The following condition \((H_2)\) about a homomorphism \( \rho \) of \( \mathfrak{g} \) into \( \mathfrak{g}' \), which is stronger than \((H_1)\), will play an important rôle for the reduction of our problem:

\[ \rho(H_0) = H_0'. \]

Now that all the Cartan subalgebras of \( \mathfrak{g}' \) in \( \mathfrak{v} \) are mutually \((k)\)-conjugate, any homomorphism satisfying \((H_1)\) must be \((k)\)-equivalent to a homomorphism \( \rho \) such that

\[ \rho(\mathfrak{h}) \subseteq \mathfrak{h}'. \]

Hence, in the following, we shall always assume without mentioning specifically that a homomorphism \( \rho \) satisfying \((H_1)\) or \((H_2)\) also satisfies \((15)\).

2.2. Let \( \mathfrak{g}_c \) be a complex semi-simple Lie algebra, \( \mathfrak{g}_u \) a compact real form of \( \mathfrak{g}_c \). The usual complex conjugation acts on \( \mathfrak{g}_c \) by

\[ Z = X + \sqrt{-1} Y \rightarrow \bar{Z} = X - \sqrt{-1} Y \quad (X, Y \in \mathfrak{g}_u). \]
It is well known that, twisting this action in the following manner, we have all (sets of real points of) real forms of \( g_{0} \): if \( \tau_{u} \) is an involutive automorphism of \( g_{0} \) which preserves \( g_{u} \) and if we put

\[
(16) \quad f_{\alpha} = \begin{cases} 
1 & (\alpha = \text{id.}) \\
\tau_{u} & (\alpha \neq \text{id.})
\end{cases} \quad \text{for} \quad \alpha \in \text{Aut}_{R}(C),
\]

we can define another action of \( \text{Aut}_{R}(C) \) on \( g_{0} \) by

\[
(17) \quad [\alpha](X) = f_{\alpha}(\bar{X}).
\]

The set of ' \( R \)-rational points' \( \gamma \) of \( g_{0} \) corresponding to this action is usually called a real form of \( g_{0} \), and \( \sigma = [\alpha](\alpha \neq \text{id.}) \) a conjugation of \( g_{0} \) w.r.t. \( g \). Let \( g'_{0} \) be another complex semi-simple Lie algebra with a real form \( g' \), \( \sigma' \) the conjugation of \( g'_{0} \) w.r.t. \( g' \). A homomorphism \( \rho \) of \( g_{0} \) into \( g'_{0} \) is defined over \( R \), or in other words \( \rho \) induces a homomorphism of \( g \) into \( g' \), if and only if

\[
(18) \quad \rho \circ \sigma = \sigma' \circ \rho.
\]

Suppose that the compact real form \( g'_{u} \) of \( g'_{0} \) is taken as \( \rho(g_{u}) \subset g'_{u} \). Then, in view of the relations (16) and (17), the condition (18) is equivalent to

\[
(19) \quad \rho \circ \tau_{u} = \tau'_{u} \circ \rho.
\]

Now, let \( g = \mathfrak{t} + \mathfrak{p} \) (resp. \( g' = \mathfrak{t}' + \mathfrak{p}' \)) be a Lie algebra of hermitian type with a Cartan decomposition, \( H_{0} \) (resp. \( H'_{0} \)) a complex structure of \( (g, \mathfrak{t}) \) (resp. \( (g', \mathfrak{t}') \)). Taking the compact duals of \( g \) and \( g' \) for \( g_{u} \) and \( g'_{u} \) respectively, i.e.

\[
\tau_{u} = \exp \pi \text{ad}_{c}(H_{0}) \quad \text{and} \quad \tau'_{u} = \exp \pi \text{ad}_{c}(H'_{0})
\]

respectively, where \( \text{ad}_{c} \) denotes the adjoint representations of the complex Lie algebras \( g_{0} \) and \( g'_{0} \). If \( \rho \) is a homomorphism of \( g_{0} \) into \( g'_{0} \) such that \( \rho(g_{u}) \subset g'_{u} \) and that

\[
(20) \quad \rho \circ \text{ad}_{c}(H_{0}) = \text{ad}_{c}(H_{0}) \circ \rho,
\]

it satisfies the condition (19), hence also (18). Then the image of \( g \) by \( \rho \) is in \( g' \), and according to (21) the restriction of \( \rho \) to \( g \) satisfies \((H_{1})\). Conversely, if \( \rho \) is a homomorphism of \( g \) into \( g' \) satisfying \((H_{1})\), its extension to that of \( g_{0} \) into \( g'_{0} \) carries \( g_{u} \) into \( g'_{u} \) and has the property (21). Thus we have proved the following

**PROPOSITION 2.** Let \( g \) and \( g' \) be of hermitian type. Then a homomorphism \( \rho \) of \( g_{0} \) into \( g'_{0} \) induces one of \( g \) into \( g' \) satisfying \((H_{1})\) (resp. \((H_{2})\)) if and only if \( \rho(g_{u}) \subset g'_{u} \) and \( \rho \circ \text{ad}_{c}(H_{0}) = \text{ad}_{c}(H_{0}) \circ \rho \) (resp. \( \rho(H_{0}) = H'_{0} \)), where \( g_{u} \) (resp. \( g'_{u} \)) denotes the compact dual of \( g \) (resp. \( g' \)).
2.3. In this paragraph, we shall define a certain type of subalgebras of a Lie algebra of hermitian type.

Let the notations be as in 1.4. A subset \( \Delta \) of the root system \( \rho \) is called by Dynkin a \( \Pi \)-system if it has the following properties:

(i) If \( \alpha = \Delta \) and \( \beta \in \Delta \), \( \alpha - \beta \) is not in \( \rho \).

(ii) \( \Delta \) is a linearly independent system in \( \sqrt{-1} \).

If \( \Delta \) is a \( \Pi \)-system, the subset \( \mathfrak{g}_\Delta \) of \( \mathfrak{g} \) defined by

\[
\mathfrak{g}_\Delta = \sum_{\alpha \in \Delta} CH_\alpha + \sum_{\alpha \in \rho \setminus \Delta} \mathfrak{g}_\alpha; \quad \mathfrak{r}(\Delta) = \left( \sum_{\alpha \in \Delta} Z\alpha \right) \cap \mathfrak{r},
\]

is a regular semi-simple subalgebra of \( \mathfrak{g} \); \( \mathfrak{r}(\Delta) \) can be considered as the root system of \( \mathfrak{g}_\Delta \) relative to the Cartan subalgebra \( \sum_{\alpha \in \Delta} CH_\alpha \), in which the \( \Pi \)-system \( \Delta \) forms a simple root system ([1], Chap. II).

A \( \Pi \)-system, however, will be needed in this paper, only when it has moreover the following property:

(iii) Each connected component of the Dynkin diagram of \( \Delta \) contains at most one positive non-compact root besides compact roots.

Therefore we shall say that a subset \( \Delta \) of \( \rho \) is a \( \Pi \)-system if it has the property (iii) in addition to (i) and (ii).

**PROPOSITION 3.** Let \( \Delta \) be a \( \Pi \)-system in \( \rho \). Then the regular subalgebra \( \mathfrak{g}_\Delta \) of \( \mathfrak{g} \) is defined over \( \mathbb{R} \) whose real form \( \mathfrak{g}(\Delta) = \mathfrak{g}_\Delta \cap \mathfrak{g} \) is a Lie algebra of hermitian type, and there is a unique complex structure of \( (\mathfrak{g}(\Delta), \mathfrak{r}(\Delta)) \) by which the injection of \( \mathfrak{g}(\Delta) \) into \( \mathfrak{g} \) satisfies (H1), where \( \mathfrak{r}(\Delta) = \mathfrak{r}(\Delta) \cap \mathfrak{r} \).

**PROOF.** Let \( \Delta = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_d \) be the decomposition of \( \Delta \) into a disjoint union, where \( \Delta_0 \) contains only compact roots, and \( \Delta_i = \{ \gamma, \beta_1, \cdots, \beta_t \} \) (\( 1 \leq i \leq d \)) has a connected Dynkin diagram, in which \( \gamma \) denotes the unique positive non-compact root. Then all the roots in \( \mathfrak{r}(\Delta_i) (\subset \mathfrak{r}) \) are of the form

\[
\pm \left( \gamma + \sum_{j=1}^d c_j \beta_j \right) \in \mathfrak{g}_\Delta \quad \text{or} \quad \pm \sum_{j=1}^t d_j \beta_j \in \mathfrak{u},
\]

where \( c_j \) and \( d_j \) are non-negative integers. Define an element \( H_{0i} \) of \( \mathfrak{g}_\Delta \) by

\[
\frac{1}{\sqrt{-1}} \gamma(H_{0i}) = 1, \quad \frac{1}{\sqrt{-1}} \beta_j(H_{0i}) = 0 \quad (1 \leq j \leq t),
\]

and an element \( H_{0d} \) of \( \mathfrak{g}_\Delta \) by

\[
H_{0d} = \sum_{i=1}^d H_{0i}.
\]

In view of (12), (23), (24), and (25), we have

\[
\alpha(H_{0d}) = \alpha(H_{0i}) \quad \text{for} \quad \alpha \in \mathfrak{r}(\Delta).
\]

Now, let \( \mathfrak{g}_0(\Delta) \) be a compact real form of \( \mathfrak{g}_\Delta \) contained in \( \mathfrak{g}_0 \), i.e.

\[
\mathfrak{g}_0(\Delta) = \mathfrak{g}_\Delta \cap \mathfrak{g}_0.
\]
The equations (24) and (25) show that $H_{ad}$ is an element of the center of the intersection $\mathfrak{k}(\mathcal{A})$ of $\mathfrak{g}_0(\mathcal{A})$ and $\mathfrak{t}$, and that the inner automorphism $\tau_{ad}$ of $\mathfrak{g}_0(\mathcal{A})$ defined by

$$\tau_{ad} = \exp \pi \text{ad}_\mathfrak{g}(H_{ad})$$

is involutive and keeps $\mathfrak{g}_0(\mathcal{A})$ invariant. The real form $\mathfrak{g}(\mathcal{A})$ of $\mathfrak{g}_0(\mathcal{A})$ defined by $\tau_{ad}$ is a Lie algebra of hermitian type, in which $\mathfrak{k}(\mathcal{A})$ is a maximal compact subalgebra, and $H_{ad}$ is a complex structure of $(\mathfrak{g}(\mathcal{A}), \mathfrak{t}(\mathcal{A}))$. Since the equation (26) means that the injection $\iota$ of $\mathfrak{g}_0(\mathcal{A})$ into $\mathfrak{g}_0$ satisfies

$$\iota \circ \text{ad}_\mathfrak{g}(H_{ad}) = \text{ad}_\mathfrak{g}(H_0) \circ \iota,$$

the proposition follows from Proposition 2, q. e. d.

A subalgebra of a Lie algebra $\mathfrak{g}$ of hermitian type will also be called in the following a regular subalgebra (of $\mathfrak{g}$), if it is a real form $\mathfrak{g}(\mathcal{A})$ of a regular semi-simple subalgebra $\mathfrak{g}_0(\mathcal{A})$ of $\mathfrak{g}_0$, as defined in Proposition 3 for a $\Pi$-system $\mathcal{A}$ in our sense. Two regular subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$ (w. r. t. the fixed Cartan subalgebra) will be called $(k)$-equivalent if they are $(k)$-conjugate. (Note that our definition of a regular subalgebra depends on the choice of a Cartan subalgebra in $\mathfrak{t}$.)

2.4. In this paragraph, we shall give a condition of $(k)$-equivalence between regular subalgebras of a Lie algebra of hermitian type.

Let $\mathfrak{g}$ be a Lie algebra of hermitian type. Let further $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{r}$, and $\Pi$ be same things as in 1.4, $K$ the subgroup of $\text{Int}(\mathfrak{g})$ corresponding to $\mathfrak{t}$. The subgroup $W_K$ of the Weyl group of $\mathfrak{g}_0$ generated by all 'reflections' associated to compact roots can be considered as the Weyl group of the reductive Lie algebra $\mathfrak{t}$: a reflection $w_\beta$ (in the dual vector space of $\sqrt{-1} \mathfrak{h}$) associated to a root $\beta$ is defined by

$$w_\beta(\alpha) = \alpha - \alpha(H_\beta)\beta \quad (\alpha \in \mathfrak{t}).$$

One knows that $W_K$ is isomorphic to the quotient group of the normalizer $N_K(\mathfrak{h})$ of $\mathfrak{h}$ in $K$ by the centralizer $Z_K(\mathfrak{h})$ of $\mathfrak{h}$ in $K$. Now that an element of $W_K$ carries a compact root to a compact root, positive non-compact root to a positive non-compact root respectively, every image of a $\Pi$-system by it must be also a $\Pi$-system.

Now we shall show the following

**Theorem 1.** Let $\mathfrak{g}$ be a Lie algebra of hermitian type, $\mathfrak{k}$ a maximal compact subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ in $\mathfrak{k}$. Take an order of the root system $\mathfrak{r}$ of $\mathfrak{g}_0$ relative to $\mathfrak{h}_0$ w. r. t. a complex structure of $(\mathfrak{g}, \mathfrak{t})$, and let $\Delta_1, \Delta_2$ be $\Pi$-systems in $\mathfrak{r}$ w. r. t. this order. Then the regular subalgebras $\mathfrak{g}(\Delta_1)$ and $\mathfrak{g}(\Delta_2)$ of $\mathfrak{g}$ are conjugate in $\mathfrak{g}$, if and only if there is an element $w$ of the Weyl group $W_K$ of $\mathfrak{t}$ such that $\Delta_2 = w(\Delta_1)$.
By Lemma 1, it is easy to see that two regular subalgebras are mutually
cjugate if and only if they are $(k)$-equivalent.

To begin with, we establish some lemmas.

**LEMMA 2.** Two subalgebras $h_1$ and $h_2$ of $g$ contained in $h$ are $(k)$-conjugate,
if and only if there is an element $k$ of $N_k(h)$ such that $h_2 = k(h_1)$.

**PROOF.** Suppose that $h_1$ and $h_2$ are $(k)$-conjugate, and let $k$ be an element
of $K$ such that $h_2 = k(h_1)$. Then we have another Cartan subalgebra $h^* = k(h)$
of $t$, and $h_2$ is contained in $h \cap h^*$. The normalizer of $h \cap h^*$ in $t$ is a direct
sum

$$n_0(h \cap h^*) = a + f_0$$

of an abelian subalgebra $a$ and a compact semi-simple subalgebra $f_0$. The
Cartan subalgebras $h$ and $h^*$ are clearly in $n_0(h \cap h^*)$, hence there are Cartan
subalgebras $h_0$ and $h_0^*$ of $f_0$ such that

$$h = a + h_0, \quad h^* = a + h_0^*.$$  \(29\)

Since all Cartan subalgebras of a compact semi-simple Lie algebra are mutually
conjugate, there is an element $k_1$ of a subgroup of $\text{Int}(g)$ corresponding to $f_0$
such that

$$k_1(h_0) = h_0, \quad k_1 = \text{id. on } a, \quad k_1 | f_0 \in \text{Int}(f_0).$$  \(30\)

Since the intersection $h \cap h^*$ is contained in $a$, the automorphism $k_1$ preserves
each element of $h_0$. Therefore the composition $k_1 k$ is an element of $N_k(h)$ (by
(29) and (30)), which maps $h_0$ onto $h_0$, q.e.d.

**LEMMA 3.** Let $g$ be a non-compact simple Lie algebra of hermitian type, $t$
a maximal compact subalgebra of $g$, and let $h$ be a Cartan subalgebra of $g$
contained in $t$. Take an order of the root system of $g_0$ relative to $h_0$ as above
for a fixed complex structure of $(g, t)$. Then for each fundamental system
$\{\beta_1, \beta_2, \ldots, \beta_{l-1}\}$ of compact roots of $g_0$, there is a uniquely determined positive
non-compact root $\gamma$ such that $\{\gamma, \beta_1, \ldots, \beta_{l-1}\}$ becomes a fundamental root system
of $g_0$.

**PROOF.** Let $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be the simple root system of $g_0$ w.r.t. the
fixed order, in which $\alpha_1$ is the simple non-compact root. Then $\{\alpha_2, \alpha_3, \ldots, \alpha_l\}$
becomes a fundamental system of compact roots, hence there is an element $w$ of $W_0$ such that $w(\{\alpha_1\}) = \{\beta_j\}$. Thus we have a positive non-compact root $\gamma = w(\alpha_1)$, which is required. Suppose that we get another positive non-compact root $\gamma$, such that $\{\gamma, \beta_1, \ldots, \beta_{l-1}\}$ becomes a fundamental root system. Then
it is easy to see that the following equations should be held:

$$\gamma = \gamma_1 + \sum_{i=1}^{l-1} c_i \beta_i, \quad \gamma = \gamma_1 + \sum_{j=1}^{l-1} d_j \beta_j,$$

where $c_i, d_j$ are all non-negative integers. But since $\beta_1, \ldots, \beta_{l-1}$ are linearly
independent, we can see at once
\[ c_i = d_i = 0, \quad \text{and hence} \]
\[ \gamma_i = \gamma, \quad \text{q. e. d.} \]

**Proof of Theorem 1.** Suppose that there is an element \( w \in W_K \), such that \( \Delta = w(\Delta) \). Then we have an element \( k \) of \( N_K(h) \) such that
\[ k \left( \frac{1}{\sqrt{-1}} H_\alpha \right) = \frac{1}{\sqrt{-1}} H_{w(\alpha)} \quad (\alpha \in \Gamma). \] Extending \( k \) linearly to \( g \), we can see that \( k(g_\alpha) = g_{w(\alpha)} \) where \( g_\alpha \) (resp. \( g_{w(\alpha)} \)) is the eigen space of the adjoint representation belonging to a root \( \alpha \) (resp. \( w(\alpha) \)). Thus we have
\[ k(g_\alpha(\Delta)) = g_{w(\alpha)}(\Delta). \] Since \( k \) preserves \( g \), (32) means that \( g(\Delta) = g(g(\Delta)) \). Conversely, let \( g(\Delta) \) and \( g(\Delta) \) be conjugate in \( g \). Then they are \((k)\)-equivalent, i.e. there is an element \( k \) of \( K \) such that \( k(g(\Delta)) \) is equal to \( g(\Delta) \). Let \( \mathfrak{h}_1 \) and \( \mathfrak{h}_2 \) be the Cartan subalgebras of \( g(\Delta) \) and \( g(\Delta) \) generated by \( \{ \sqrt{-1}^{-1} H_\alpha : \alpha \in \Delta \} \) and \( \{ \sqrt{-1}^{-1} H_\alpha : \alpha \in \Delta \} \) respectively; both of them are contained in \( \mathfrak{h} \). Then \( \mathfrak{h}_1 \) and \( \mathfrak{h}_2 \) are \((k)\)-conjugate, since \( k(\mathfrak{h}_1) \) is, as well as \( \mathfrak{h}_2 \), a Cartan subalgebra of \( g(\Delta) \) contained in \( g(\Delta) \). Hence, by Lemma 2, there is an element \( k_0 \) of \( N_K(\mathfrak{h}) \) such that \( k_0 = k_0(\mathfrak{h}_1) \). Taking \( w_0 \) in \( W_K \) which corresponds to \( k_0 \) modulo \( Z_K(\mathfrak{h}) \), we have
\[ k_0(H_\alpha) = H_{w_0(\alpha)} \quad \text{for all} \ \alpha \in \Gamma; \ \text{in particular,} \]
\[ w_0(g(\Delta)) = g(\Delta) . \]

Therefore we have another fundamental system \( w_0(\Delta) \) of \( g(\Delta) \) besides \( \Delta \). Using Lemma 3, it is easy to see that there is an element \( w \) in the subgroup of \( W_K \) generated by \( w(\beta) \) where \( \beta \) runs over all compact roots in \( \Delta \), such that \( w(\Delta) = \Delta \), q. e. d.

**2.5.** The following theorem corresponds to the Proposition 1 in [4], when the Lie algebra \( g' \) in our problem is \( (III)_p \). It will give a reduction of our problem.

**Theorem 2.** Let \( g \) and \( g' \) be Lie algebras of hermitian type, \( \mathfrak{t} \) and \( \mathfrak{t}' \) maximal compact subalgebras of \( g \) and \( g' \), and let \( H_0 \) and \( H_0 \) be complex structures of \( (g, \mathfrak{t}) \) and \( (g', \mathfrak{t}') \) respectively. Let further \( \rho \) be a homomorphism of \( g \) into \( g' \) satisfying \((H_1)\) w.r.t. \( H_0 \) and \( H_0 \). Then there is a regular subalgebra \( g'' \) of \( g' \) such that the image \( \rho(g) \) is contained in \( g'' \) and that \( \rho : g \rightarrow g'' \) satisfies \((H_2)\) with respect to \( H_0 \) and the natural complex structure of \( (g'', \mathfrak{t}'') \) defined by \( H_0 \). In particular, \( g'' \) can be taken as a proper subalgebra if \( \rho : g \rightarrow g' \) does not satisfies \((H_2)\).

**Proof.** The condition \((H_1)\) implies that the image \( \rho(g) \) is contained in the
centralizer $g_1$ of $H_0 - \rho(H_0)$ in $g'$. Clearly $g_1 = g'$, if and only if $\rho$ satisfies (H2).
Let $r'$ be the root system of $g'_c$ relative to $\mathfrak{h}_0'$, and $r''$ the subset of $r'$ consisting of roots such as

\[(34) \quad \alpha'(H_0 - \rho(H_0)) = 0.\]

It is easily seen that $r''$ is a subsystem of $r'$, and

\[g_{1C} = \mathfrak{h}' + \sum_{\alpha' \in r''} \mathfrak{g}_{a'\alpha'}, \quad g_1 = g_{1C} \cap \mathfrak{g},\]

where $\mathfrak{g}_{a'\alpha'}$ is the eigen space of the adjoint representation of $\mathfrak{h}_0'$ belonging to $\alpha' \in r'$. Let $g''$ denote the semi-simple part of $g_1$, and put $H'' = \rho(H_0)$. The image $\rho(g)$ is semi-simple, so that $\rho(g) \subset g''$. Since $r''$ can be considered as the root system of $g''_c$ relative to the Cartan subalgebra $\mathfrak{h}''_c = g''_c = \mathfrak{h}_0''$, the subalgebra $\mathfrak{g}''_{a''} = \mathfrak{g}''_c \cap \mathfrak{g}''_a$ is a compact real form of $g''_c$. The equation (34) implies that $H''_c$ is an element of the center of $t'' = g''_c \cap t' = g'' \cap g''_a$, and that the inner automorphism $\tau''$ of $g''$ defined by

\[\tau'' = \exp \pi \text{ad}(H''_c)\]

is the restriction of the Cartan involution $\tau'$ of $(g', t')$ to $g''$. Hence $\tau''$ is a Cartan involution of $g''$, and $t''$ is the maximal compact subalgebra of $g''$ defined by $\tau''$ which contains $\mathfrak{h}'' = \mathfrak{h}''_c \subset g''$. Therefore the subalgebra $g''_c$ is of hermitian type, and $H''_c$ is the complex structure of $(g''_c, t'')$ by which the homomorphism $\rho : g \to g''$ and the injection : $g'' \to g'$ satisfy (H2) and (H3) respectively. Moreover the equation (34) also implies that a compact (resp. non-compact) root of $g''_c$ relative to $\mathfrak{h}''_c$ is (the restriction to $\mathfrak{h}''_c$ of) a compact (resp. non-compact) root in $r'$. Hence a simple root system of $r''$ w.r.t. a suitable order (by which a root $\alpha' \in r''$ satisfying $\alpha'(H''_c) = \sqrt{-1}$ is positive) is a $\Xi$-system, q.e.d.

2.6. In virtue of Theorem 2, our study of the problem can be divided generally into two steps: the first is to determine all $(k)$-equivalence classes of regular subalgebras of a given Lie algebra $g'$ of hermitian type; the second is to find out, for a fixed regular subalgebra $g''$ of $g'$, all equivalence classes of pairs $(g, \rho)$ of a Lie algebra of hermitian type and a homomorphism of $g$ into $g''$ satisfying (H2).

Now, we make here a little more reductions. When we say that $\rho$ is a homomorphism of $g$ into $g'$ satisfying (H2), we always assume that $H_0$ and $H''_c$, complex structures of $(g, \mathfrak{f})$ and $(g', \mathfrak{f}')$ respectively, are given, where $\mathfrak{f}$ (resp. $\mathfrak{f}'$) is a maximal compact subalgebra of $g$ (resp. $g'$). In the first place,

*We may consider only the case $g'$ is simple.*

In fact, if $g'$ is a direct sum

\[g' = \sum_{i=1}^{n} g'_i\]
of simple ideals $g'_i$, we have a projection $\rho_i$ of $\rho$ and $H'_0$ of $H'_0$ on $g'_i$; it is clear that $H'_0$ is a complex structure of $(g_0, t'_0)$ ($t'_0 = t' \cap g'_0$), and that the homomorphism $\rho_i$ of $g$ into $g'$ satisfies (H$_i$). If $\rho$ satisfies (H$_0$), each $\rho_i$ satisfies also (H$_i$). It is easy to see that two homomorphisms $\rho^{(1)}$ and $\rho^{(2)}$ of $g$ into $g'$ satisfying (H$_i$) are $(k)$-equivalent, if and only if the projections $\rho^{(1)}_{|g'_i}$ and $\rho^{(2)}_{|g'_i}$ of them on $g'_i$ are $(k)$-equivalent in $g'_i$ for all $i$.

Secondly, suppose that $\rho : g \to g'$ satisfies the stronger condition (H$_2$), and that $g$ is a direct sum of simple ideals $g_i$ ($1 \leq i \leq d$). By the above remark, we may assume again that $g'$ is simple. Let $\rho_{|g_i}$ be the restriction of $\rho$ to $g_i$, and $H'_0$ the projection of $H'_0$ on $g'_i$. The restriction $\rho_{|g}$ is a homomorphism of $g_i$ into $g'$ satisfying (H$_1$) w. r. t. $H_{g_i}$ and $H'_0$. Since $g_i$'s are simple, some of $\rho_{|g_i}$'s are injective and others are trivial. Let $\rho^{(1)}, \ldots, \rho^{(e)}$ ($e \leq d$) be injective. Then

$$\rho^{(1)} \oplus \cdots \rho^{(e)} : g_1 \oplus \cdots \oplus g_e \to g'$$

is a homomorphism satisfying (H$_2$). If one of $\rho^{(i)}$'s satisfies (H$_2$), say $\rho^{(i)}(H'_{g_i}) = H'_{g_i}$, it is easy to see by injectivity that $H_{g_i} \ldots H_{g_e}$ are equal to zero, i.e. $g_1, \ldots, g_e$ are compact. On the other hand, if non of $\rho_{|g_i}$'s satisfies (H$_2$), we see from Theorem 2 that each $\rho_{|g_i}$ maps $g_i$ into a proper regular subalgebra $g'_i$ of $g'$ satisfying (H$_2$). We can see that $[g'_i, g'_j] = 0$ if $i \neq j$). Even if $g'_i$ is not simple, we can again apply the above consideration. If $g$ is compact and if $\rho$ satisfies (H$_2$), $g'$ must obviously be compact.

Thus our problem has been reduced to the following three:

a) Determine all $(k)$-equivalence classes of regular subalgebras of each non-compact simple Lie algebra of hermitian type.

b) Find out, for each non-compact simple Lie algebra $g'$ of hermitian type, all $(k)$-equivalence classes of pairs $(g, \rho)$ of a simple Lie algebra of hermitian type and a homomorphism of $g$ into $g'$ satisfying (H$_2$).

c) Let $g'_1$ and $g'_2$ be non-compact simple regular subalgebras of simple $g'$, $\rho_1$ and $\rho_2$ representatives of $(k)$-equivalence classes of homomorphisms of a simple Lie algebra $g$ of hermitian type into $g'_1$ and $g'_2$ satisfying (H$_2$) respectively. Examine whether $\iota_1 \circ \rho_1$ and $\iota_2 \circ \rho_2$ are $(k)$-equivalent in $g'$, where $\iota_i$ is the injection of $g'_i$ into $g'$ ($i = 1, 2$).

The solutions to the problem a) will be given in §4. The problem b) is already solved by Satake in [4] (if the results in it and our certain solutions for a) are combined) when $g'$ is (I)$_p$, (II)$_p$, or (III)$_p$; while for the remaining cases that $g'$ is (IV)$_p$, (EIII), or (EVII), it will be done in §5 of this paper. We shall never describe in this paper the solutions to c), because they would depend too much on the respective circumstances. But, for each case specified, it will be not difficult to carry it out by applying Theorem 3 in §3.

1) Proof of this property will be given elsewhere.
Example. There is a regular subalgebra $g''$ of type $(IV)_{2l-2}$ in $g'=(IV)_{2l-1}$. Take this $g''$ for both of $g_1'$ and $g_2'$ in $\mathfrak{c})$. If $g=(IV)_{2l-n}$, there are two $(k)$-equivalence classes of homomorphisms of $g$ into $g''$ satisfying $(H_2)$ (see 5.1). Let $\rho_1$ and $\rho_2$ be their representatives. Then we can see that $\tau \circ \rho_1$ and $\tau \circ \rho_2$ are equivalent in $g'$. On the other hand, there is also a regular subalgebra $g''$ of type $(IV)_{n}$ $(l=5)$ in $g'=(EIII)$. But, in this case, we can see that $\tau \circ \rho_1$ and $\tau \circ \rho_2$ are not equivalent in $g'$. Hence we have two classes of homomorphisms of $(IV)_{n}$ into $(EIII)$ satisfying $(H_2)$.

§ 3. Some properties of homomorphisms of Lie algebras.

In this section, we prove some propositions which will be needed in § 5. Throughout this section, $\rho$ denotes always (the complexification of) a monomorphism of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying $(H_2)$, where both $\mathfrak{g}$ and $\mathfrak{g}'$ are non-compact simple Lie algebras of hermitian types; hence we may assume moreover the property (15). Other notations are the same as those used before. Especially,

$$H_{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle} \in \sqrt{-1} \mathfrak{h} \ (\alpha \in \mathfrak{r}), \quad H_k = H_{\alpha_l};$$

the root $\alpha_1$ means always the non-compact simple root. All these notations are used with prime for $\mathfrak{g}'$.

Recall that there are at most two sorts of the length of roots for each simple Lie algebra $\mathfrak{g}_C$. A root of the maximal (resp. minimal) length will be called simply a longer (resp. shorter) root. Let $\alpha$ (resp. $\beta$) be a longer (resp. shorter) root. Then, if a non-compact real form $\mathfrak{g}$ of $\mathfrak{g}_C$ is of hermitian type, one knows that

$$\langle \alpha, \alpha \rangle > \langle \beta, \beta \rangle = \frac{2}{\langle H_\beta, H_\beta \rangle} = \left\{ \begin{array}{ll} 2 \ (g=(III), (IV)_{2n+1}), \\ 1 \ (\text{others}). \end{array} \right.$$ 

3.1. Let $\phi$ and $\phi'$ be the Killing forms of $\mathfrak{g}_C$ and $\mathfrak{g}'_C$ respectively. Since $\phi'(\rho(X), \rho(Y)) \ (X, Y \in \mathfrak{g}_C)$ is again an invariant non-degenerate bilinear form of simple Lie algebra $\mathfrak{g}_C$, there is a constant $c_\rho$ depending only on $\rho$ such that

$$\phi'(\rho(X), \rho(Y)) = c_\rho \phi(X, Y).$$

Therefore we have

$$\langle \rho(X), \rho(Y) \rangle' = c_\rho \langle X, Y \rangle \quad (X, Y \in \sqrt{-1} \mathfrak{h}).$$

Now, since $H_1', \ldots, H_{l'}$ forms a basis of $\sqrt{-1} \mathfrak{h}'$, we can put

$$\rho(H_k) = \sum_{j=1}^{l'} \mu_j \rho H_j' \quad (k=1, \ldots, l).$$

Proposition 4. All the coefficients $\mu_j$ of the equation (36) are rational integers. In particular, if $\rho$ satisfies $(H_2)$,
PROOF. Let \( \lambda'_j \) be the dominant weight defined by \( \lambda'_j(H_0) = \delta_{j0} \), the usual Kronecker's delta. Then we have

\[
\lambda'_j \circ \rho(H_0) = \mu^j_{i0}.
\]

If \( A_j \) is the irreducible representation of \( g_0 \) defined by \( \lambda'_j \), the linear form \( \lambda'_j \circ \rho \) on \( h_0 \) is a weight of the representation \( A_j \circ \rho \) of \( g_0 \) relative to \( h_0 \). Therefore the equation (37) implies that \( \mu^j_{i0}' \)'s must be rational integers. If \( i \geq 2 \) (resp. \( j \geq 2 \)), \( \alpha_i \) (resp. \( \alpha'_j \)) is a compact root; hence we have \( \alpha_i(H_0) = 0 \) (resp. \( \alpha'_j(H_0) = 0 \)), or equivalent to saying \( \langle H_0, \sqrt{-1} H_0 \rangle = 0 \) (resp. \( \langle H'_j, \sqrt{-1} H'_0 \rangle' = 0 \)). If \( \rho \) satisfies \( (H_2) \), we have therefore

\[
\mu^j_{i0} = \langle \rho(H_0), \sqrt{-1} H_0 \rangle' = \langle \rho(H'_0), \sqrt{-1} \rho(H'_0) \rangle' = c_0 \langle H_0, -\sqrt{-1} H_0 \rangle = 0,
\]

q. e. d.

3.2. A root \( \alpha \) is said to be strongly orthogonal to another root \( \beta \), if \( \alpha \pm \beta \) are not roots. A subset \( S \) of the root system \( \tau \) of \( g_0 \) relative to \( h_0 \) is called a strongly orthogonal system of roots, if any root in \( S \) is strongly orthogonal to all the other roots in \( S \). Each connected component of a strongly orthogonal system contains only one root, so it is a \( \Pi \)-system (in our sense). One knows that there is a strongly orthogonal system \( \Delta_0 \) consisting of positive non-compact roots \( \gamma_1, \gamma_2, \ldots, \gamma_r \), where \( r \) is the dimension of a maximal abelian subalgebra contained in \( p \) of \( g \). The dimension \( r \) is called the rank of the pair \( (g, \tau) \). The regular subalgebra \( g(\Delta_0) \) is isomorphic to a direct sum of \( r \)-copies of 3-dimensional Lie algebras of type \( (I)_{1,1} \); hence, in view of Proposition 3, we have a monomorphism

\[
\kappa: (I)_{1,1} + \cdots + (I)_{1,1} \rightarrow g
\]

satisfying \( (H_1) \), called a Hermann map (see [5]).

One knows that all the roots in \( \Delta_0 \) are longer roots; \( \gamma_i \) may be considered as the non-compact simple root \( \alpha_i \), and \( \gamma_r \) the highest root \( \gamma \). We denote by \( \gamma'_p, r', \Delta'_0, \ldots \) the similar things belonging to \( g' \).

In the rest of this paper, we shall denote by \( E_\alpha \) a basis of \( g_\alpha \) for each root \( \alpha \) of \( g_0 \) (and similarly by \( E'_\alpha \) for \( \alpha' \) of \( g'_0 \)) such that

\[
\tilde{E}_\alpha = -E_\alpha, \quad [E_\alpha, E_\alpha] = H_\alpha,
\]

where \( - \) denotes the conjugation of \( \rho \) w. r. t. the compact dual \( g_\alpha \) of \( g \) (see e. g. [2], pp. 219-221). Then we get a basis of \( g_\alpha \):

\[
\sqrt{-1} H_i \quad (1 \leq i \leq l), \quad E_\alpha - E_\alpha, \sqrt{-1} (E_\alpha + E_\alpha) \quad (\alpha \in \tau).
\]

Since \( g_\alpha \) is the compact dual of \( g \), a basis of \( p \) is given by \( \sqrt{-1}(E_\alpha - E_\alpha), E_\gamma + E_{-\gamma}, \) where \( \gamma \) runs over all positive non-compact roots.
**PROPOSITION 5.** The homomorphism \( \rho \) satisfying \( (H_1) \) can be modified by an element of \( K' \) preserving the condition \( (15) \) in such a way that there is a subset \( S(\gamma) \) of \( \Delta_0 \) for every \( \gamma \in \Delta_0 \) such that

\[
\rho(H_\gamma) = \sum_{\gamma' \in S(\gamma)} H_{\gamma'}, \quad \rho(E_{\gamma'}) = \sum_{\gamma' \in S(\gamma)} E_{\gamma'}, \quad \rho(E_{-\gamma'}) = \sum_{\gamma' \in S(\gamma)} E_{-\gamma'}.
\]

**Proof.** (Cf. Satake [5], Proposition 3.) Put \( X_\gamma = E_\gamma + E_{-\gamma} \), \( X'_{\gamma'} = E'_{\gamma'} + E_{-\gamma'} \). Then one knows that \( \{X_\gamma : \gamma \in \Delta_0 \} \) (resp. \( \{X'_{\gamma'} : \gamma' \in \Delta'_0 \} \) spans a maximal abelian subalgebra \( a \) (resp. \( a' \)) in \( p \) (resp. \( p' \)). Let \( b \) (resp. \( b' \)) be the centralizer of \( a \) (resp. \( a' \)) in \( h \) (resp. \( h' \)). Then we get Cartan subalgebras \( h_0 = b + a \) and \( h'_0 = b' + a' \) of \( g \) and \( g' \) respectively, and direct sum decompositions

\[
h = b + \text{ad}(H_{\gamma}) a, \quad h' = b' + \text{ad}(H_{\gamma}) a',
\]

so that the mutually orthogonal subspaces of \( h \) and \( h' \) respectively w.r.t. the Killing form. Since all the maximal abelian subalgebras in \( p' \) are \( (k) \)-conjugate to \( a' \), there is an element \( k_1 \) of \( K' \) such that \( k_1 \circ \rho(a) \subseteq a' \). As \( \rho \) satisfies \( (H_1) \), it is easy to see that \( k_1 \circ \rho(a) \subseteq a' \). Let \( h'_0 = b' + a' \) be another Cartan subalgebra of \( g' \) which is conjugate to \( h_0 \) and contains \( k_1 \circ \rho(h_0) \); hence clearly \( k_1 \circ \rho(h) \subseteq h' \). One knows ([6], § 1, Proposition 5) that there is an element \( k_2 \) of \( K' \) such that

\[
k_2(X) = X \quad \text{for all } X \in a,
\]

and

\[
k_2(h') = h'.
\]

Clearly \( (k_2, k_1) \circ \rho(h) \subseteq h' \) and \( (k_2, k_1) \circ \rho(a) \subseteq a' \). Therefore we may assume that \( \rho(h) \subseteq h' \) and \( \rho(a) \subseteq a' \). Now, let \( \gamma \) be a root in \( \Delta_0 \). Then we can write

\[
\rho(X_\gamma) = \sum_{\gamma' \in \Delta_0} \kappa_{\gamma', X_{\gamma'}}, \quad \kappa_{\gamma', X_{\gamma'}} \in R.
\]

Since \( \rho \) satisfies \( (H_1) \), \( \sqrt{-1} \)-eigenspace of \( \text{ad}(H_{\gamma}) | p_0 \) is carried by \( \rho \) to \( \sqrt{-1} \)-eigenspace of \( \text{ad}(H_{\gamma}) | p_0' \); hence we have

\[
\rho(E_{\gamma'}) = \sum_{\gamma' \in \Delta_0} \kappa_{\gamma', E_{\gamma'}}, \quad \rho(E'_{\gamma'}) = \sum_{\gamma' \in \Delta'_0} \kappa_{\gamma', E'_{\gamma'}}.
\]

The relation \( [E_{\gamma'}, E_{-\gamma'}] = H_{\gamma'} \) implies that \( \rho(H_{\gamma'}) = \sum |\kappa_{\gamma'}|^2 H_{\gamma'} \), and hence the relation \( [H_{\gamma'}, E_{\gamma'}] = 2E_{\gamma'} \) implies \( |\kappa_{\gamma'}|^2 = 1 \) if \( \kappa_{\gamma'} \neq 0 \). Putting \( S(\gamma') = \{\gamma' \in \Delta_0 : |\kappa_{\gamma'}| = 0 \} \), we have therefore

\[
\rho(H_{\gamma'}) = \sum_{\gamma' \in S(\gamma)} H_{\gamma'}, \quad \rho(E_{\gamma'}) = \sum_{\gamma' \in S(\gamma')} \kappa_{\gamma', E_{\gamma'}} , \quad \kappa_{\gamma'} = \pm 1.
\]

Since \( S(\gamma') \) forms a linearly independent system in \( \sqrt{-1} h' \), we can find an element \( H' \) of \( \sqrt{-1} h' \) such that \( \gamma'(H') = \pi(|\kappa_{\gamma'}| - \kappa_{\gamma'})/2 \) for all \( \gamma' \in S(\gamma') \). Then the inner automorphism \( h = \exp \text{ad}(\sqrt{-1} H') \) of \( g' \) is contained in \( K' \) and keeps
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every element of \( \delta \)' invariant; the homomorphism \( h \circ \rho \) is what we are looking for, q. e. d.

Hence, in the following, we assume that \( \rho \) satisfies also (39) for all \( \gamma \in \Delta_\rho \).

3.3. For the non-compact simple root \( \gamma_1 = \alpha_1 \), we shall denote in the following by \( S(1) \) the subset \( S(\gamma_1) \) of \( \Delta_\rho \) given in Proposition 5. By the equation (35), \( S(\gamma) \) and \( S(\gamma_1) \) are disjoint if \( i \neq j \). On the other hand, every element \( \gamma \) in \( \Delta_\rho \) is transposed to \( \alpha_1 \) by a suitable element of \( W_\rho \), hence it is easy to see that the number of elements of \( S(\gamma) \) is equal to that of \( S(1) \) which will be denoted in the following by \( m_\rho \). The number \( m_\rho \) depends only on the \((k)\)-equivalence class containing \( \rho \). Clearly,

\[
m_\rho \delta' \leq r'.
\]

PROPOSITION 6. i) The positive integer \( m_\rho \) is equal to the coefficient \( \mu_{10} \) in the equation (36).

ii) Put

\[
\delta = \langle H_1, H_1 \rangle, \quad \delta' = \langle H_1', H_1' \rangle.
\]

Then \( c_\rho = m_\rho \delta'/\delta \), or in other words,

\[
(1/\delta')\langle \rho(X), \rho(Y) \rangle' = (m_\rho/\delta)\langle X, Y \rangle \quad (X, Y \in \sqrt{-1}\, \delta).
\]

iii) Let \( H_\rho = \sqrt{-1} \sum a_i H_i \) and \( H_\rho' = \sqrt{-1} \sum a_i' H_i' \) be the expression as in 1.4 of the given complex structures. Suppose \( \rho \) satisfies the stronger condition \( (H_2) \). Then \( m_\rho = a_i'/a_i \).

PROOF. Every positive non-compact root \( \gamma' \) is of the form

\[
\gamma' = \alpha_1 + \sum_{j=2}^s c_j \alpha_j.
\]

If \( \gamma' \) is in \( \Delta_\rho \), it is a longer root; accordingly, we see that

\[
H_{\gamma'} = H_1' + \sum_{j=2}^s d_j H_j'.
\]

Therefore the property i) follows from Proposition 5. The property iii) follows from i) easily. For ii), we see from (35) and (39) that

\[
\langle \rho(H_\rho), \rho(H_\rho) \rangle' = c_\rho \langle H_1, H_1 \rangle
\]

\[
= \sum_{r \in S(1)} \langle H_{\gamma'}, H_{\gamma'} \rangle' = m_\rho \langle H_1', H_1' \rangle',
\]

hence if follows that \( c_\rho = m_\rho \delta'/\delta \), q. e. d.

3.4. PROPOSITION 7. Suppose \( m_\rho \) is equal to 1, and let \( \alpha \) be a longer root of \( \Delta_\rho \). Then there is a root \( \alpha' \) of \( \Delta_\rho \) such that \( \rho(H_\alpha) = H_\alpha' \), \( \rho(E_\alpha) = \kappa_\alpha E_\alpha' \), \( \rho(E_{-\alpha}) = \kappa_\alpha E_{-\alpha}' \), \( |\kappa_\alpha'| = 1 \). The root \( \alpha' \) is compact or positive non-compact according as the root \( \alpha \) is.

PROOF. Put \( \rho(E_\alpha) = \sum \kappa_\alpha E_\alpha' \). Since \([H_\alpha, E_\alpha] = 2E_\alpha\), we have \( \alpha'(\rho(H_\alpha)) = 2 \)
if $\kappa_{\alpha'} \neq 0$, and so
\[
\langle H', \rho(H_{\alpha}) \rangle' = \langle H', H' \rangle'
\]
where $H' = H_{\alpha'}$.

On the other hand, from the Schwartz inequality, it follows that
\[
\langle H', \rho(H_{\alpha}) \rangle'^2 \leq \langle H', H' \rangle' \cdot \langle \rho(H_{\alpha}), \rho(H_{\alpha}) \rangle',
\]
where the equality holds if and only if $\rho(H_{\alpha})$ is a scalar multiple of $H'$. If $m_{\beta} = 1$ and $\alpha$ is a longer root, it follows from (43), (44), and (42) that
\[
\frac{1}{\delta'} \langle H', H' \rangle' = \frac{\langle \alpha'_1, \alpha'_1 \rangle'}{\langle \alpha', \alpha' \rangle'} = 1.
\]
Therefore the equality should be hold. Hence, again by (43), we have $\rho(H_{\alpha}) = H_{\alpha'}$, and so $\rho(E_{\alpha}) = \kappa_{\alpha} E_{\alpha'}$. Now that $\rho$ satisfies (H1), it must also satisfy $\rho(\delta_{\alpha}) \subset \delta_{\alpha'}$, i.e. $\rho(\delta) = \rho(\delta_{\alpha}) (X \in \mathfrak{g})$, where $-$ denotes both of the conjugations of $\mathfrak{g}_C$ and $\mathfrak{g}_C$ w.r.t. $\delta$ and $\delta'$ respectively. Thus, from (38), we see that
\[
\rho(E_{-\alpha}) = \delta_{\alpha} E_{\alpha'}, \quad |\kappa_{\alpha'}| = 1.
\]
The remaining part of the proposition is clear, q.e.d.

**COROLLARY.** If $\mathfrak{g}$ is not of type (III)$_p$ or (IV)$_{2l-1}$ and if $m_{\beta} = 1$, then the image $\rho(\mathfrak{g})$ is a regular subalgebra of $\mathfrak{g}'$. Let $\beta_1$ be the root corresponding to the simple root $\alpha_1$. Then $\rho$ is $(k)$-equivalent to $\rho_0$ defined by
\[
\rho(H_{1}) = H_{\beta_1}, \quad \rho(E_{\pm \alpha_1}) = E_{\pm \beta_1}.
\]
**PROOF.** The first part is a direct consequence from Proposition 7. In view of the proof of the proposition, we may put
\[
\rho(E_{\alpha}) = e^{\theta_i E_{\beta_i}}, \quad 0 \leq \theta_i < 2\pi \ (1 \leq i \leq l).
\]
Since $\beta_1, \ldots, \beta_l$ are linearly independent, there is an element $H'$ of $\sqrt{-1} H'$ such that $\beta(H') = -\theta_1$. Then we have
\[
\rho_0 = \exp \text{ad}(\sqrt{-1} H') \circ \rho, \quad \text{q.e.d.}
\]

3.5. Two homomorphisms $\rho_1$ and $\rho_2$ of a complex semi-simple Lie algebra $\mathfrak{g}_0$ into another $\mathfrak{g}_0$ are said to be $L$-equivalent, if $A \circ \rho_1$ and $A \circ \rho_2$ are equivalent representations of $\mathfrak{g}_0$ for any representation $A$ of $\mathfrak{g}_0$. It is known that, if $\mathfrak{g}_0$ is of type (A), (B), or (C), two $L$-equivalent homomorphisms of $\mathfrak{g}_0$ into $\mathfrak{g}_0$ are equivalent in $\mathfrak{g}_0$ (Dynkin [1], Chap. 1).

**PROPOSITION 8.** Let $\mathfrak{g}'$ be of type (I)$_{p,q}$, (II)$_{p}$, (III)$_{p}$, or (IV)$_{2l-1}$. Let $\rho_1$ and $\rho_2$ be two homomorphisms of a non-compact simple Lie algebra $\mathfrak{g}$ of hermitian type into $\mathfrak{g}'$ satisfying (H2). Then, if $\rho_1$ and $\rho_2$ coincide on the fixed Cartan subalgebra of $\mathfrak{g}$, $\rho_1$ and $\rho_2$ are $(k)$-equivalent.

**PROOF.** We denote the complexifications of $\rho_1$ and $\rho_2$ by the same notations respectively. By the assumption, we have $\rho_1|_{\mathfrak{g}_0} = \rho_2|_{\mathfrak{g}_0}$. Then $\rho_1$ and $\rho_2$ are
L-equivalent homomorphisms of \( g_c \) into \( g' \) ([1], Chap. I, Theorem 1.1), so they are equivalent if \( g'_c \neq (\Pi)_p \). Suppose \( g'_c \neq (\Pi)_p \). Then there is an inner automorphism \( s \in \text{Int}(g'_c) \) such that \( \rho_s = s \circ \rho_s \). Since \( \rho_s(g_u) \subseteq g'_u \) \((i = 1, 2)\), we may assume that \( s \) is in \( \text{Int}(g_u) \) (cf. Lemma 1; \( g_u \) and \( g'_u \) are maximal compact subalgebras of semi-simple Lie algebras \( g_c \) and \( g'_c \) considered over \( \mathbb{R} \)). As \( \rho_1 \) and \( \rho_2 \) satisfy (H2), it follows that \( \rho(H_0) = H'_0 \). In fact,

\[
H'_0 = \rho_s(H_0) = s \circ \rho_s(H_0) = s(H_0).
\]

Therefore it follows from Proposition 2 that \( s(g'_c) = g'_c \); so the restriction of \( s \) to \( g'_c \) gives an inner automorphism of \( g'_c \). Hence it follows from Proposition 1 that \( \rho_1 \) and \( \rho_2 \) are \( (k) \)-equivalent in \( g'_c \). Now, let \( g'_c \) be of type \( (\Pi)_p \). It is known that there is an automorphism \( a \) of \( g'_c \) such as \( \rho_s = a \circ \rho_s \) ([1], Chap. I, § 1, NO. 5). As above, it is easy to see that \( a \) can be taken as an automorphism of \( g'_u \) and that \( a(H_0) = H'_0 \). Therefore \( a \) induces an automorphism on \( g_u \). On the other hand, it is well known that \( a \) mod \( \text{Int}(g_u) \) corresponds to an automorphism \( \omega \) of the Dynkin diagram. Now that \( a(H_0) = H'_0 \), \( \omega \) must keep \( \alpha'_i \) invariant. Hence \( \omega \) is the identity; in fact, an automorphism of the Dynkin diagram of \( g'_c \) is either the identity or that giving the permutation between \( \alpha'_i \) and \( \alpha'_j \). Therefore \( a \) must be inner, q.e.d.

**REMARK.** This proposition is also proved in [4] for the case \( g'_c = (I)_p,q, (II)_p, \) and \( (III)_p \) (p. 439 Lemma 2).

The following proposition seems to hold good more generally, but the author does not know a general proof (for the case \( m_p = 1 \), see the corollary to Proposition 7).

**PROPOSITION 9.** Let \( g'_c \) be of type \( (IV)_p, (EIII), \) or \( (EVII), \) and \( \rho \) a homomorphism of \( g \) into \( g'_c \) satisfying (H2), where \( g \) is a non-compact simple Lie algebra of hermitian type. Then, modifying by \( (k) \)-equivalence, \( \rho \) can be determined by the following equations:

\[
\rho(H_i) = \sum_{\alpha' \in S(i)} H'_{\alpha'}, \quad \rho(E_{\pm\alpha}) = \sum_{\alpha' \in S(i)} E'_{\pm\alpha'}.
\]

where \( S(i) \) is a subset of \( \Delta_u \), \( S(i) \) \((i \geq 2)\) a strongly orthogonal system of compact roots, which is uniquely determined up to \( W_{K'} \)-equivalence for each fixed \( S(i) \).

**PROOF** (Verifications of some properties in it are reserved to § 5). In view of Proposition 5, we can modify \( \rho \) so as to satisfy

\[
\rho(H_i) = \sum_{\alpha' \in S(i)} H'_{\alpha'}, \quad \rho(E_{\pm\alpha}) = \sum_{\alpha' \in S(i)} E'_{\pm\alpha'}.
\]

For a fixed \( S(i) \) in every case, we shall get in § 5 uniquely determined \( W_{K'} \)-class of strongly orthogonal system \( S(i) \) of compact roots for each index \( i \) such that \( \rho(H_i) = \sum_{\alpha' \in S(i)} H'_{\alpha'} \). Put \( \rho(E_{\alpha'}) = \sum_{\alpha' \in S(i)} E'_{\pm\alpha'} \). Then, in view of (38), it is easy to see that \( \rho(E_{-\alpha}) = \sum_{\alpha' \in S(i)} E'_{-\alpha} \). From the equation

\[
\rho(H_i) = \sum_{\alpha' \in S(i)} H'_{\alpha'}, \quad \rho(E_{\pm\alpha}) = \sum_{\alpha' \in S(i)} E'_{\pm\alpha'},
\]
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\[ \alpha'(H) \rho(E_{\alpha}) = [\rho(H), \rho(E_{\alpha})] = \sum \kappa_{\alpha'} \alpha'(\rho(H)) E_{\alpha'}, \]

it follows that, if \( \kappa_{\alpha'} \neq 0, \)

\[ (47) \quad \alpha'(\rho(H)) = 2, \quad \alpha'(\rho(H)) = \alpha(H). \]

We shall see in § 5 for every case that the set of compact roots satisfying (47) coincides with \( S(i). \) Then, from the relation \( [E_{\alpha}, E_{-\alpha}] = H_{\alpha}, \) we have

\[ \rho(E_{\alpha}) = \sum_{\alpha' \in S(i)} \kappa_{\alpha'} E_{\alpha'}, \quad \rho(E_{-\alpha}) = \sum_{\alpha' \in S(i)} \overline{\kappa_{\alpha'}} E_{-\alpha'}, \]

\[ \kappa_{\alpha'} = e^{-i \theta_{\alpha'}}, \quad 0 \leq \theta_{\alpha'} < 2\pi. \]

It will be easily seen (in § 5) that the union of \( S(i) \) (\( 1 \leq i \leq l \)) forms a linearly independent system in \( \sqrt{1} \gamma \). Therefore there is an element \( H' \) of \( \sqrt{1} \gamma \) such as \( \alpha'(H') = -\theta_{\alpha'} \) (\( \alpha' \in S(i), 1 \leq i \leq l \)). The inner automorphism \( h = \exp ad(\sqrt{1} H') \) of \( \gamma' \) is contained in \( K' \); if it is extended linearly to the automorphism of \( \gamma_0 \), it keeps every element of \( \sqrt{1} \gamma \) invariant, and \( h(E_{\alpha'}) = e^{-i \theta_{\alpha'}} E_{\alpha'}. \) Therefore the homomorphism \( h \circ \rho \), instead of \( \rho \), satisfies clearly all the relations in (46), q. e. d.

**THEOREM 3.** Let \( \gamma \) and \( \gamma' \) be non-compact simple Lie algebras of hermitian types. Let \( \rho_1 \) and \( \rho_2 \) be homomorphisms of \( \gamma \) into \( \gamma' \) satisfying (H2) and (15). Let further \( \Theta_i \) be the restriction of \( \rho_i \) onto the fixed Cartan subalgebra \( \gamma \) of \( \gamma \). Then \( \rho_1 \) and \( \rho_2 \) are \( (k) \)-equivalent if and only if there is an element \( k \) of \( N_K(b') \) such as \( \Theta_2 = k \circ \Theta_1. \)

**PROOF.** Firstly, suppose there is \( k \in N_K(b') \) such that \( \Theta_2 = k \circ \Theta_1. \) Then, from Proposition 8 and 9, it follows easily that \( \rho_2 \) and \( k \circ \rho_1 \) are \( (k) \)-equivalent, so that \( \rho_1 \) and \( \rho_2 \) are. Conversely, let \( \rho_1 \) and \( \rho_2 \) be \( (k) \)-equivalent: \( \rho_2 = k \circ \rho_1, \) \( k \in K'. \) Putting \( \gamma = k(\gamma') \), we can find an element \( k_2 \) of \( K' \), by the same way as the proof of Lemma 2, such that the restriction of it to \( \rho_2(\gamma) = k_1 \circ \rho_1(\gamma) \) is the identity and that \( k_2(\gamma) = k' \gamma \). Clearly the element \( k = k_2 k_1 \) of \( K' \) normalizes the Cartan subalgebra \( \gamma' \), and we have \( \Theta_2 = k \circ \Theta_1, \) q. e. d.

§ 4. Determination of regular subalgebras.

In this section, we shall determine all regular subalgebras of each non-compact simple Lie algebra \( \gamma \) of hermitian type. But actually, it is sufficient to find all proper maximal regular subalgebras (i.e. proper subalgebras which are maximal among regular subalgebras) of all simple \( \gamma \), because others are determined automatically. In fact, if a regular subalgebra \( \gamma_i \) of \( \gamma \) is contained in another regular subalgebra \( \gamma_i \) of \( \gamma \), we may consider by definition \( \gamma_i \) to be a regular subalgebra of \( \gamma_i \). Especially, the injection \( \iota_1 \) of \( \gamma_i \) into \( \gamma \) satisfies (H2) if and only if both \( \iota_1 : \gamma_i \rightarrow \gamma \) and \( \iota_2 : \gamma_i \rightarrow \gamma \) satisfy (H2).
Moreover, we have only to determine all $W_K$-equivalence classes of maximal $\Pi$-systems because of Proposition 3 and Theorem 1.

We shall refer freely the following known facts:

i) All positive non-compact roots of a same length are mutually permutable by translations of $W_K$.

ii) The highest root $\gamma$ is a positive non-compact root, and has the maximal length.

iii) Let $A_i$ be a system obtained from the extended fundamental system $\{\Pi, -\gamma\}$ (i.e. the union of the simple root system $\Pi$ and the lowest root $-\gamma$) by omitting a compact simple root $\alpha_i$, and by replacing all elements $\alpha$ in $<-\gamma>$ by $-\alpha$, where $<-\gamma>$ denotes the connected component which contains $-\gamma$. Then, except for the case $g=(l)_p q\Pi$, $A_i$ is a proper maximal $\Pi$-system in our sense (cf. [1], Chap. II, Theorem 5.5).

We shall say that $A_i$ in (iii) is obtained from the simple root system by an elementary transformation. A regular subalgebra $g_i$ of $g$ will be called of class $(H_2)$ if the injection of $g_i$ into $g$ satisfies $(H_2)$.

4.1. The case $g=(l)_p q\Pi$ ($1 \leq p \leq q$). In this case, the extended Dynkin diagram is given as follows:

```
    a_{q+1}  a_i  a_1  a_2  a_q
      \downarrow     \downarrow     \downarrow
       \gamma
```

All positive non-compact roots are in the $W_K$-orbit of $a_i$. If $p \geq 2$, all compact roots are divided into two $W_K$-classes: one is the $W_K$-orbit of $a_{q+1}$, the other of $a_i$. If $p=1$, all compact roots are in the $W_K$-orbit of $a_i$. Let $\beta = \sum_{i=2}^{q} c_i \alpha_i$ be a compact root. If $\beta(H_1)$ is equal to $-1$, it is easily seen that $c_i=1$ or, when $p \geq 2$, $c_i=1$. Hence $\beta$ is really of the form

$$\beta = \alpha_1 + \sum_{i=2}^{t} \alpha_i \quad (t \leq q)$$

or, if $p \geq 2$,

$$\beta = \alpha_0 + \sum_{i=2}^{t} \alpha_i \quad (s \geq q+1).$$

All such compact roots $\beta$ are transposed to $\alpha_s$ or $\alpha_i$ by the subgroup of $W_K$ generated by $\{w_1, \ldots, w_q\}$ or $\{w_{q+1}, \ldots, w_l\}$ respectively, where $w_i$ denotes the reflection $w_{\alpha_i}$ (defined by (27)); this subgroup keeps $\alpha_i$ invariant. If $\beta(H_i) = \beta(H_{i+1}) = \cdots = \beta(H_{q-1}) = 0$ and $\beta(H_q) = -1$, it is easy to see by a similar argument that $\beta$ is transposed to $\alpha_i$ by the
subgroup of $W_K$ generated by $\{w_{i+1}, \ldots, w_q\}$ which keeps $\alpha_1, \alpha_3, \ldots, \alpha_{t-1}$ invariant. Similarly, when $p \geq 2$, a connected Dynkin diagram $\{\beta, \alpha_i, \alpha_{i+1}, \ldots, \alpha_t, \alpha_1\}$ of type (A) with a compact root $\beta$ is equivalent under $W_K$ to $\{\alpha_{s-1}, \alpha_s, \ldots, \alpha_t, \alpha_1\}$. Moreover, we can find an element of $W_K$ which gives the following permutations:

$$
\begin{align*}
\alpha_1 & \mapsto \gamma, \\
\alpha_2 & \mapsto -\alpha_q, \\
\alpha_3 & \mapsto -\alpha_{q+1}.
\end{align*}
$$

Under these considerations, we see that every proper maximal $\Pi$-system is equivalent to one of the following, those obtained by omitting two roots from the above diagram:

- $(I)_{p,q} + (A_{p-r-1}) (1 \leq r < p)$: $\{\alpha_{i-r+1}, \ldots, \alpha_{i-1}, \alpha_i, \alpha_i, \ldots, \alpha_q\}$
  $\cup \{\alpha_{q+1}, \alpha_{q+2}, \ldots, \alpha_{q+r}\}$.

- $(I)_{s,p} + (A_{q-s-1}) (p < s < q)$: $\{\alpha_{s-1}, \alpha_s, \alpha_s, \ldots, \alpha_{q+1}\}$
  $\cup \{\alpha_{q+2}, \alpha_{q+3}, \ldots, \alpha_q\}$.

- $(I)_{s,r} + (A_{q-s-1}) (1 \leq s \leq r < q)$: $\{\alpha_{s-1}, \alpha_s, \alpha_s, \ldots, \alpha_{q+1}\}$
  $\cup \{\alpha_{q+2}, \alpha_{q+3}, \ldots, \alpha_q\}$.

In this table, the following notations are used:

- The elements in a pair of brace $\{\}$ form a connected component of a $\Pi$-system of roots;
- $(A_0)$ denotes the empty set;
- $(I)_{r,s}$ means the regular subalgebra of type $(I)_{r,s}$, but not conjugate to that corresponding to the $\Pi$-system

$$
\{\alpha_{i-s+1}, \ldots, \alpha_{i-1}, \alpha_i, \alpha_s, \alpha_2, \ldots, \alpha_r\}.
$$

This will be denoted also by $(I)_{r,s} (r > s)$.

Using the expressions of the complex structures as described in 14, we can find regular subalgebras of class $(H_2)$ in the above table; we see at once that they must be of a form $g_i = (I)_{r,s} + (I)_{p-r,q-s}$. The complex structures of $(I)_{r,s}$ and $(I)_{p-r,q-s}$ are

$$
\sqrt{-1} \left[ sH_{i-r+1} + 2sH_{i-r+2} + \cdots + rsH_q + r(s-1)H_q + \cdots + rH_s \right],
$$
As \( \mathfrak{g}_1 \) is of class \((H_2)\), we have
\[
\frac{rs}{r+s} + \frac{(p-r)(q-s)}{(p+q)-(r+s)} = \frac{pq}{p+q},
\]
and hence
\[
(p-ar)(q-as) = 0, \quad a = \frac{p+q}{r+s}.
\]
Thus we have
\[
(48) \quad r:s = p:q.
\]
Conversely, it is easy to see that \( \mathfrak{g}_1 \) is of class \((H_2)\), if \((r, s)\) satisfies the relation \((48)\). We have seen that
\[
(I)_{r,s}+(I)_{p-r,q-s} : r : s = p : q
\]
are maximal regular subalgebras of class \((H_2)\) of \((I)_{p,q}\).

Maximal regular subalgebras without compact factors not being of class \((H_2)\) (i.e. non-compact regular subalgebras not of class \((H_2)\) which is maximal w.r.t. this property) are \((k)\)-equivalent to one of the following:
\[
(I)_{p,q-1}, \quad (I)_{p-1,q}, \quad (I)_{r,s}+(I)_{p-r,q-s}(r:s \neq p:q).
\]

4.2. The case \( g = (II)_p \). In this case, the extended Dynkin diagram is given as follows:

![Diagram](attachment:diagram.png)

All positive non-compact roots are in the \( W_\mathcal{K}\)-orbit of \( \alpha_1 \), all compact roots in that of \( \alpha_2 \). By elementary transformations, we get some \( \Pi \)-systems representing \((k)\)-equivalence classes of maximal regular subalgebras. Omitting \(-\gamma\) and \(\alpha_{p-1}\) from the above diagram, we get a \( \Pi \)-system corresponding to \((II)_{p-1}\), which is also proper maximal. If we omit \(\alpha_1\) and \(-\gamma\) from the diagram, we get also a \( \Pi \)-system corresponding to \((A_{p-1})\) which is a maximal compact subalgebra, so we can see at once that there is no compact root \( \alpha \) such that
\[
\alpha(H_1) = \cdots = \alpha(H_{s-1}) = \alpha(H_{s+1}) = 0, \quad \alpha(H_i) = -1.
\]
Hence, by arguments similar to those in 4.1, we get all maximal \( \Pi \)-systems such that positive non-compact roots in them have extreme positions in con-
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netted components of their Dynkin diagrams; it is easy to see that there is no regular subalgebra of type \((\text{III})_r, (\text{IV})_r, (r \geq 7), (\text{EIII}), \text{ or } (\text{EVII})\). The remaining cases to examine are those that some simple factors of regular subalgebras are of type \((\text{I})_r\). Let \(\beta = \sum_{2 \leq i \leq p} c_i \alpha_i\) be a compact root such that

\[
\beta(H_i) = -1, \quad \beta(H_j) = 0, \quad \beta(H_k) = 0,
\]

where \(t\) is a positive integer smaller than \(p-1\). The equations (49) are equivalent to the relation

\[
c_p = c_2 = \ldots = c_{t+1} = 1.
\]

Such a root \(\beta\) is transposed to \(\alpha_2 + \alpha_3 + \ldots + \alpha_p\) by an element of the subgroup of \(W_K\) generated by \(\{w_1, \ldots, w_{p-1}\}\); this subgroup keeps \(\alpha_1, \alpha_2, \ldots, \alpha_t\) invariant. In the rest of this paragraph, we put

\[
\mathbb{R}^{\alpha_2 + \alpha_3 + \ldots + \alpha_p}
\]

Clearly \(4(H_{p-1}) \neq 0\). Let \(\beta_0 = \sum d_i \alpha_i\) be another compact root such that

\[
\beta_0(H_i) = \ldots = \beta_0(H_{p-2}) = 0, \quad \beta_0(H_{p-1}) = -1.
\]

Then we see at once that \(\beta_0 = -\alpha_{p-1}\). By a same manner, we have finally a maximal \(\Pi\)-system for each positive integer \(r \leq \left\lfloor \frac{p}{2} \right\rfloor\):

\[
\{-\alpha_{p-r+2}, \ldots, -\alpha_{p-2} - \alpha_{p-1}, \beta, \alpha_1, \ldots, \alpha_{p-r}\}
\]

which corresponds to a regular subalgebra of type \((\text{I})_{r,r-p-r}\). There is no root orthogonal to it.

Thus we have got all \((k)\)-equivalence classes of maximal regular subalgebras; they are represented by the following \(\Pi\)-systems:

\[(\text{I})_{r,r-p-r} \quad (1 \leq r \leq \left\lfloor \frac{p}{2} \right\rfloor): \quad \{-\alpha_{p-r+2}, \ldots, -\alpha_{p-2} - \alpha_{p-1}, \beta, \alpha_1, \ldots, \alpha_{p-r}\}.
\]

\[(\Pi)_{r} + (\Pi)_{p-r} \quad \left(\left\lfloor \frac{p}{2} \right\rfloor \leq r \leq p-2\right): \quad \left\{\frac{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{r-1}}{\alpha_p}\right\}
\]

\[
\cup \left\{\gamma, -\alpha_{p-2} - \alpha_{p-3} - \ldots - \alpha_{r+1}\right\},
\]

\[(\Pi)_{p-1} : \quad \left\{\frac{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{p-2}}{\alpha_p}\right\},
\]

\[(A_{p-1}) : \quad \left\{\alpha_p, \alpha_2, \alpha_3, \ldots, \alpha_{p-1}\right\},
\]

\[(\Pi)_{p-1} = (\text{I})_{p-1} + (\Pi)_{p-1}, \quad (\Pi)_{p-1} = (\text{I})_{p-1} + (\Pi)_{p-1}, \quad (\Pi)_{p-1} = (\Pi)_{p-1} + (\Pi)_{p-1} + (\Pi)_{p-1}.
\]

Since \(H_r = H_1 + 2(H_2 + \ldots + H_{p-2}) + H_{p-1} + H_p\), we see at once that the regular subalgebras \((\Pi)_{r} + (\Pi)_{p-r} \quad \left(\left\lfloor \frac{p}{2} \right\rfloor \leq r \leq p-2\right)\) are of class \((\Pi)\). Clearly, the regular subalgebra \((\Pi)_{p-1}\) is not of class \((\Pi)\). The complex structure of a regular subalgebra \((\text{I})_{r,r-p-r}\) is
\[
\frac{1}{p} \left[ -(p-r)H_{p-r+2} - 2(p-r)H_{p-r+3} - \cdots + (r-1)(p-r)(H_2 + \cdots + H_p) + r(p-r)H_1 + \cdots + rH_{p-r} \right].
\]

Hence, if \((I)_{r,p-r}\) is of class \((H_2)\), we see that
\[
\frac{r(p-r)}{p} = \frac{p}{4}, \quad \text{so that} \quad p = 2r.
\]

Conversely, if \(p = 2r\), the regular subalgebra \((I)_{r,r}\) is of class \((H_2)\). Thus all \((k)\)-equivalence classes of maximal regular subalgebras of class \((H_2)\) are represented by
\[
(I)_{r,r} \quad \text{(if} \quad p = 2r\text{),} \quad (II)_{r} + (II)_{p-r}.
\]

maximal non-compact regular subalgebras not of class \((H_2)\) are \((k)\)-equivalent to one of
\[
(I)_{r,p-r} \quad \text{(if} \quad p = 2r\text{),} \quad (II)_{r} + (I)_{s,p-r-s} \quad (s \neq \frac{p-r}{2}), \quad (II)_{p-1}.
\]

4.3. The case \( \pi = (III)_p \). The extended Dynkin diagram of \( \varrho_0 \) is
\[
\begin{array}{c}
\alpha_1 \quad \alpha_2 \quad \alpha_p \quad \gamma \\
\end{array}
\]

In this case, there are two sorts of the length of roots; all longer roots are non-compact. Hence there are two \( W_K \)-classes of positive non-compact roots represented by the longer root \( \alpha_1 \) and by the shorter root \( \gamma_1 = \alpha_1 + \alpha_2 \) respectively. The orbit \( W_K(\alpha_1) \) contains \( p \) elements and forms a strongly orthogonal system. On the other hand, all compact roots are mutually permutable by elements of \( W_K \). Every compact root \( \beta \) satisfying \( \alpha_5(H_5) = -2 \) is equivalent to \( \alpha_2 \) under an element of \( W_K \) which preserves \( \alpha_1 \) fixed. Moreover, if a compact root \( \beta \) satisfies
\[
\beta(H_1) = \beta(H_2) = \cdots = \beta(H_{p-1}) = 0, \quad \beta(H_p) = -1,
\]
it is transposed to \( \alpha_{t+i} \) by an element of the subgroup of \( W_K \) generated by \( \{w_{i+2}, \cdots, w_p\} \); this subgroup preserves \( \alpha_1, \alpha_2, \cdots, \alpha_i \) fixed. Thus we have seen that all maximal \( \Pi \)-systems containing longer roots are equivalent under \( W_K \) to one of those given by elementary transformations. Now, let \( \alpha = \sum_{\leq \frac{p}{2}} c_i \alpha_i \) be a compact root such as \( \alpha(H_1) = -1 \), where \( \gamma_1 \) is the positive non-compact root \( \alpha_1 + \alpha_2 \). Since \( H_1 = 2H_1 + H_p \), we can see at once that \( c_1 = 1 \). Hence \( \alpha \) is of the form \( \alpha_3 + \sum_{t \geq \frac{p}{2}} \alpha_t \) or \( \alpha_3 + \sum_{t \geq \frac{p}{2}} \alpha_t \). Thus, by the same considerations as for the case of \( (II)_p \), we have a representative
\[
\{-\alpha_{p-r+2}, \cdots, -\alpha_{p-1}, -\alpha_p, \beta, \gamma_1, \alpha_5, \cdots, \alpha_{p-r+1}\}
\]
of an equivalence class of \( \Pi \)-systems which corresponds to a regular subalgebra.
of type \((I)_{r,p-r}\), where
\[ \beta = \alpha_2 + \alpha_3 + \cdots + \alpha_p. \]

It is easy to see that there is no compact root \(\alpha\) such as
\[ \alpha(H_2) = 0, \quad \alpha(H_3) = \cdots = \alpha(H_{t-1}) = \alpha(H_{t+1}) = 0, \quad \alpha(H_t) = -1. \]

Hence there is no regular subalgebra of type \((II)_{r}, (IV)_{r}, (EIII)\) or \((EVIII)\).

We have seen that all \((k)\)-equivalence classes of maximal regular subalgebras of \(\mathfrak{g}\) are represented by the following:

\[ (I)_{r,p-r} \quad (1 \leq r \leq \left\lfloor \frac{p}{2} \right\rfloor): \quad \{ -\alpha_{p-r+2}, \ldots, -\alpha_{p-1}, -\alpha_p, \beta, \gamma, \alpha_3, \alpha_4, \ldots, \alpha_{p-r+1} \}. \]

\[ (II)_{r}+(III)_{p-r} \quad (\left\lfloor \frac{p}{2} \right\rfloor \leq r \leq p-1): \quad \{ \alpha_1, \ldots, \alpha_r \} \cup \{ \gamma, -\alpha_m, -\alpha_{p-1}, -\alpha_{r+2} \}. \]

\[ (A_{p-1}): \quad \{ \alpha_2, \alpha_3, \ldots, \alpha_p \}. \]

\[ (III)_{p} \quad (r = \left\lfloor \frac{p}{2} \right\rfloor). \]

\[ A_{maximal} \quad regular \ subalgebra \ of \ class \ (H_{s}) \ is \ (k)-equivalent \ to \ one \ of \ \]
\[ (I)_{r,p-r} \quad if \ \ p = 2r, \quad (III)_{r}+(III)_{p-r}; \]

a maximal non-compact regular subalgebra not of class \((H_{s})\) to one of
\[ (I)_{r,p-r} \quad (r \neq \left\lfloor \frac{p}{2} \right\rfloor), \quad (II)_{p-1}, \quad (III)_{r}+(I)_{s,p-r-s} \quad (s \neq \left\lfloor \frac{p-r}{2} \right\rfloor). \]

4.4. The case \(g = (IV)_{p}\).

4.4.1. The case \(p = 2l-2\). The extended Dynkin diagram is as follows:

\[ \begin{array}{c}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_{l-2} & \alpha_{l-1} \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
-\gamma & \alpha_t \\
\end{array} \]

\[ \gamma = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_t. \]

All positive non-compact roots (resp. all compact roots) are in the \(W_{K}\)-orbit of \(\alpha_1\) (resp. \(\alpha_2\)). Let \(\alpha\) be a compact root. If \(\alpha\) satisfies
\[ \alpha(H_1) = \alpha(H_2) = \cdots = \alpha(H_{l-1}) = 0, \quad \alpha(H_l) = -1, \]
where \(t\) is a positive integer smaller than \(l-3\), then \(\alpha\) is transposed to \(\alpha_{l+t}\) by an element \(w\) of the subgroup of \(W_{K}\) generated by \(w_{l+t}, \ldots, w_t\), keeping \(\alpha_1, \ldots, \alpha_t\) invariant. On the other hand, if \(\alpha\) satisfies
\[ \alpha(H_1) = \cdots = \alpha(H_{l-1}) = 0, \quad \alpha(H_{l-1}) = 1, \]
then \(\alpha\) is either \(\alpha_{l-1}\) or \(\alpha_t\); there is no element of \(W_{K}\) that causes a permutation between \(\alpha_{l-1}\) and \(\alpha_t\) keeping \(\alpha_1, \ldots, \alpha_{l-2}\) invariant. The root
\[ \beta_1 = \alpha_2 + 2(\alpha_3 + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_t. \]
is the only one compact root which satisfies
\[ \beta_1(H_1) = -1, \quad \beta_2(H_2) = 0; \]
the root
\[ \beta_2 = \alpha_{l-2} + \alpha_{l-1} + \alpha_l \]
is the unique compact root such as
\[ \beta_2(H_1) = \cdots = \beta_2(H_{l-1}) = \beta_2(H_{l-2}) = 0, \quad \beta_2(H_{l-1}) = -1. \]
Thus we have the following table of \((k)\)-equivalence classes of maximal regular subalgebras:

\[
\begin{align*}
(I)_{1,1}+(D_{l-2}) & : \{ \alpha_1 \} \cup \{ \gamma \} \cup \{ \alpha_{l-2}, \alpha_{l-1} \}, \\
(I)_{1,r}+(D_{l-r}) & : \{ \alpha_1, \cdots, \alpha_r \} \cup \{ \alpha_{r+1}, \cdots, \alpha_{l-2}, \alpha_{l-1} \}, \\
(I)_{l-1} & : \{ \alpha_1, \cdots, \alpha_{l-2}, \alpha_{l-1} \}, \\
(\gamma)_{l-2} & : \{ \alpha_2, \alpha_3, \cdots, \alpha_{l-2}, \alpha_{l-1} \}, \\
(IV)_{p-2} & : \{ \alpha_1, \alpha_2, \cdots, \alpha_{l-2}, \alpha_{l-1} \}, \\
(D_{l-1}) & : \{ \alpha_2, \alpha_3, \cdots, \alpha_{l-2}, \alpha_{l-1} \}.
\end{align*}
\]

Among them, the regular subalgebras
\[ (I)_{1,1}+(D_{l-2}), (I)_{2,2}+(A_{l-6}), (IV)_{p-2} \]
are of class \((H_2)\); a maximal non-compact regular subalgebra not of class \((H_2)\) is \((k)\)-equivalent to one of
\[ (I)_{1,r}, (I)_{l-1}, \quad (D_{l-1}) \cdot \]

4.4.2. The case \( p = 2l - 1 \). In this case, the diagram is

\[
\begin{tikzpicture}
\node (a1) at (0,0) {$\alpha_1$};
\node (a2) at (1,0) {$\alpha_2$};
\node (a3) at (2,0) {$\alpha_{l-1}$};
\node (a4) at (3,0) {$\alpha_l$};
\node (y) at (0,-1) {$-\gamma$};
\draw[->] (a1) -- (a2);
\draw[->] (a2) -- (a3);
\draw[->] (a3) -- (a4);
\end{tikzpicture}
\]

\[ \gamma = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_l). \]

There is only one positive non-compact shorter root \( \gamma_1 \):
\[ \gamma_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_{l-1} + \alpha_l. \]
The root \( r_1 \) is orthogonal to every compact root; hence it is stable under the action of \( W_K \). All the other positive non-compact roots are in the orbit \( W_K(\alpha_i) \), and have the maximal length. All the compact roots are divided into two orbits under \( W_K \); one is \( W_K(\alpha_i) \) and another \( W_K(\alpha_i) \). Putting
\[
\beta_1 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_i), \\
\beta_2 = \alpha_{i-1} + 2\alpha_i, \\
\beta_3 = \alpha_{i-1} + \alpha_1,
\]
we get easily the following table of \((k)\)-equivalence classes of maximal regular subalgebras:

\[
\begin{align*}
(I)_{1,1} + (I)_{1,1} + (B_{l-2}) & : \{\alpha_1\} \cup \{\gamma\} \cup \{\alpha_3, \alpha_4, \cdots, \alpha_i\} , \\
(I)_{1,1} + (B_{l-1}) (2 \leq r \leq l-2) & : \{\alpha_1, \alpha_2, \cdots, \alpha_r\} \cup \{\alpha_{r+2}, \alpha_{r+3}, \cdots, \alpha_i\} , \\
(I)_{1,1} + (B_{l-2}) & : \{\beta_1, \alpha_{i-1}\} \cup \{\alpha_4, \alpha_5, \cdots, \alpha_i\} , \\
(I)_{1,1} + (B_{l-1}) & : \{\gamma\} \cup \{\alpha_2, \alpha_3, \cdots, \alpha_i\} , \\
(IV)_{p-1} & : \{\alpha_1, \alpha_2, \cdots, \alpha_{i-2}, \alpha_{i-1}\} \cup \{\alpha_i\} , \\
(IV)_{p-2} + (A_1) & : \{\alpha_1, \alpha_2, \cdots, \alpha_{i-2}, \beta_3\} \cup \{\alpha_i\} .
\end{align*}
\]

In them, the regular subalgebras
\[
(IV)_{p-1}, (IV)_{p-2} + (A_1)
\]
is of class \((H_2)\). (Observe that \( H_{1} = H_1 + H_7 = 2\sqrt{-1}H_0 \).) Every maximal non-compact regular subalgebra not of class \((H_2)\) are \((k)\)-equivalent to \((I)_{1,1}\).

4.4.3. REMARKS. i) Let \( p = 2l-2 \). There is a monomorphism \( \rho \), unique up to within \((k)\)-equivalence, of \((IV)_{p-1}\) into \((IV)_{p}\) satisfying \((H_2)\) (see \(\S\ 5, \text{ Theorem 4}\)). It will be easily seen that the maximal regular subalgebra \((IV)_{p-2}\) of \((IV)_{p}\) is \((k)\)-equivalent in \((IV)_{p}\) to the image \(\rho((IV)_{p-2})\) of the maximal regular subalgebra \((IV)_{p-2}\) of \((IV)_{p-1}\). Hence the maximal regular subalgebra \((IV)_{p-2}\) of \((IV)_{p}\) is not a maximal subalgebra.

ii) \( p = 2l-1 \). The regular subalgebras \((I)_{1,3} + (B_{l-2})\) and \((I)_{1,3} + (B_{l-3})\) are \((k)\)-equivalent in \((IV)_{p}\) if we consider \((IV)_{p-1}\) as the regular subalgebra of \((IV)_{p}\) corresponding to the \(\Pi\)-system given in the table in 4.4.2. In fact, \(\beta_2 = \alpha_{l-1} + 2\alpha_i\) is transposed to \(\alpha_{l-1}\) by \( w_i \) keeping \(\alpha_1, \alpha_2, \cdots, \alpha_{l-2}\) invariant.

iii) In view of Proposition 5, we can see at once that the regular subalgebra \((I)_{1,3} + (B_{l-1})\) is \((k)\)-conjugate to the diagonal subalgebra (in a natural sense) of the regular subalgebra \((I)_{1,1} + (I)_{1,1}\) corresponding to the \(\Pi\)-system.
\{\alpha_i\} \cup \{\gamma\}. In fact, \(H_{r_1} = H_1 + H_r\).

4.5. The case \(\mathfrak{g} = (EIII)\). The extended Dynkin diagram of \(\mathfrak{g}_0\) is as follows:

\[\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
\odot & & & & \odot \\
\gamma & \odot & \odot & \odot & \odot
\end{array}\]

\[\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6.\]

All positive non-compact roots (resp. all compact roots) are transposed to \(\alpha_1\) (resp. \(\alpha_2\)) by elements of \(W_K\). Putting

\[\beta_1 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \quad \beta_2 = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 ,\]

we have the following table of \((k)\)-equivalence classes of maximal regular subalgebras by the same considerations as above:

- \((I)_{1,5} + (I)_{1,1}\): \(\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \cup \{\gamma\}\); of class \((H_2)\).

- \((I)_{1,5} + (I)_{1,2} + (A_2)\): \(\{\alpha_1, \alpha_2\} \cup \{\gamma, -\alpha_5\} \cup \{\alpha_4, \alpha_6\}\); of class \((H_2)\).

- \((I)_{2,4} + (A_2)\): \(\{\beta_1, \alpha_1, \alpha_2, \alpha_3, \alpha_6\} \cup \{\alpha_5\}\); of class \((H_2)\).

- \((I)_{2,5}\): \(\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\).

- \((IV)_8\): \(\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\).

- \((D_5)\): \(\{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}\).

Maximal non-compact regular subalgebras not of class \((H_2)\) are \((k)\)-equivalent to

\((I)_{1,5}, (I)_{1,4} + (I)_{1,1}, (II)_5, (IV)_8\).

Remark. There are two \((k)\)-equivalence classes of regular subalgebras of type \((I)_{1,4}\); one is

\((I)_{1,4}\): \(\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\), and the other

\((I)'_{1,4}\): \(\{\alpha_1, \alpha_2, \alpha_3, \alpha_6\}\).

Both of them are contained in the regular subalgebra \((IV)_8\).

4.6. The case \(\mathfrak{g} = (EVII)^0\). The extended Dynkin diagram is as follows:

\[\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
\odot & & & & \odot \\
\gamma & \odot & \odot & \odot & \odot
\end{array}\]

\[\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6.\]

2) The table in [3] given for this case was wrong; some \(\Pi\)-systems in it are not maximal, which are omitted here.
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\[ \gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7. \]

By the same considerations as above, we get the following list of \((k)\)-equivalence classes of maximal regular subalgebras; here \(\beta_1, \beta_2, \text{ and } \beta_3\) denote the compact roots as follows:

\[ \beta_1 = \alpha_3 + 2\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \]
\[ \beta_2 = \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \]
\[ \beta_3 = \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7. \]

\((I)_{1,6} + (I)_{1,2} : \{\alpha_4, \alpha_3, \alpha_2, \alpha_5, \alpha_7\} \cup \{\gamma, -\alpha_6\}; \text{ of class } (H_2).\)
\((I)_{1,6} + (I)_{1,8} + (A_1) : \{\alpha_4, \alpha_3, \alpha_2, \alpha_5\} \cup \{\gamma, -\alpha_6, -\alpha_5\}; \text{ of class } (H_2).\)
\((I)_{2,6} : \{\beta_2, \alpha_4, \alpha_3, \alpha_2, \alpha_5, \alpha_6, \alpha_7\}; \text{ of class } (H_2).\)
\((I)_{2,6} + (A_2) : \{\alpha_4 + \alpha_5, \alpha_3, \alpha_2, \alpha_5\} \cup \{\alpha_7, \alpha_6\}; \text{ of class } (H_2).\)
\((II)_{5} + (A_1) : \{\alpha_5, \alpha_2, \alpha_3, \alpha_4, \alpha_7\}; \text{ of class } (H_2).\)
\((IV)_{10} + (I)_{1,11} : \{\alpha_4, \alpha_3, \alpha_2, \alpha_5, \alpha_6\}; \text{ of class } (H_2).\)
\((E_{III}) : \{\alpha_5, \alpha_3, \alpha_2, \alpha_4, \alpha_7\}.\)
\((E_{\alpha}) : \{\alpha_4, \alpha_3, \alpha_2, \alpha_5, \alpha_7\}.\)

Maximal non-compact regular subalgebras not of class \((H_2)\) are \((k)\)-equivalent to one of the following:

\((I)_{1,4} + (I)_{1,2}, (I)_{1,5} + (I)_{1,11}, (I)_{2,8} + (I)_{1,11}, (I)_{2,9}, (IV)_{10}, (E_{III}),\)
where \((I)_{1,5}\) and \((I)_{1,11}\) are not \((k)\)-equivalent to each other:

\((I)_{1,5} : \{\alpha_4, \alpha_3, \alpha_2, \alpha_5, \alpha_7\},\)
\((I)_{1,11} : \{\alpha_4, \alpha_3, \alpha_2, \alpha_5, \alpha_7\}.\)

Both \((I)_{1,5}\) and \((I)_{1,11}\) are contained in the regular subalgebra \((IV)_{10}\) in the above table.
§ 5. Determination of homomorphisms satisfying (H2).

In this section, we determine all pairs \((g, \rho)\) of a simple Lie algebra \(g\) of hermitian type and a monomorphism \(\rho\) of \(g\) into a simple Lie algebra \(g'\) of type (IV)_p, (EIII), and (EVII) satisfying the condition (H2).

5.1. In the first place, we consider generally the case \(g\) is isomorphic to \(g'\). Identifying \(g'\) with \(g\) by the isomorphism, our problem becomes to determine all automorphisms \(\rho\) of \(g\) such as \(\rho(H_g) = H'_g\), where \(H_g\) and \(H'_g\) are complex structures of \((g, t)\) and \((g', t')\) respectively. Modifying \(t'\) by an inner automorphism of \(g\), we may assume that \(\rho(t) = t' = t\); then \(H'_g\) becomes \(H_g\) or \(-H_g\) since \(g\) is simple. Furthermore, we may also assume that \(\rho(\bar{t}) = \bar{t}\), i.e. that \(\rho\) induces an automorphism of \(\mathfrak{h}\). Under these assumptions, it is easy to see the following facts (cf. Corollary to Proposition 7):

i) If \(H_g = H'_g\), \(\rho\) is \((k)\)-equivalent to an automorphism \(a\) of \(g\) caused by an automorphism of the Dynkin diagram of \(g_C\) keeping the non-compact simple root invariant;

ii) If \(H_g = -H'_g\), \(\rho\) is \((k)\)-equivalent to an automorphism \(a\) given by a composition of an automorphism of \(g\), associated with the permutation of roots such as \(\alpha \rightarrow -\alpha\) for any root \(\alpha\), and an automorphism \(a\) got in i) for the case \(H_g = H'_g\).

Hence we have got the following trivial solutions:

A) \(g = g' = (I)_{p,q}, (II)_p, (III)_p, (IV)_p (p = \text{odd}), (EIII), (EVII). \)

\[ \rho \sim \text{id.}, \quad \overline{\text{id.}}. \quad \text{(according as } H'_g \sim H_g \text{ or } -H_g). \]

B) \(g = g' = (I)_{p,q}, (IV)_p (p = \text{even}). \)

\[ \rho \sim \begin{cases} \text{id.}, & \text{if } H'_g \sim H_g, \\ \rho_0, & \text{if } H'_g \sim -H_g. \end{cases} \]

In this table, \(\rho_0\) denotes the outer automorphism caused by the automorphism of the Dynkin diagram different from the identity: in view of Theorem 3, it can be said that \(\rho_0\) is determined (up to \((k)\)-equivalence) by the automorphism \(\Theta\) of \(\mathfrak{h}_C\) such as

\[ \Theta(H_g) = H_g, \quad \Theta(H_{p+1}) = H_{p+1} \quad (0 \leq i \leq p-2) \quad \text{for (I)_p,} \]

\[ \Theta(H_g) = H_g(1 \leq i \leq l-2), \quad \Theta(H_{l-1}) = H_{l-1}, \quad \Theta(H_l) = H_l. \quad \text{for (IV)_p.} \]

In the following, we shall consider only the case \(g\) is not isomorphic to \(g'\).

5.2. The case \(g' = (IV)_p\). Let \((g, \rho)\) be a pair satisfying the above conditions, and \(H_g = \sum_{1 \leq i \leq l} a_i H_i\) the fixed complex structure of \((g, t)\). The complex structure of \((g', t')\) are expressed as
Hence, from Proposition 6 iii), we have $a_m = 1$. Since the number of elements of $\mathcal{D}'_0$ (see 3.2) is 2, $m_\rho$ is equal to 1 or 2. Now that $\rho$ is injective, the dimension of $\mathfrak{h}$ can not be larger than that of $\mathfrak{h}'$, and neither the number of elements of $\mathcal{D}_0$ (i.e. the rank of $(\mathfrak{g}, \mathfrak{f})$) than that of $\mathcal{D}'_0$.

In the first place, we treat the case $\mathfrak{g} = (IV)_p$. For convenience, $p$ is admitted to be 1, 2, 3, or 4, and $p'$ to be 2, 3, or 4 by the following identifications:

\[
(IV)_1 = (I)_{1,1}, \quad (IV)_2 = (I)_{1,1} + (I)_{1,1}, \\
(IV)_3 = (III)_{2,2}, \quad (IV)_4 = (I)_{2,2}
\]

(though $(IV)_2$ is not simple). If $\mathfrak{g} \neq (IV)_1$, we have $a_1 = a'_1 = 1$, and hence $m_\rho = 1$.

For $\mathfrak{g} = (IV)_1$, $a_1 = \frac{1}{2}$, so $m_\rho = 2$.

1. Let $p' = 2l'-2$, $p = p'-1$, and suppose that we have a monomorphism $\rho$ satisfying (H2). Clearly, $l = l'-1$. By Proposition 7 (and Proposition 5 for the case $p' = 2$), we can modify $\rho$ by $(k)$-equivalence so as to make it to satisfy the following relations:

\[
\rho(H_i) = H_i (1 \leq i \leq l'-1 = l'-2), \quad \rho(H_l) = \sum_{j=1}^{l'} \mu_j H'_j,
\]

where $\mu_j$'s are integers and $\mu_1 = 0$ if $l \neq 1$. Proposition 6 ii) implies

\[
\frac{1}{\delta'} \langle H'_i, \rho(H_i) \rangle' = \begin{cases} 
0 & (1 \leq i \leq l'-3) \\
-1 & (i = l'-1) \\
1 & (i = l'-2)
\end{cases},
\]

\[
\frac{1}{\delta'} \langle \rho(H'_i), \rho(H'_l) \rangle' = 2.
\]

Hence we have

\[
\mu_2 = \mu_3 = \cdots = \mu_{l'-2} = 0,
\]

\[
\frac{1}{2} \langle \mu_{l'-1} + \mu_{l'} \rangle' = 1, \quad \mu_{l'-2} + \mu_{l'} = 2,
\]

and so $\rho(H_l) = H_{l'-1} + H_{l'}$. Now we determine all compact roots satisfying the relation (47) in §3. If $l \leq l-1$, it is easy to see by induction that the root $\alpha' = \alpha'_i$ is the unique compact root such as

\[
\alpha' = \alpha'(\rho(H_{l-1})) = 0,
\]

\[
\alpha'(\rho(H_l)) = -1, \quad \alpha'(\rho(H_{l-1})) = 2.
\]

Let $\alpha' = \sum c_j \alpha_j$ be again a compact root satisfying now $\alpha'(\rho(H_i)) = 0$ ($i \leq l-2$). Then we can see at once that $c_j = 0$ ($j \leq l'-2$); hence $\alpha'$ must be equal to $\pm \alpha'_{l'-1}$ or $\pm \alpha'_{l'}$. If $\alpha'$ satisfies moreover
\[ \alpha'(p(H_{l-1})) = -1, \quad \alpha'(p(H_{l-1})) = 2, \]
then we have \( \alpha' = \alpha_{l-1} \) or \( \alpha_l' \). Thus, for the present case, we have proved Proposition 9 completely:

\[ \rho(H) = H', \quad \rho(E_{a_i}) = E'_{a_i}, \quad (1 \leq i \leq l-1 = l'-2), \]
(50)

\[ \rho(H) = H'_{l-1} + H'_l, \quad \rho(E_{a_i}) = E'_{a_{l-1}} + E'_{a_l}, \]
and all the other homomorphisms satisfying \( (H_2) \) are equivalent to \( \rho \). (Observe that \( \rho \) is invariant under the automorphism \( \rho_0 \) of \( g' \) given in 5.1.)

ii) Let \( p' = 2l'-1, \quad p = p'-1 \) (hence \( l = l' \)), and suppose that we have a monomorphism \( \rho \) satisfying \( (H_2) \). From Corollary to Proposition 7, it is easy to see that \( \rho \) can be modified by \((k)\)-equivalence to satisfy

\[ \rho(H) = H', \quad \rho(E_{a_i}) = E'_{a_i}, \quad (1 \leq i \leq l-1 = l'-1), \]
(51)

\[ \rho(H) = H'_{l-1} + H'_l, \quad \rho(E_{a_i}) = E'_{a_{l-1}} + E'_{a_l}, \]

where \( \beta' = \alpha_{l-1} + 2\alpha_l' \) is a compact root of the maximal length. Hence there is one and only one \((k)\)-equivalence class of homomorphisms of \( g = (IV)_{2l'-2} \) into \( g' = (IV)_{2l'-1} \) satisfying \( (H_2) \). The image of \( g \) under the homomorphism \( \rho \) defined by (51) is the regular subalgebra \( (IV)_{p-1} \) of \( (IV)_{p'} \).

Combining the results of i) and ii), it can be easily seen that there is a unique \((k)\)-equivalence class of homomorphism of \( g = (IV)_p \) \((1 \leq p \leq p'-1)\) into \( g' = (IV)_{p'} \) \((p' \geq 2)\) satisfying \( (H_2) \). Particularly, \( g = (I)_{l,l} \) is imbedded in \( g' = (IV)_{p'} \) by the monomorphism defined by

\[ \rho(H) = H'_{l-1} + H'_l, \quad \rho(E_{a_i}) = E'_{a_{l-1}} + E'_{a_l}, \]

where \( \gamma' \) is the highest root of \( g' \); the image \( \rho((I)_{l,l}) \) is the diagonal subalgebra of the regular subalgebra \((I)_{l+1} + (I)_{l,l}\) of \( g' \); if \( p' = 2l'-1 \), there is a homomorphism \( \rho_i \), \((k)\)-equivalent to \( \rho \), of \( (I)_{l,l} \) into \( (IV)_{p'} \) such that the image is the regular subalgebra \((I)_{l,l}\) given in the table in 4.4.2:

\[ \rho_i(H) = H'_{l-1} + H'_l, \quad \rho_i(E_{a_i}) = E'_{a_{l-1}} + E'_{a_l}. \]

Now, let \( g \) be of type \((I)_{p,q} \) \((p \leq q)\), and suppose that we get a \( \rho \). Clearly, \( p \) is at most 2 and \( p+q \leq l'+1 \). By the expression of \( H_\rho \), we have

\[ \frac{p-1}{p+q-1} = 1, \quad \text{i.e.} \quad (m_p - 1)(m_q - 1) = 1. \]

For this equation, we get only two solutions of positive integers:

\[ m_p = 1, \quad p = q = 2; \quad m_q = 2, \quad p = q = 1. \]

But these two cases are already examined above as the cases \( g = (IV)_{4} \) and \( g = (IV)_{1} \) respectively.

In the second place, let \( g \) be of type \((III)_p \) \((p \geq 2)\). Since the rank of \((g, f)\)
is $p$, the only possible case for the existence of $\rho$ is that $p = 2$. But that is already examined above as the case $g = (IV)^p$.

For the remaining cases $g = (II)^p$ ($p \geq 5$), (EIII), and (EVII), it is easy to see from the relation $a_r m_p = 1$ that there is no homomorphism satisfying $(H_2)$.

Thus we have proved the following

**Theorem 4.** Let $(g, \rho)$ be a pair of a simple Lie algebra of hermitian type and a homomorphism into $g' = (IV)^p$ satisfying $(H_2)$ w.r.t. a fixed complex structures of $g$ and $g'$ respectively. Then $g$ is of type $(IV)^p$ ($1 \leq p \leq p' - 1$) and $\rho$ is equivalent to that derived inductively from (50) and (51). For the case $p = 1$, $m_\rho$ is equal to 2, and for others, $m_\rho = 1$.

5.3. The case $g' = (EIII)$. In this case, the rank of $(g', t')$ is equal to 2. Let $(g, \rho)$ be a pair what we are looking for and $H_0 = \sum_{r < \ell} H_1$ the fixed complex structure of $(g, t)$. Then Proposition 6 iii) implies $a_r m_\rho = \frac{4}{3}$, where $m_\rho$ is at most 2. Hence it is easy to see that $g$ can not be other than of type $(I)^p q$. Firstly, let $m_\rho$ be equal to 1. Then the image $\rho(g)$ must be a regular subalgebra of $g'$. On the other hand, we have

$$\frac{pq}{p+q} = \frac{4}{3},$$

and so $(3p - 4)(3q - 4) = 16$.

This equation has a unique solution of positive integers: $p = 2, q = 4$. We have seen in 4.5 that there is certainly a regular subalgebra $(I)^{2,4}$ of class $(H_2)$ represented by the $\Pi$-system \(\{\beta_1, \alpha_1', \alpha_1^{'}, \alpha_6^{'}, \alpha_6\}\), where \(\beta_1 = \alpha_1' + 2\alpha_1^{'}, 2\alpha_1^{'}, \alpha_5^{'}, \alpha_6^{'}, \alpha_6\). Therefore we can see that there is a monomorphism of $(I)^{2,4}$ into $(EIII)$, unique up to $(k)$-equivalence: for instance, a representative $\rho$ is defined by

$$(52)$$

$$\rho(H_1) = H_1, \quad \rho(E_{z\alpha_1}) = E_{z\alpha_1}' (1 \leq i < 5),$$

$$\rho(H_2) = H_2, \quad \rho(E_{z\alpha_6}) = E_{z\alpha_6}' .$$

Secondly, let $m_\rho$ be equal to 2. Then we have

$$\frac{2pq}{p+q} = \frac{4}{3},$$

and so $(3p - 2)(3q - 2) = 4$.

Hence we can see at once that $g$ must be of type $(I)^{2,4}$. By Proposition 5, the homomorphism $\rho$ can be modified by $(k)$-equivalence so as to satisfy

$$\rho(H_1) = H_1 + H_1', \quad \rho(H_2) = \sum_{2 \leq j \leq 6} \mu_j H_j',$$  

where $\gamma'$ denotes the highest root of $g'_0$. By Proposition 6 ii), we have

$$\mu_2 - \mu_6 = 2,$$

$$\mu_2^2 + \cdots + \mu_6^2 - (\mu_2 \mu_3 + \mu_2 \mu_4 + \mu_3 \mu_4 + \mu_4 \mu_5 + \mu_5 \mu_6) = 2.$$

Since $\mu_j$'s are rational integers, it follows easily from (53) that
Thus we have got a homomorphism \( \rho \) defined by

\[
\begin{align*}
\rho(H_1) &= H_1 + H_7, \\
\rho(E_{ \pm a_1}) &= E_{ \pm a_1} + E_{ \pm a_7}, \\
\rho(H_3) &= H_3 - H_6 = H_3 + H_{-a_6}, \\
\rho(E_{ \pm a_2}) &= E_{ \pm a_2} + E_{ \pm a_6}.
\end{align*}
\]

(54)

It can be easily seen that \( \rho \) satisfies (H2). Now, let \( \alpha' = \sum c_j \alpha_j \) be a compact root satisfying

\[
\begin{align*}
\alpha'(\rho(H_1)) &= -1, \\
\alpha'(\rho(H_2)) &= 2.
\end{align*}
\]

(55)

The first condition in (55) implies

\[
\begin{align*}
\alpha'(H_1) &= -1, \\
\alpha'(H_7) &= 0,
\end{align*}
\]

or

\[
\begin{align*}
\alpha'(H_3) &= 0, \\
\alpha'(H_7) &= -1,
\end{align*}
\]

and hence \((c_2, c_7) = (1, 0)\) or \((0, -1)\). Then, from the second condition of (55), it follows that \( \alpha' = \alpha_3 \) or \(-\alpha_3\). Thus, for the present case, we have proved Proposition 9 completely: all homomorphisms of \((I)_{12}\) into \((E_{III})\) satisfying (H2) are equivalent to \( \rho \) defined by (54).

We have proved the following

**Theorem 5.** Let \( g' \) be of type (EIII). Then every pair \((g, \rho)\) of a simple Lie algebra of hermitian type and a homomorphism into \( g' \) satisfying (H2) is equivalent to one of the following:

i) \( g \) is of type \((1)_{34}\), \( \rho \) is determined by (52); 
   \( m_\rho = 1, \rho(g) = \text{the regular subalgebra } (I)_{24} \) of (EIII).

ii) \( g \) is of type \((1)_{12}\), \( \rho \) is determined by (54); 
   \( m_\rho = 2, \rho(g) = \text{the diagonal subalgebra of the regular subalgebra } (D)_{12} + (D)_{12} \).

5.4. The case \( g' = (EVII) \). Since the rank of \((g', V)\) is 3, the integer \( m_\rho \) defined as before is at most 3; the coefficient \( a_i' \) in the expression of \( \frac{1}{\sqrt{-1}} H_0 \) is equal to \( \frac{3}{2} \).

Firstly, let \( g \) be of type \((I)_{p,q}\). We have the following four solutions of the equation \( m_\rho \frac{pq}{p+q} = \frac{3}{2} \):

i) \( m_\rho = 1, p = 2, q = 6. \)

ii) \( m_\rho = 1, p = q = 3. \)

iii) \( m_\rho = 2, p = 1, q = 3. \)

iv) \( m_\rho = 3, p = q = 1. \)

If \( m_\rho = 1 \), it follows from Corollary to Proposition 7 that the image \( \rho(g) \) must be \((k)\)-conjugate to a regular subalgebra of \( g' \) given in the table in 4.6. There are really regular subalgebras \((I)_{2,s} \) and \((I)_{s,3} \) of class \((H_2)\), in it. Hence we
get a homomorphism $\rho$ of $\mathfrak{g}$ into $\mathfrak{g'}$ defined uniquely up to $(k)$-equivalence by

$$\rho(H_i) = H'_i \quad (1 \leq i \leq 6), \quad \rho(H_7) = H'_8 \quad \text{for } \mathfrak{g} = (1)_{2,6},$$

and

$$\rho(H_i) = H'_i \quad (1 \leq i \leq 3), \quad \rho(H_4) = -H'_7, \quad \rho(H_5) = H'_8 \quad \text{for } \mathfrak{g} = (1)_{3,3},$$

where $\beta'$ denotes the compact root $\alpha'_2 + 2\alpha'_4 + 3\alpha'_5 - 2\alpha'_6 - 2\alpha'_7$. The automorphism $\rho_0$ of the regular subalgebra $(1)_{3,3}$ defined in 1.2 is induced by an inner automorphism of $\mathfrak{g}'$ corresponding to an element $w$ of $W_K$, such as

$$w : \alpha'_1 \rightarrow \alpha'_1, \quad \alpha'_2 \rightarrow \beta', \quad \alpha'_3 \rightarrow -\alpha'_1;$$

for instance, $w = w_7(w_0)^2w_2w_3w_4w_5w_4w_3w_2w_0$, where $w_i (i \neq 0)$ denotes the reflection associated with $\alpha'_i$ (cf. (27)) and $w_0 = w_2w_3w_4w_5$. Hence $\rho_0 \circ \rho$ is $(k)$-equivalent to $\rho$. For other homomorphisms satisfying (H2), it is easy to see that they are $(k)$-equivalent to $\rho$. Now, for iv), it follows from Proposition 5 that all homomorphisms of $\mathfrak{g} = (1)_{1,1}$ into $\mathfrak{g'}$ satisfying (H2) are $(k)$-equivalent to $\rho$ determined by

$$\rho(H_i) = H'_i + H'_{\gamma_1} + H'_{\gamma'}; \quad \rho(H_7) = E_{\pm \alpha_1} + E_{\pm \gamma_1} + E_{\pm \gamma'};$$

where $\gamma'$ is the highest root, and $\gamma'_1$ is a positive non-compact root strongly orthogonal to both $\alpha'_1$ and $\gamma'$. The image $\rho(\mathfrak{z})$ is the diagonal subalgebra of the regular subalgebra whose $\Pi$-system is $\mathcal{A}_6 = \{\alpha'_1, \gamma'_1, \gamma'\}$. Now, let $\mathfrak{g}$ be of type $(1)_{1,3}$ (the case iii)), and suppose we get a homomorphism of it into $\mathfrak{g'}$ satisfying (H2). Modifying $\rho$ by $(k)$-equivalence, we may put

$$\rho(H_i) = H'_i, \quad \rho(H_7) = \sum_{j=2}^{7} \mu_j H'_j, \quad \rho(H_8) = \sum_{j=2}^{7} \nu_j H'_j.$$
tion (57) implies \( \mu_2 - \mu_6 = 2 \). Putting \( x_2 - x_6 = 2 \), we get the following 9 integral solutions of (59):

\[
N_1 = (1, 0, 0, 0, -1, 0), \quad N_2 = (1, 1, 0, 0, -1, 0), \quad N_3 = (2, 1, 1, 0, -1, 0), \\
N_4 = (1, 1, 1, 0, -1, 1), \quad N_5 = (1, 0, 0, -1, -1, 0), \quad N_6 = (1, 0, -1, -1, -1, 0), \\
N_7 = (1, 0, -1, -1, -1, 1), \quad N_8 = (1, 1, 0, -1, -1, 0), \quad N_9 = (2, 2, 2, 0, 0, 1).
\]

All \( N_i \) but \( N_9 \) are transposed to \( N_1 \) by elements of \( N_{K'}(y') \) keeping \( H_1 \) invariant. In fact, denoting by \( k_i \) an element of the class in \( N_K(y') \) modulo \( Z_K(y') \) corresponding to the reflection \( w_i \in W_K \), we have the following diagram.

\[
\begin{array}{cccccc}
N_4 & \xrightarrow{k_7} & N_5 & \xrightarrow{k_6} & N_1 \\
N_6 & \xleftarrow{k_3} & N_7 & \xleftarrow{k_4} & N_1 & \xleftarrow{k_5} & N_8
\end{array}
\]

Therefore we may assume that \( \rho(H_2) \) is equal to \( N_1 = H_2 - H_6 \) or \( N_9 \). By (57), we have \( \nu_2 = \nu_6 \). On the other hand, the relation (58) implies that \( \nu_2 - \nu_6 = 2 \) if \( \rho(H_2) = N_9 \), or that \( -2\nu_2 + \nu_6 = 2 \) if \( \rho(H_2) = N_9 \). Under the relations \( x_2 = x_6 \) and \( x_3 - x_5 = 2 \), we have only one integral solution \( (0, 1, 0, -1, 0, 0) \) of (59). On the other hand, there is no solution of (59) satisfying \( x_2 = x_6 \) and \( 2x_2 + x_6 = 2 \). Hence we get a \((k)\)-equivalence class of homomorphisms of \( g = (1)^{(1,3)} \) into \( g' = (EVII) \) satisfying \((H_2)\) represented by that defined by

\[
\begin{align*}
\rho(H_1) &= H_1 + H'_1, \quad \rho(E_{s}a_1) = E'_{s}a_1 + E'_{s}r_1, \\
\rho(H_2) &= H_2 + H'_2, \quad \rho(E_{s}a_2) = E'_{s}a_2 + E'_{s}a'_2, \\
\rho(H_3) &= H_3 + H'_3, \quad \rho(E_{s}a_3) = E'_{s}a_3 + E'_{s}a'_3,
\end{align*}
\]

the sets \( S(1) = \{a', \gamma'\}, S(2) = \{a'_3, -a'_6\}, S(3) = \{a'_5, -a'_6\} \) are strongly orthogonal systems. Let \( \alpha' \) be a compact root satisfying

\[
\alpha'(\rho(H_1)) = -1, \quad \alpha'(\rho(H_2)) = 2.
\]

Then, by the same way as before, we can see that \( \alpha' \) is equal to \( \alpha'_3 \) or \( -\alpha'_6 \) and hence an element of \( S(2) \). Let further \( \beta' = \sum c_{j} \alpha'_j \) be a compact root satisfying

\[
\beta'(\rho(H_1)) = 0, \quad \beta'(\rho(H_2)) = -1, \quad \beta'(\rho(H_3)) = 2.
\]

The first condition in (61) implies

\[
(c_3, c_6) = (0, 0), \quad (1, 1), \quad \text{or} \quad (-1, -1),
\]

and the second is equivalent to say

\[
\beta'(H_3) = -1 \quad \text{(resp. 0)}, \quad \beta'(H_5) = 0 \quad \text{(resp. -1)}.
\]
Hence, if \((c_2, c_6) = (0, 0)\), we see from the last condition that \(\beta'\) is equal to \(\alpha'_5\) or \(-\alpha'_6\), i.e. an element of \(S(3)\); for other cases (62) implies \(2(c_2-c_6) = -1\), which can not happen. Therefore, by Proposition 9, we conclude that there is no other \((k)\)-equivalence class than that represented by \(\rho\). The image \(\rho(g)\) is clearly the diagonal subalgebra of the regular subalgebra \((I)_{5,6}+(I)_{5,6}\) of \(g'\).

For the case \(g\) is of type \((II)_p\) \((p \geq 5)\), it is easy to see that \(p\) must be equal to 6, \(m_p\) to 1. Hence we get a unique \((k)\)-equivalence class represented by a homomorphism \(\rho\) such that the image \(\rho(g)\) is the regular subalgebra \((II)_6\) of \((EVII)\).

Now, let \(g\) be of type \((III)_p\). The possible case is only that \(m_p = 1\) and \(p = 3\). Suppose we get a homomorphism \(\rho\) of \(g = (III)_3\) into \(g' = (EVII)\) satisfying (\(H_2\)). We may put

\[
\rho(H_1) = H', \quad \rho(H_2) = \sum_{j=2}^7 \mu_j H'_j, \quad \rho(H_3) = \sum_{j=2}^7 \nu_j H'_j.
\]

From Proposition 6 ii), we have the following relations:

\[
\begin{align*}
\frac{1}{\delta'} \langle \rho(H_2), \rho(H_2) \rangle' &= \frac{1}{\delta'} \langle \rho(H_2), \rho(H_2) \rangle' = 2, \\
\frac{1}{\delta'} \langle H', \rho(H_2) \rangle' &= \frac{1}{\delta'} \langle \rho(H_2), \rho(H_2) \rangle' = -1, \\
\frac{1}{\delta'} \langle H', \rho(H_2) \rangle' &= 0.
\end{align*}
\]

The relation (63) implies that the rational integers \(\mu_j (2 \leq j \leq 7)\) and \(\nu_j (2 \leq j \leq 7)\) are solutions of (59). Then we can see that \(\rho(H_2)\) should be equal to one of \(N_i\) \((1 \leq i \leq 10)\) defined as follows:

- \(N_1 = (2, 2, 3, 2, 1, 2)\), \(N_2 = (2, 2, 3, 2, 1, 1)\), \(N_3 = (2, 2, 2, 1, 1, 1)\), \(N_4 = (2, 2, 1, 1, 1, 1)\), \(N_5 = (2, 2, 1, 0, 1, 1)\), \(N_6 = (2, 3, 3, 2, 1, 2)\), \(N_7 = (2, 3, 4, 2, 1, 2)\), \(N_8 = (2, 3, 4, 3, 1, 2)\), \(N_9 = (2, 3, 4, 3, 2, 2)\), \(N_{10} = (2, 3, 3, 2, 1, 1)\).

Taking \(k_i \in N_{k_i}(b')\) as above, we get easily the following diagram:

\[
\begin{array}{cccccc}
N_6 & \xrightarrow{k_6} & N_4 & \xrightarrow{k_4} & N_2 & \xrightarrow{k_2} & N_1 \\
& & \downarrow{k_3} & & \downarrow{k_3} & & \\
N_{10} & \xrightarrow{k_{10}} & N_6 & \xrightarrow{k_6} & N_7 & \xrightarrow{k_6} & N_8 & \xrightarrow{k_6} & N_9
\end{array}
\]

Hence in any case, we can modify \(\rho\) by \((k)\)-equivalence keeping \(H'_1\) invariant so as to satisfy
\[\rho(H_2) = N_1 = 2H_4 + 2H_5 + 3H_6 + 2H_7 + H_8 + 2H_9 = H_4 + H_8,\]

where

\[\gamma' = \alpha'_2 + 2\alpha'_3 + 3\alpha'_4 + 2\alpha'_5 + 2\alpha'_6 + 2\alpha'_7\]
is a compact root of \(q' = (\text{EVII})\). Then the relations (64) and (65) with (66) implies

\[\nu_9 = 0, \quad \nu_9 = (\nu_9/2) + (\nu_7/2) = -1.\]

There is only one solution \(H_3 - H_7\) of (59) satisfying (68). Thus we have a homomorphism satisfying \((H_2)\) defined by

\[\rho(H_1) = H_1, \quad \rho(H_2) = H_2 + H_8, \quad \rho(H_3) = H_3 - H_7, \quad \rho(E_{\alpha_1}) = E_{\alpha_1}, \quad \rho(E_{\alpha_2}) = E_{\alpha_2} + E_{\alpha_7}, \quad \rho(E_{\alpha_3}) = E_{\alpha_3} + E_{\alpha_7};\]

the sets \(S(1) = \{\alpha_i\}, S(2) = \{\alpha_i, \beta'\}, S(3) = \{\alpha_i, -\alpha_i\}\) are strongly orthogonal systems. Let \(\alpha' = \sum \epsilon_i \alpha'_i\) be a compact root satisfying

\[\alpha' = -1, \quad \alpha'(H_2) = 2, \quad \alpha'(H_1) = 0, \quad \alpha'(H_3) = 0.\]

Recall that \(\beta'\) given by (67), is contained in the \(\Pi\)-system corresponding to the regular subalgebra \((\Pi)_{13,6}\) of \(q'\) given in the table in 4.6. Then we see that \(\beta'\) is strongly orthogonal to all simple roots but \(\alpha'_1\) and \(\alpha'_5\), and that \(\beta'(H_1) = -1, \beta'(H_2) = 1\). Then it follows from the condition (70) that \(\alpha'\) is equal to \(\alpha'_2\) or \(\beta'\). By the same way, we can easily see that a compact root \(\alpha'\) satisfying

\[\alpha'(H_1) = 0, \quad \alpha'(H_2) = -1, \quad \alpha'(H_3) = 0\]
is an element of \(S(3)\). Therefore, for the present case, we have proved Proposition 9 completely, and so we see that there is a unique equivalence class of homomorphisms of \(q = (\Pi)_{13}\) into \(q'\) satisfying \((H_2)\) whose representative is given by the above \(\rho\). The image of \(q\) by \(\rho\) is clearly contained in the regular subalgebra \((\Pi)_{13,6}\) of \(q'\); we see at once that \(\rho\) is invariant by the automorphism \(\rho_{\alpha}\), defined in 5.1, of the regular subalgebra \((\Pi)_{13,6}\).

There is no solution such that \(q = (\Pi)_{13} (p \geq 5)\) or \((\Pi)_{13,11}\).

Thus we have proved the following

**Theorem 6.** Let \(q'\) be of type \((\text{EVII})\). Then every pair \((q, \rho)\) of a simple Lie algebra of hermitian type and a homomorphism into \(q'\) satisfying \((H_2)\) is equivalent to one of those given as follows:

i) \(q = (\Pi)_{11,6}, \rho(H_1) = H_1 (1 \leq i \leq 6), \rho(H_8) = H_8; \quad m_8 = 1, \quad \rho(q) = \text{the regular subalgebra } (\Pi)_{11,6} \text{ of } q'.\)

ii) \(q = (\Pi)_{11,6}, \rho(H_1) = H_1 (1 \leq i \leq 3), \rho(H_8) = -H_7, \rho(H_9) = H_8; \quad m_8 = 1, \quad \rho(q) = \text{the regular subalgebra } (\Pi)_{11,6} \text{ of } q'.\)
iii) \( g = (I)_{1,2} \), \( \rho \) is determined by (60);
\( m_\rho = 2 \), \( \rho(g) \) is the diagonal subalgebra of the regular subalgebra
\( (I)_{1,2} + (I)_{1,2} \).

iv) \( g = (I)_{1,3}, \rho(H_1) = H_1 + H_{r_1} + H_r \);
\( m_\rho = 3 \), \( \rho(g) \) is the diagonal subalgebra of the regular subalgebra
\( (I)_{1,3} + (I)_{1,3} + (I)_{1,3} \).

v) \( g = (II)_6 \), \( \rho \) is determined naturally;
\( m_\rho = 1 \), \( \rho(g) \) is the regular subalgebra (II) of \( g' \).

vi) \( g = (III)_3 \), \( \rho \) is determined by (69);
\( m_\rho = 1 \), \( \rho(g) \) is the regular subalgebra (III) of \( g \).

Appendix

We give here some numerical results about correspondences of boundary components w.r.t. a holomorphic imbedding of a symmetric domain into another, supplementary to those given by Satake ([5]).

We recall at first the general theory given in [5]. Let \( \varphi \) be a holomorphic imbedding of symmetric domain \( D \) into another \( D' \) such as \( \varphi(o) = o' \), where \( o \) and \( o' \) are fixed origin of \( D \) and \( D' \) respectively. Let further \( g \) (resp. \( g' \)) be the Lie algebra of hermitian type corresponding to \( D \) (resp. \( D' \)), \( \rho \) the homomorphism (and its complexification) of \( g \) into \( g' \) satisfying (H1) which comes from \( \varphi \). Suppose that \( D \) (resp. \( D' \)) is imbedded in \( p\varphi \) (resp. \( p'\varphi \)) by the Harish-Chandra imbedding, where
\[
p_\varphi = \sum_{\alpha \in \Delta_+} g_\alpha \quad (\text{resp. } p'_\varphi = \sum_{\alpha \in \Delta'_+} g'_\alpha).
\]

Then \( \varphi \) is the restriction on \( D \) of the \( C \)-linear map \( \rho : p_\varphi \to p'_\varphi \). For each boundary component \( F \) of \( D \), there is a uniquely determined boundary component \( F' \) of \( D' \) in which \( \rho(F) \) is contained. If \( \rho \) is injective and \( F \) is proper boundary component (i.e. not \( D \) itself), \( F' \) is also proper. We may consider only the case both \( D \) and \( D' \) are irreducible; hence we assume so in the following. Let \( \Delta_0 = \{ \gamma_k \}_{k \in R} \) be a maximal strongly orthogonal system of positive non-compact roots of \( g_0 \) ordered by \( R = \{ 1, \ldots, r \} \), where \( r \) is the rank of \( (g, I) \). For a number \( i \) in \( R \), \( F_i \) denotes the \( i \)-th boundary component of \( D \); that is defined as the boundary component of \( D \) containing the image under the Hermann map \( \kappa \) of the boundary component
\[
\{ 1 \} \times \cdots \times \{ 1 \} \times U \times \cdots \times U
\]
\( i \)-copies \( (r-i) \)-copies
of \( U^r \), where \( U \) is the unit disc (symmetric domain of type (I)_{1,1}) and hence \( \{ 1 \} \) is a boundary component of \( U \). Every boundary component of \( D \) is equivalent under \( K \) to one of \( F_i \)'s. Similarly \( \Delta_0', \gamma', r', R', F'_j \) are defined for
D'. Modifying \( \rho \) by \((k)\)-equivalence, one can find \( j \in R' \) for each \( i \in R \) such that
\[
(71) \quad \rho(F_i) \subseteq F_j.
\]
The number \( j \) is really a multiple of \( i \):
\[
(72) \quad j = mi.
\]
Thus the correspondence of boundary components is considered to be determined by the multiplier \( m \) in (72).

We may take the non-compact simple root \( \alpha_i \) for \( \gamma_1 \). Then it can be seen that the properties (71) and (72) are equivalent to saying (modifying \( \rho \) by \((k)\)-equivalence if necessary) that 
\[
\rho(H_i) = \sum_{1 \leq s \leq m} H'_{r_s},
\]
Therefore we see at once that the number \( m_\rho \) defined in 3.3 is equal to \( m \) of (72).

Let \( \mathfrak{g} \) be a simple regular subalgebra of \( \mathfrak{g}' \) corresponding to a \( \Pi \)-system \( A' \). The injection of \( \mathfrak{g} \) into \( \mathfrak{g}' \) is denoted by \( \iota \). Let further \( \alpha \) be the positive non-compact root in \( A' \). Then, by Proposition 6 ii), we have
\[
m_i = \frac{1}{\delta} \langle H'_\alpha, H'_\alpha \rangle' = \frac{-\langle \alpha'_i, \alpha'_i \rangle'}{\langle \alpha, \alpha \rangle'}.
\]
Hence \( m \) is equal to either 1 or 2 according as \( \alpha \) is longer or shorter root of \( \mathfrak{g}' \).

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References