Prolongations of tensor fields and connections to tangent bundles III

—Holonomy groups—

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(Received Feb. 24, 1967)

1. Introduction

In our previous paper [3] we introduced the notion of complete lift of an affine connection. Let $M$ be a manifold $T(M)$ its tangent bundle space. Then every affine connection $\nabla$ of $M$ induces in a natural manner an affine connection, called the complete lift $\nabla^c$ of $\nabla$, of the manifold $T(M)$. We shall show in this paper that the linear holonomy group $\Phi(\nabla^c)$ of the connection $\nabla^c$ coincides with the tangent group $T(\Phi(\nabla))$ of the linear holonomy group $\Phi(\nabla)$ of the connection $\nabla$, i.e.,

$$\Phi(\nabla^c) = T(\Phi(\nabla)).$$

This confirms one of the conjectures we stated at the end of [3].

2. Tangent connection

Let $P$ be a principal fibre bundle over a manifold $M$ with Lie structure group $G$ and projection $\pi$. Then $T(P)$ is a principal fibre bundle over $T(M)$ with group $T(G)$ and projection $\pi_*$, where $\pi_*$ denotes the differential of $\pi$, (see [1]). (Perhaps the notation $T(\pi)$ instead of $\pi_*$ would make the whole thing more functorial.) One of the present authors has shown that every connection $\nabla$ in $P$ induces in a natural manner a connection, called the connection tangent to $\nabla$ and denoted by $T(\nabla)$, in the bundle $T(P)$.

We apply these constructions to a subbundle $P$ of the bundle $L(M)$ of linear frames, i.e., a $G$-structure $P$ on $M$. The tangent group $T(G)$ is a semi-direct product of $G$ with its Lie algebra $\mathfrak{g}$. If we represent an element of $G$ by a matrix $X \in GL(n; R)$, then we may represent also an element of $T(G)$ by a matrix of the form

$$\begin{pmatrix} X & 0 \\ X_\xi & X \end{pmatrix} \in GL(2n; R),$$

*) Supported partially by NSF Grant GP-5798.
where $\xi$ is an element of $\mathfrak{gl}(n; R)$. In this way we may consider $T(G)$ as a subgroup of $GL(2n; R)$. In a natural manner we may consider also the bundle $T(P)$ as a $T(G)$-structure on the manifold $T(M)$.

Let $\mathcal{F}$ be a connection in $P$. We view it as an affine connection of $M$. Similarly, we consider the tangent connection $T(\mathcal{F})$ in the bundle $T(P)$ as an affine connection of the manifold $T(M)$. We assert

$$T(\mathcal{F}) = \mathcal{F}^e.$$ 

The verification of this fact is straightforward; see the last formula of § 4 and the last formula of § 6 of Chapter IV in [1].

3. Holonomy theorem

In general, let $\mathcal{F}$ be a connection in a principal fibre bundle $P$ over $M$ with group $G$ and let $\Phi(\mathcal{F})$ be its holonomy group. Then the holonomy group $\Phi(T(\mathcal{F}))$ of the connection $T(\mathcal{F})$ in $T(P)$ coincides with $T(\Phi(\mathcal{F}))$, i.e.,

$$\Phi(T(\mathcal{F})) = T(\Phi(\mathcal{F})).$$

This fact was proved in [1] and is essentially equivalent to the so-called holonomy theorem of Ambrose-Singer.

This fact together with the assertion made in § 2 establishes the theorem;

$$\Phi(\mathcal{F}^e) = T(\Phi(\mathcal{F})).$$

4. Concluding remarks

It is probably possible to prove the equality $\Phi(\mathcal{F}^e) = T(\Phi(\mathcal{F}))$ more directly (i.e., without the use of $T(\mathcal{F})$ and equality $T(\mathcal{F}) = \mathcal{F}^e$) in the frame work of our previous paper [3]. In this respect, the paper of Nijenhuis [2] could be useful. As a matter of fact, in the case of real analytic affine connection, results of Nijenhuis in [2] together with our results in [3] give a simple proof of the theorem above. But it would be more important to find a better definition of $T(\mathcal{F})$ (a definition as simple as that of $\mathcal{F}^e$) which yields a simple proof of $T(\mathcal{F}) = \mathcal{F}^e$.

Finally, the equality $T(\mathcal{F}) = \mathcal{F}^e$ implies immediately that, if $\Phi(\mathcal{F})$ consists of matrices of the form

$$\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} \in GL(n; R),$$

then $\Phi(\mathcal{F}^e)$ consists of matrices of the form
In particular, the existence of a parallel distribution (i.e., parallel field of tangent subspaces) on $M$ implies the existence of certain parallel distributions on $T(M)$. This fact, of course, can be shown more directly in the frame work of [3].

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Bibliography