Some results on $l$-extensions of algebraic number fields

Dedicated to Professor Iyanaga on his 60th birthday

By Sige-Nobu KURODA

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Let $l$ denote a prime number which we fix throughout the present paper. Let $F_0$ be an algebraic number field of finite degree, and let $F/F_0$ be a $l$-extension over $F_0$. Namely $F/F_0$ is a Galois extension whose Galois group is isomorphic to the additive group of $l$-adic integers. In the following we shall consider a $l$-module $A(K/F)$, attached to $F/F_0$, which will be defined analogously to the cyclotomic case considered by Iwasawa [8]. After the preliminaries in § 1 we shall give in § 2 a necessary and sufficient condition for the regularity of $A(K/F)$ (as $l$-module) in terms of characters of idèle groups of intermediate fields of $F$ and $F_0$ (Theorems 1 and 2). The $l$-module $A(K/F)$ is intimately related to $l$-adic behaviour of global unit groups of algebraic number fields (Theorem 3 in § 2).

Now let in particular the ground field $F_0$ be an imaginary quadratic extension of the rational number field. In such a case there exist, in a fixed algebraic closure of $F_0$, two independent $l$-extensions over $F_0$ (with respect to our fixed prime number $l$). Under additional conditions on $F_0$ the regularity of $A(K/F)$ will be obtained in § 3 (Theorem 4 in § 3).

General notations. We denote by $l$ a prime number which we fix throughout the present paper. $\mathbb{Z}$ and $\mathbb{Q}$ stand for the ring of rational integers and the rational number field, respectively. We denote by $\mathbb{Z}_l$ and $\mathbb{Q}_l$ the ring of $l$-adic integers and the $l$-adic completion of $\mathbb{Q}$, respectively. $\mathbb{Z}/(d)\mathbb{Z}$ means the additive group of integers modulo $d$, where $d \in \mathbb{Z}$.

§ 1. Preliminaries.

1.1.2) Now let in general $E$ be a field and $K/E$ a Galois extension. Then the Galois group of $K/E$ equipped with the Krull topology will be denoted by $G(K/E)$. Let $F$ be an intermediate field of $K$ and $E$ which is also a Galois extension over $E$. Then the Galois group $G(F/E)$ is canonically isomorphic to $G(K/E)$.

1) We reserve the notations $p$, $p$, etc. for general prime numbers or prime divisors.
2) Cf. Iwasawa [8], § 1. The purpose of the descriptions in § 1.1 and § 1.2 is to introduce notations.
the factor group $G(K/E)/G(K/F)$, $G(K/F)$ being of course a closed normal subgroup of $G(K/E)$. Furthermore if $K/F$ is abelian then every inner automorphism $x \mapsto s^{-1}xs$ of $G(K/E)$ induces a topological automorphism of $G(K/F)$ which depends only upon the coset $\sigma$ of $s \mod G(K/F)$. $G(K/F)$ is thus made into a $G(F/E)$-group on which $G(F/E)$ acts unitarily (i.e. $1 \cdot x = x$) and continuously. The discrete character group of the compact abelian group $G(K/F)$ will be denoted by $A(K/F)$. The action of $G(F/E)$ on $A(K/F)$ is defined by setting

$$a^\sigma(x) = a(x^\sigma) = a(s^{-1}xs), \quad x \in G(K/F),$$

where $a \in A(K/F)$ and $s$ is any element of $G(F/E)$ such that $\sigma$ is the coset of $s \mod G(K/F)$. We notice that $G(F/E)$ acts on $G(K/F)$ from the right, and $G(F/E)$ acts on $A(K/F)$ from the left: e.g. $(a^\sigma)^{\tau} = a^\tau$.

Let $a$ be a character of $A(K/F)$, and let $K_a$ denote provisionally the fixed field of the kernel of $a$. If $G(F/E)$ is cyclic or contains a dense cyclic subgroup, then $K_a/E$ is abelian if and only if $a^\sigma = a$ for every $\sigma$ in $G(F/E)$.

1.2. Let $F$ denote a multiplicative topological group isomorphic to the additive group of $\mathbb{Z}_l$. For an integer $m \geq 0$ we denote by $\Gamma_m$ the unique open subgroup with index $l^m$ in $\Gamma$. Let $F_0$ be a finite algebraic extension over $\mathbb{Q}$, and let $F/F_0$ be a $\Gamma$-extension. Namely $F/F_0$ is a Galois extension whose Galois group is isomorphic to $\Gamma^{\infty}$. We identify the Galois group $G(F/F_0)$ with $\Gamma$, and we denote by $F_m$ the fixed field of $\Gamma_m$. $F$ is the union of the increasing sequence of all $F_m$ ($m \geq 0$). We denote by $S$ the set of all prime divisors of $F$ which divide the rational prime divisor $l$. If $K'$ and $K''$ are two algebraic extensions of $F$ in which no prime divisor of $F$ outside $S$ is ramified, then the same holds good for the composite field $K' \cdot K''$. Thus there exists the unique maximal $l$-primary abelian extension $K$ over $F$ in which no prime divisor of $F$ outside $S$ is ramified. Furthermore if $I$ is an element in $S$ then the prime divisors conjugate to $I$ with respect to $F/F_0$ are also contained in $S$. Thus $K/F_0$ is a Galois extension, and we are in the situation described in § 1.1 with $E = F_0$. In particular the Galois group $\Gamma$ of $F/F_0$ acts on $A(K/F)$, the dual of the Galois group $G(K/F)$, as described in § 1.1.

A discrete group $A$ is said to be $l$-primary if $A$ is the direct limit of a family of finite $l$-groups, and a compact group $G$ is said to be $l$-primary if $G$ is the inverse limit of a family of finite $l$-groups. A discrete $l$-primary (additive) abelian group $A$ is said to be a discrete $\Gamma$-module if $\Gamma$ acts on $A$ unitarily and continuously. Similarly a compact $l$-primary (additive) abelian group

3) Cf. Iwasawa [6], where the fixed prime number is denoted by $p$.
4) 'l-primary abelian' means here that the Galois group $G(K/F)$ is an inverse limit of a family of finite $l$-abelian groups. Cf. also Remark at the end of § 2.3.
G is said to be a compact \( \Gamma \)-module if \( \Gamma \) acts on \( G \) unitarily and continuously\(^5\).

Thus \( G(K/F) \) and \( A(K/F) \) mentioned above are compact and discrete \( \Gamma \)-modules, respectively.

1.3. Let in general \( A \) be a discrete \( \Gamma \)-module. Then, as usual, we denote by \( A_m \) the submodule of \( A \) which consists of all the elements \( a \) in \( A \) such that \( \sigma a = a \) for every \( \sigma \in \Gamma_m \). Let \( m \) and \( n \) be integers such that \( m \geq n \geq 0 \). Then \( A_m \) is naturally made into a \( \Gamma_n/\Gamma_m \)-module. It is known by Iwasawa \([7]\) that a discrete \( \Gamma \)-module \( A \) is regular (as \( \Gamma \)-module) if and only if we have

\[
H^i(\Gamma_n/\Gamma_m, A_m) \cong (0), \quad i = 1, 2,
\]

for every \( m \geq n \geq 0 \). It is also known that \( A \) is regular if only we have

\[
H^i(\Gamma_n/\Gamma_m, A_m) \cong (0), \quad m \geq n,
\]

whenever both \( m \) and \( n \) are sufficiently large.

We shall make use of the following

**Lemma 1.** Let \( A \) be a discrete \( \Gamma \)-module. Then \( A \) is regular if we have

\[
H^i(\Gamma/\Gamma_m, A_m) \cong (0), \quad i = 1, 2,
\]

for every sufficiently large integer \( m \).

**Proof.** We choose \( m \) so large that the assumption in Lemma 1 is satisfied. Since the order of \( \Gamma/\Gamma_m \) is a power of a prime number \( l \), and since the cohomology groups of \( \Gamma/\Gamma_m \) in \( A_m \) vanish for two consecutive dimensions \( i = 1, 2 \), we get, by Nakayama's theorem on cohomological triviality\(^6\), \( H^i(G, A_m) \cong (0) \) for every dimension \( i \) and for every subgroup \( G \) of \( \Gamma/\Gamma_m \). Thus all the cohomology groups of \( \Gamma_n/\Gamma_m \) in \( A_m \) vanish for every \( m \) and \( n \) such that \( m \geq n \geq 0 \) and that \( m \) is sufficiently large, which together with the above referred facts proves Lemma 1.

1.4. Remark. For an integer \( m (\geq 0) \) let \( \zeta_m \) denote a primitive \( l^m \)-th root of unity, and let \( F_m = \mathbb{Q}(\zeta_{m+1}) \) for \( m \geq 0 \). Let \( F \) denote the union of the increasing sequence of all \( F_m (m \geq 0) \). If \( l \) is an odd prime number then \( F/F_0 \) is a \( \Gamma \)-extension. In such a case the \( \Gamma \)-module \( A(K/F) \) has already been considered by Iwasawa \([8]\), and it is known that \( A(K/F) \) is regular if and only if the group of principal units\(^7\) of the local cyclotomic field \( \Phi_m = \mathbb{Q}(\zeta_m + i) \) contains \( l^m(l-1)/2-1 \) global units in \( F_m \) which are independent over \( \mathbb{Z} \) for every \( m \geq 0 \). The regularity of \( A(K/F) \) is known to be the case when the class

\(^5\) The structure theorems on \( \Gamma \)-modules are given by Iwasawa \([6]\) and \([7]\). For the definition of the regularity of \( \Gamma \)-modules, see \([6]\), p. 187.

\(^6\) Cf. e.g. Serre \([13]\), p. 152.

\(^7\) A local unit \( u \) in \( \Phi_m \) is said to be principal if \( u = 1 (1_m) \), where \( 1_m \) stands for the valuation ideal of \( \Phi_m \).
number of $F_o$ is prime to $l^{10}$.

§ 2. Formulation in terms of characters of idèles.

2.1. Let $F/F_o$ be, as in § 1.2, a $\Gamma$-extension over an algebraic number field $F_o$ of finite degree, and let $A(K/F)$, $F_m$ etc. be as in § 1.2. In this section we give a necessary and sufficient condition for the regularity of $A(K/F)$ in terms of characters of idèle group of $F_m$. We denote the idèle group and the principal idèle group of $F_m$ by $I_m$ and $P_m$, respectively. Let $C_m = I_m/P_m$, and let $D_m$ denote the connected component of the identity of the idèle class group $C_m$ of $F_m$. We denote by $\mathcal{D}_m$ the group of all continuous characters of $I_m$ with finite orders which are trivial (i.e. take the value 1) on $P_m$. Then $\mathcal{D}_m$ may be naturally regarded as the dual of the compact abelian group $C_m/D_m$. Now let $\mathfrak{B}$ be a prime divisor of $F_m$, and let $\chi$ be an element in $\mathcal{D}_m$. Then the local component $\chi_{\mathfrak{B}}$ of $\chi$ at $\mathfrak{B}$ is defined by means of the local component of idèles. $\chi$ is said to be unramified at $\mathfrak{B}$ if $\chi_{\mathfrak{B}}$ is trivial on the unit group of the $\mathfrak{B}$-completion of $F_m$, and, if otherwise, said to be ramified at $\mathfrak{B}$. $\chi$ is ramified at $\mathfrak{B}$ if and only if $\mathfrak{B}$ is ramified by the cyclic extension over $F_m$ with which $\chi$ is associated in the sense of class field theory$^{10}$.

We define the action of the Galois group $\Gamma_n/\Gamma_m$ of $F_m/F_n$ on $\mathcal{D}_m$ by setting

$$\chi^\sigma(\bar{a}) = \chi(\bar{a}^\sigma), \quad \sigma \in \Gamma_n/\Gamma_m,$$

where $\bar{a} \in I_m$ and $\bar{a}^\sigma$ is an idèle conjugate to $\bar{a}$ by $\sigma$.$^{11}$

Let $\mathcal{A}_m$ denote the subgroup of $\mathcal{D}_m$ which consists of all the elements in $\mathcal{D}_m$ whose orders are powers of $l$. We denote by $S_m$ the set of all prime divisors of $F_m$ which divide the rational prime divisor $l$. Now we define two subgroups of $\mathcal{A}_m$ by setting

$$\mathcal{A}_m = \{ \chi \in \mathcal{A}_m | \chi \text{ is unramified at every prime divisor of } F_m \text{ outside } S_m \},$$

and

$$\mathcal{A}_m^e = \{ \chi \in \mathcal{A}_m | \exists m' \geq m : \ker \chi = P_m \cdot N'(I_m) \},$$

where $N'$ stands here for the norm mapping from $I_m$ to $I_m$. Namely elements of $\mathcal{A}_m^e$ are associated with sub-extensions of $F/F_m$ in the sense of class field

8) Cf. also Ax [2], Iwasawa and Sims [9], Jehne [10], where other results on this subject are found.

9) In the present paper no sign condition is imposed on real infinite components of idèles.

10) For the class field theory used in the present paper without references, see Chevalley [3], Weil [14], § 1 and Whaples [15], Theorem 3. As is well-known, the differentials of $F_m$ defined by Chevalley loc. cit. are nothing but elements of $\mathcal{D}_m$.

11) The Galois group acts on idèles from the right.
theory, and therefore $\mathcal{N}_m^L$ is isomorphic to the dual of the Galois group $G(F/F_m)$. Now, since no prime divisors of $F_m$ outside $S_m$ are ramified by $F/F_m$, we have $\mathcal{N}_m^L \subset \mathcal{N}_m^\prime$. It is easy to observe that $\mathcal{N}_m^L$ and $\mathcal{N}_m^\prime$ are $A_n/F_m$-subgroups of $A_m$.

2.2. Let $M$ denote the maximal abelian extension over $F_m$. For a while we put $g = G(M/F_m)$ and $A = A(M/F_m)$, the dual of $g$. By class field theory the dual of $C_m/D_m$ is canonically isomorphic to $A$. Namely $D_m$ is canonically isomorphic to $A$. Let $m$ and $n$ be integers such that $m \geq n \geq 0$. Then, by our convention, the Galois group of $F_m/F_n$ is $\Gamma_n/\Gamma_m$. The above mentioned canonical isomorphism gives a canonical $\Gamma_n/\Gamma_m$-isomorphism of $D_m$ and $A$ by (2) and by a well-known property of the reciprocity map\(^{12}\).

Let $B$ be a $\Gamma_n/\Gamma_m$-subgroup of $D_m$, and let $B$ denote the canonical image of $B$ in $A$. We denote by $\Phi(\mathfrak{g}, B)$ the annihilator of $B$ in $\mathfrak{g}$, which is a $\Gamma_n/\Gamma_m$-subgroup of $g$. Then $B$ is the dual of $g/\Phi(\mathfrak{g}, B)$, and thus $B$ is canonically $\Gamma_n/\Gamma_m$-isomorphic to the dual of $g/\Phi(\mathfrak{g}, B)$.

2.3. Now let $K_m$ denote the unique maximal $l$-primary abelian extension over $F_m$ in which no prime divisor of $F_m$ outside $S_m$ is ramified. Then, putting $\mathfrak{g} = \mathfrak{g}_m$, we have, by § 2.2,

$$A(K_m/F_m) \cong \mathcal{N}_m^L, \quad (\Gamma_n/\Gamma_m \text{-isomorphism}),$$

$$A(F_m/F) \cong \mathcal{N}_m^L, \quad (\Gamma_n/\Gamma_m \text{-isomorphism}).$$

These isomorphisms may be regarded as $\Gamma$-isomorphisms in which the action of $\Gamma_m$ is trivial. On the other hand we have a canonical $\Gamma$-isomorphism $A(K_m/F) \cong A(K_m/F_m)/A(F/F_m)$. Hence we have a canonical $\Gamma$-isomorphism $A(K_m/F) \cong \mathcal{N}_m^L/\mathcal{N}_m^\prime$. Since, by the remark at the end of § 1.1, we have $A(K_m/F) = A(K/F)_L$, the following lemma is obtained.

**Lemma 2.** $A(K/F)_L$ is $\Gamma_n/\Gamma_m$-isomorphic to $\mathcal{N}_m^L/\mathcal{N}_m^\prime$ for every $m \geq n \geq 0$.

**Remark.** 1. The meaning of the suffix $m$ of $A(K/F)$ is described at the beginning of § 1.3.

2. As far as the extension $K/F$ is concerned, only Lemma 2 and the formula (9) in § 2.7 will be necessary for our later argument. Thus we may rather define $K$ as the union of the increasing sequence of all $K_m$ ($m \geq 0$).

2.4. The following Proposition will be proved in the next § 2.5.

**Proposition.** We have

$$H^1(\Gamma_n/\Gamma_m, \mathcal{N}_m^L/\mathcal{N}_m^\prime) \cong (0)$$

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13) Cf. e.g. the last formula in Chap. XI, 3 of Serre [13], in which the Galois group $G$ acts on the $G$-modules (in ‘class formation’) from the left, contrary to our convention.
if and only if the natural homomorphism
\[ H^i(I^n/I_m, \mathcal{A}_{m}) \to H^i(I^n/I_m, \mathcal{A}_m) \]
is injective.

Combining Lemmas 1, 2 and Proposition, we get immediately the following

**Theorem 1.** Let \( F/F_0 \) be a \( I^- \)-extension over an algebraic number field \( F_0 \) of finite degree. Then the \( I^- \)-module \( \mathcal{A}(K/F) \) is regular if and only if the natural homomorphisms
\[ H^i(I^n/I_m, \mathcal{A}_m) \to H^i(I^n/I_m, \mathcal{A}_m), \quad i = 1, 2, \]
are both injective for every sufficiently large integer \( m \), where \( I^n/I_m \) stands for the Galois group of \( F_m/F_0 \).

**2.5.** For the proof of Proposition we prepare some lemmas.

**Lemma 3.** We have \( H^1(I^n/I_m, \mathcal{A}_m) \cong \mathbb{Z}/(l^{m-n})\mathbb{Z} \) and \( H^2(I^n/I_m, \mathcal{A}_m) \cong (0) \) for every \( m \geq n \geq 0 \).

**Proof.** Let \( D_m \) denote the connected component of the identity in the idèle class group \( C_m \) of \( F_m \). Then we have
\[ H^1(I^n/I_m, C_m) = (0), \quad H^2(I^n/I_m, C_m) \cong \mathbb{Z}/(l^{m-n})\mathbb{Z}, \]
the first three of which are of general character, and we have the last, because no infinite prime divisor of \( F_n \) is ramified by \( F_m/F_n \). From the exact sequence
(1) \( \to D_m \to C_m \to C_m/D_m \to (1) \), we get the exact sequence
\[ H^1(I^n/I_m, D_m) \to H^1(I^n/I_m, C_m) \to H^1(I^n/I_m, C_m/D_m) \]
\[ H^2(I^n/I_m, D_m) \to H^2(I^n/I_m, C_m) \to H^2(I^n/I_m, D_m), \]
because \( I^n/I_m \) is cyclic. Then we get by (5) \( H^1(I^n/I_m, C_m/D_m) \cong (0) \) and \( H^2(I^n/I_m, C_m/D_m) \cong \mathbb{Z}/(l^{m-n})\mathbb{Z} \). Since \( D_m \) is dual to the compact abelian group \( C_m/D_m \), and since \( I^n/I_m \) is cyclic, \( H^1(I^n/I_m, D_m) \) is dual to \( H^2(I^n/I_m, C_m/D_m) \), and \( H^2(I^n/I_m, D_m) \) is dual to \( H^1(I^n/I_m, C_m/D_m) \). On the other hand we have \( H^2(I^n/I_m, D_m) \cong H^1(I^n/I_m, \mathcal{A}_m) \), because the order of \( I^n/I_m \) is a power of a prime number \( l \). Now Lemma 3 follows from the above mentioned duality.

Now we prepare some notations. Let the element \( \nu \) in the group ring \( \mathbb{Z}[I^n/I_m] \) be defined by \( \nu = 1 + \sigma + \cdots + \sigma^{l^{n-m}-1} \), where \( \sigma \) is a generator of \( I^n/I_m \). Let in general \( M \) be a multiplicative abelian \( I^n/I_m \)-group. Then we put
\[ B^1(M) = \{ a^{1-\sigma} | a \in M \}, \quad C^1(M) = \{ a \in M | a^{\sigma} = 1 \}, \]
\[ B^2(M) = \{ a^{\sigma} | a \in M \}, \quad C^2(M) = \{ a \in M | a^{\sigma} = a \}. \]

14) Cf. Artin and Tate [1], Chevalley [4], Hochschild and Nakayama [5], Weil [14].
These notations will be retained in the following. Moreover we identify $H^i(G_n/\Gamma_m, M)$ with $C^i(M)/B^i(M)$, where $i = 1, 2$.

**Lemma 4.** $H^1(G_n/\Gamma_m, \mathcal{M}^m_n)$ and $H^2(G_n/\Gamma_m, \mathcal{N}^m_m)$ are canonically isomorphic for every $m \geq n \geq 0$.

**Proof.** Since $F/F^m$ is abelian, every element in $\mathcal{M}^m_n$ is invariant under the action of the Galois group $\Gamma_n/\Gamma_m$. Let $\chi_{m,2m-n}$ be an element in $\mathcal{M}^m_n$ whose order is equal to $l^{m-n}$. Then $\chi_{m,2m-n}$ generates $\mathcal{M}^m_n$. Put now $\chi_{m,m+1} = (\chi_{m,2m-n})^{m-n-i}$. Then $\chi_{m,m+1}$ is associated with the class field $F_{m+1}/F_m$. Namely $F_{m+1}/F_m$ is the class field defined over the kernel of $\chi_{m,m+1}$. Let $\chi_{n,n+1}$ be an element in $\mathcal{M}^m_n$ which is associated with $F_{m+1}/F_n$. Then by the translation theorem in class field theory we have

$$\ker \chi_{m,m+1} = \{ \bar{a} \in I_m | N(a) \in \ker \chi_{n,m+1} \},$$

where $N$ stands here for the norm mapping from $I_m$ to $I_n$. The factor group $I_n/\ker \chi_{n,m+1}$ is cyclic and of order $l^{m-n+1}$. We denote by $\iota$ the natural injective homomorphism of $I_n$ into $I_m$, and let $\bar{b}$ be an idele of $F_n$ which belongs to a generating coset of $I_n/\ker \chi_{n,m+1}$. Then $N(\iota(\bar{b})) = \bar{b}^{m-n} \in \ker \chi_{n,m+1}$; namely we have

$$\chi_{m,m+1}(\iota(\bar{b})) \neq 1.$$

If there exists an element $\chi$ in $\mathcal{N}^m_n$ such that $\chi_{m,m+1} = \chi^{1-n}$, then we have, by (2), $\chi_{m,m+1}(\iota(\bar{b})) = \chi(1) = 1$, which contradicts (6). Thus we get $B^1(\mathcal{M}^m_n) \cap \mathcal{M}^m_n = B^1(\mathcal{M}^m_n) = (1)$, which together with Lemma 3 proves the assertion in Lemma 4 for $i = 1$ (and also for odd $i$ by the periodicity of cyclic cohomologies). For $i = 2$ it is easily observed that $\mathcal{M}^m_n = C^2(\mathcal{M}^m_n) = B^2(\mathcal{M}^m_n)$, which together with Lemma 3 proves the assertion for $i = 2$. Lemma 4 is proved.

We notice in particular that the natural homomorphism

$$H^i(G_n/\Gamma_m, \mathcal{M}^m_n) \longrightarrow H^i(G_n/\Gamma_m, \mathcal{N}^m_n)$$

is injective.

Now we prove Proposition stated in § 2.4. Assume that the natural homomorphism for $i = 1$ in Proposition is injective. Then it follows from the injectiveness of (7) and Lemmas 3, 4, that $H^1(G_n/\Gamma_m, \mathcal{M}^m_n)$ and $H^2(G_n/\Gamma_m, \mathcal{N}^m_n)$ are canonically isomorphic. From the exact sequence

$$H^1(G_n/\Gamma_m, \mathcal{M}^m_n) \longrightarrow H^1(G_n/\Gamma_m, \mathcal{N}^m_n) \longrightarrow H^2(G_n/\Gamma_m, \mathcal{M}^m_n)$$

it then follows that $i_1$ in the above sequence is a surjective isomorphism. Then the isomorphism (3) in Proposition for $i = 1$ follows from the fact that $H^1(G_n/\Gamma_m, \mathcal{M}^m_n) \cong (0)$. Conversely assume now (3) for $i = 1$, then by the above
sequence we get $H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m) \cong \mathbb{Z}/(l^{m-n})\mathbb{Z}$, which means by Lemmas 3 and 4 that the natural homomorphism for $i=1$ in Proposition is injective. For $i=1$ this completes the proof of Proposition. Our proposition for $i=2$ follows similarly from Lemmas 3 and 4 and the above sequence (8).

By the above and by the fact referred from [7] in § 1.3 we observe also the following

**Theorem 2.** Let the notation be as in Theorem 1. Then the natural homomorphisms

$$H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m) \longrightarrow H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m), \quad i=1, 2,$$

are both bijective for every $m \geq n \geq 0$ if and only if the $\Gamma$-module $A(K/F)$ is regular.

2.6 To introduce the next theorem we first prepare some notations concerning infinite abelian groups. Let $\mathbb{Z}(l, \infty)$ denote the group of all the roots of unity whose orders are powers of $l$. An abelian group $M$ is said to be a torsion $l$-abelian group if every element of $M$ is of order a power of $l$. Let $M^{(0)}$ be the subgroup of $M$ which consists of all the elements $x$ of $M$ with $x^l=1$. Then $M^{(0)}$ may be regarded as a vector space over the prime field of characteristic $l$, of which dimension we shall call the rank of the torsion $l$-abelian group $M$. A subgroup $N$ of $M$ is said to be divisible if, for any element $x$ of $N$ and any power $l^r$ of $l$, there exists an element $y$ in $N$ such that $x=y^{l^r}$. The torsion $l$-abelian group $M$ contains a unique largest divisible subgroup $M_m$, and $M_\infty$ is isomorphic to the direct product of finite or infinite number of $\mathbb{Z}(l, \infty)$. If the rank of $M$ is finite, then $M$ is the direct product of $M_m$ by a finite subgroup of $M$. After the terminology of Kubota [12] we shall call the rank of $M_\infty$ the dimension of $M$, and we denote it by $\dim M$.

Let $M$ and $M'$ be torsion $l$-abelian groups, and let there be given a homomorphism of $M$ onto $M'$ whose kernel is finite. Then we have $\dim M=\dim M^{(0)}$.

2.7. We next consider the ring $R_m = F_m \otimes \mathbb{Q}_l$. Let $R_m^*$ denote the multiplicative group of all the regular elements in $R_m$. Then $R_m^*$ is canonically identified with the direct product $\prod_{1 \leq i \leq m} F_{m,i}^*$, where $F_{m,i}^*$ stands for the multiplicative group of the $l$-completion of $F_m$ for $1 \leq i \leq m$. $S_m$ denotes, as before, the set of all prime divisors of $F_m$ which divide the rational prime divisor $l$. The elements in $R^*$ which are congruent 1 modulo $l$ form a multiplicative group $H_{R_m}$, and the power $u^a$ is defined for every $u \in H_{R_m}$ and $a \in \mathbb{Z}_l$. The dimension over $\mathbb{Z}_l$ of $H_{R_m}$ (modulo the finite torsion subgroup if $l=2$) is equal to the degree $d_m$ of $F_m$ over $\mathbb{Q}$, as observed from the well-known structure theo-

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rem of the local unit groups. Let \( r_1(m) \) denote the dimension of \( \mathbb{Z}_L \)-subspace of \( H_{R_m} \) spanned by units \( \varepsilon (= \varepsilon \otimes 1) \) of \( F_m \) contained in \( H_{R_m} \). Then the equality \( \dim \mathcal{A}_m = \dim \mathcal{A}'_m = d_m - r_1(m) \) is known by Kubota \[12\], Theorem 5, where \( \dim \mathcal{A}_m \) etc. are defined in \( \S \) 2.6. Then Lemma 2 entails

\[ \dim A(K/F)_m = \dim \mathcal{A}'_m - 1 = d_m - r_1(m) - 1, \]

where \( d_m = [F_m : \mathbb{Q}] \). Furthermore the \( \Gamma' \)-module \( A(K/F) \) is \( \Gamma' \)-finite; namely the rank of \( A(K/F)_m \) is finite for every \( m \geq 0 \).

**Theorem 3.** Let \( F/F_0 \) be a \( \Gamma' \)-extension over an algebraic number field \( F_0 \) of finite degree. Let \( A(K/F) \) be the \( \Gamma' \)-module described in \( \S \) 1.1, and let \( r_1(m) \) be as above. Assume that the \( \Gamma' \)-module \( A(K/F) \) is regular. Then we have

\[ r_1(m) = \lambda(m, r_1(0) + 1) - 1 \]

for every \( m \geq 0 \).

**Remark.** Let \( r_\omega(m) \) denote the usual rank of the unit group of \( F_m \). If we assume moreover \( r_\omega(0) = r_\omega(0) \) in Theorem 3, then (10) implies \( r_1(m) = r_\omega(m) \) for every \( m \geq 0 \), because no infinite prime divisor is ramified by \( F/F_0 \). The proof of Theorem 3 given below shows that the equality (10) follows if we assume only the regularity of the maximal divisible submodule of \( A(K/F) \).

Theorem 3 is a direct consequence of (9) and the following

**Lemma 5.** Let \( A \) be a discrete \( \Gamma' \)-finite \( \Gamma' \)-module. If \( A \) is regular, then

\[ \dim A_\omega = \lambda \dim A_\omega \text{ for every } n \geq 0^{17}. \]

**Proof.** If \( A \) is \( \Gamma' \)-finite and regular, then \( A \) is a sum of a divisible regular submodule \( B' \) of finite rank and a characteristic submodule \( C \) such that \( C \cong E(m_1, \ldots, m_s)/D \) for some \( 0 \leq m_i \leq \infty \) and for a finite submodule \( D \) of \( E(m_1, \ldots, m_s) \). The intersection \( B' \cap C \) is finite\(^18\). We have then the surjective homomorphisms \( f \) and \( g \) such that

\[
\begin{align*}
    \overline{A} = B \oplus E(m_1, \ldots, m_s) & \xrightarrow{f} B \oplus C \\
    & \xrightarrow{g} A = B + C
\end{align*}
\]

(where \( \oplus \) stands for the direct sum), and that the kernel \( \mathfrak{K} \) of \( g \circ f \) is finite. Let \( \sigma \) denote here a generator of \( \Gamma'/\Gamma_\omega \) and put \( (\overline{A} \sigma) = \{ \sigma \in A \mid (1-\sigma) \sigma = 1 \} \), the inverse image of \( A_\alpha \) by \( g \circ f \). Then we have \( (\overline{A} \sigma) / \mathfrak{K} = A_\alpha \). Since \( \mathfrak{K} \) is finite, we have \((\overline{A} \sigma)/A_\alpha < \infty \). Thus we get \( \dim (\overline{A} \sigma) = \dim A_\alpha \) and \( \dim (\overline{A} \sigma) = \dim A_\alpha \), and consequently \( \overline{A} = A_\alpha \). Thus the proof is reduced to the cases where \( A = B \) (divisible, regular and of finite rank) or \( A = E(m_1, \ldots, m_s) \) \( = E(m_1) \oplus \cdots \oplus E(m_s) \). Lemma 5 is then a direct consequence of the fact that

\[ 17 \] In the proof of Lemma 5 notations and terminologies are in accordance with those of Iwasawa \[6\]; cf. in particular Theorems 1 and 2 of \[6\]. Submodule, homomorphism etc. mean \( \Gamma' \)-submodule, \( \Gamma' \)-homomorphism, etc.

\[ 18 \] Because we have \( B = B' \), where \( B \) is the submodule appearing in loc. cit. Theorem 1, and \( B \cap C \) is finite. Moreover in our case we have \( B = B' \)
§ 3. \(\Gamma\)-extensions over imaginary quadratic fields.

3.1. In this section we shall prove the following

**Theorem 4.** Let \(F_0\) be an imaginary quadratic extension over \(\mathbb{Q}\) in which the fixed prime number \(l\) is not fully decomposed: namely \(S_0\) consists of a single element \(1\). Furthermore we assume that the class number of \(F_0\) is prime to \(l\) and that the \(l\)-completion of \(F_0\) contains no primitive \(l\)-th root of unity (this last assumption being always the case if \(l > 3\)). Let \(F/F_0\) be a \(\Gamma\)-extension over \(F_0\). Then the \(\Gamma\)-module \(A(K/F)\) is regular, and the natural homomorphisms

\[
H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m) ightarrow H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m), \quad i = 1, 2,
\]

are both injective for every \(m \geq n \geq 0\).

Let \(F/F_0\) be as in Theorem 4. Since \(r(0) = r(\infty) = 0\), we get, by Theorems 3 and 4, \(r(n) = r(\infty)\) for every finite intermediate field \(F_m\) of \(F/F_0\). For the proof of Theorem 4 it suffices, by § 2, to show the following Lemmas 6 and 7.

**Lemma 6.** If the ground field \(F_0\) of a \(\Gamma\)-extension \(F/F_0\) is an imaginary quadratic extension over \(\mathbb{Q}\), then the natural homomorphism

\[
H^1(\Gamma_n/\Gamma_m, \mathcal{A}_m) ightarrow H^1(\Gamma_n/\Gamma_m, \mathcal{A}_m)
\]

is injective for every \(m \geq 0\).

**Lemma 7.** Under the same assumptions in Theorem 4 the natural homomorphism

\[
H^2(\Gamma_n/\Gamma_m, \mathcal{A}_m) ightarrow H^2(\Gamma_n/\Gamma_m, \mathcal{A}_m)
\]

is injective for every \(m \geq 0\).

3.2. **Proof of Lemma 6.** In the proofs of Lemmas 6 and 7 the Galois group \(\Gamma_n/\Gamma_m\) of \(F_n/F_m\) is simply denoted by \(G\), and \(\sigma\) stands for a generator of \(G\). Let the element \(\nu\) in the group ring \(Z[G]\) be defined as in § 2.5. \(S_0\) (resp. \(S_\infty\)) is, as before, the set of all prime divisors of \(F_0\) (resp. \(F_\infty\)) which divide \(l\). Let \(\wp\) be any prime divisor of \(F_0\) outside \(S_\infty\). We put

\[
U_{\wp}^{\infty} = \prod_{\wp \in S_{\wp}} U_{\wp}, \text{ where } U_{\wp}^{\infty}
\]

stands for the unit group of the \(\wp\)-completion of \(F_\nu\) for a prime divisor \(\wp\) of \(F_0\).

19) Cf. loc. cit. in particular Lemma 5.1.

20) There exist two independent \(\Gamma\)-extensions over \(F_0\) (with respect to the fixed prime number \(l\)); cf. Kubota [12], Theorem 5. Thus our \(F/F_0\) is not necessarily 'cyclic'. Here we note also that our argument in the proof of this theorem is also applicable for \(\Gamma\)-extensions over \(\mathbb{Q}\).

21) The corresponding fact for \(\Gamma\)-extensions over \(\mathbb{Q}\) is known by Jehne [10] as '\(\sigma\)-th stability' of \(l\).
$F_m$ such that $\mathfrak{P}|p$. Since $p \in S_0$, $p$ is unramified by $F_m/F_0$, and we have\footnote{Cf. e. g. Chevalley [4], Theorem 12.1.}

\[(11) \quad H^i(G, U_m^p) \cong (0), \quad \text{for } i = 1, 2.\]

Now let $\chi_0$ be an element in $\mathcal{A}_m^p \cap B'(\mathcal{A}_m)$, where the notation $B'(\mathcal{A}_m) = \{\chi^{1-\sigma} | \chi \in \mathcal{A}_m\}$ is defined in § 2.5. Then there exists an element $\chi_1$ in $\mathcal{A}_m$ for which

\[(12) \quad \chi_0 = \chi_1^{1-\sigma}, \quad \chi_1 \in \mathcal{A}_m.\]

We consider $\chi_1$ on $U_m^p$. $U_m^p$ being regarded as imbedded in the idèle group $I_m$ of $F_m$. Since $\chi_1^{1-\sigma}$ is unramified at $\mathfrak{P}|p$, it follows from (2) and (11) for $i = 1$ that $\chi_1$ is trivial on $C^1(U_m^p)$. Thus we can define a character $\varphi_1$ of $B^p(U_m^p)$ by setting

\[(13) \quad \varphi_1(\mathfrak{a}^\tau) = \chi_1(\mathfrak{a}), \quad \mathfrak{a} \in U_m^p.\]

Then $\varphi_1$ is defined on $C^2(U_m^p)$ by virtue of (11) for $i = 2$. Let now $N$ denote the norm mapping from $I_m$ to $I_0$. We put

\[(14) \quad \varphi_p(N(\mathfrak{a})) = \varphi_1(\mathfrak{a}^\tau), \quad \mathfrak{a} \in U_m^p.\]

Then $\varphi_p$ is a character defined on the unit group $U_0^p$ of the $p$-completion of $F_0$ for $p \in S_0$. Since $\varphi_p$ is of finite order, $\varphi_p$ is continuous on $U_0^p$.

For a non-zero element $\alpha$ of $F_0$ we denote by $\tilde{\alpha}$ the element in the principal idèle group $P_0$ corresponding to $\alpha$, and let $\tau_0$ denote the endomorphism of $I_0$ given by

\[
(\tau_0(\mathfrak{a}))_i = \mathfrak{a}_i, \quad \text{for } i \in S_0, \quad (\tau_0(\mathfrak{a}))_p = 1, \quad \text{for } p \in S_0,
\]

where $\mathfrak{a} \in I_0$. Let $E_0$ denote the unit group of $F_0$. We define a character $\varphi_{S_0}$ on $\tau_0(\tilde{E}_0)$ by setting

\[(15) \quad \varphi_{S_0}(\tau_0(\tilde{\mathscr{e}})) = \prod_{\varepsilon \in E_0} \varphi_{S_0}^{-1}(\varepsilon \tau_0(\tilde{\mathscr{e}})^{-1})_0, \quad \varepsilon \in E_0.\]

Since $\tau_0(\tilde{\mathscr{e}}) \rightarrow \tilde{\mathscr{e}} \tau_0(\tilde{\mathscr{e}})^{-1}$ is an isomorphism, and since the right hand side of (15) is a character on $\prod_{b \in S_0} U_b^p$, $\varphi_{S_0}$ is an (algebraic) character defined on $\tau_0(\tilde{E}_0)$. Moreover, since $\tau_0(\tilde{E}_0)$ is a finite group, $\varphi_{S_0}$ is continuous on $\tau_0(\tilde{E}_0)$ with respect to the topology induced by that of $U_{S_0} = \prod_{1 \leq S_0} U_b^p$, where $U_b^p$ is the unit group of the $l$-adic completion of $F_b$. Since $\tau_0(\tilde{E}_0)$ is closed in $U_{S_0}$, and since $\varphi_{S_0}$ is of order a power of $l$, we can extend $\varphi_{S_0}$ onto $U_{S_0}$ as a continuous character of order a power of $l$, which we shall denote by the same notation $\varphi_{S_0}$\footnote{We note that every continuous character on $U_{S_0}$ is of finite order.}. We denote by $U_0$ the unit idèle group of $F_0$, and we define $\varphi$ on $U_0$ by setting
\[ \varphi = \varphi_s \cdot \prod_{\varphi \in S_0} \varphi_{\varphi}. \]

Then \( \varphi \) is a continuous character on \( U_0 \) with order a power of \( l \). \( \varphi \) is trivial on \( U_0 \cap P_0 \) because of \( U_0 \cap P_0 = E_0 \) and (15). Thus we can extend \( \varphi \) onto \( P_0 \cdot U_0 \) by putting \( \varphi(\hat{a}) = 1 \) for every \( \hat{a} \in P_0 \). The continuous character \( \varphi \) thus defined on \( P_0 \cdot U_0 \) extends now onto \( I_0 \), preserving the property that the order of \( \varphi \) is a power of \( l \), because the closed subgroup \( P_0 \cdot U_0 \) of \( I_0 \) is of finite index. Namely there exists an element \( \varphi \) in \( J_0 \) whose \( p \)-component on \( U_0 \) is given by (14). Then there exists an element \( \tilde{\varphi} \) in \( \mathcal{A}_m \) such that

\[ \tilde{\varphi}(\hat{a}) = \varphi(N(\hat{a})), \quad \text{for } \hat{a} \in I_m. \]

By virtue of (13) and (14), \( \chi_1 \cdot \tilde{\varphi}^{-1} \) is unramified at every prime divisor of \( F_m \) outside \( S_m \); namely \( \chi_1 \cdot \tilde{\varphi}^{-1} \in \mathcal{A}_m' \). Moreover it is observed by (16) that \( \tilde{\varphi} \) belongs to \( C(\mathcal{A}_m) \). We have thus \( (\chi_1 \cdot \tilde{\varphi}^{-1})^{1-e} = \chi_1^{1-e} = \chi_1 \). The existence of such \( \chi_1 \cdot \tilde{\varphi}^{-1} \) in \( \mathcal{A}_m \) is nothing but the assertion in Lemma 6.

**REMARK.** In the above proof the assumption that \( F_0 \) is an imaginary quadratic field is essentially used only in the form \( r_1(0) = r_\infty(0) \).

### 3.3. PROOF OF LEMMA 7

By the assumption in Theorem 4, \( F/F_0 \) contains no non-trivial unramified extension, and it follows further that \( S_m \) consists of a single element \( \mathcal{L} : S_m = \{ \mathcal{L} \} \). We denote by \( \Phi_m \) the \( \mathcal{L} \)-completion of \( F_m \) and by \( \Phi_0 \) the \( \mathcal{L} \)-completion of \( F_0 \), where \( S_0 = \{ 1 \} \). Then in our case the Galois group \( \Phi_m/\Phi_0 \) can be identified with that of \( F_m/F_0 \). Moreover, since the class number of \( F_0 \) is assumed to be prime to \( l \), it follows in particular that no non-principal ideal of \( F_0 \) becomes principal in \( F_m \).

Let \( U_m \) denote the unit idele group of \( F_m \) and \( U'_m \) the group of unit idèles of \( F_m \) whose \( \mathcal{L} \)-components are \( 1 \). We denote by \( \tau_\mathcal{L} \) the endomorphism of \( I_m \) given by

\[ \tau_\mathcal{L}(\hat{a})_\mathcal{L} = \hat{a}_\mathcal{L}, \quad \text{for } \hat{a} \in U_\mathcal{L}, \]

\[ (\tau_\mathcal{L}(\hat{a}))_\mathcal{L} = 1, \quad \text{for } \hat{a} \in S_\mathcal{L}, \]

where \( \hat{a} \in I_m \). We put \( U_\mathcal{L} = \tau_\mathcal{L}(U_m) \). We denote by \( E_m \) the unit group of \( F_m \).

Now let \( \chi_0 \) be an element in \( \mathcal{A}_m' \cap B(\mathcal{A}_m) \). Then there exists an element \( \chi_1 \) in \( \mathcal{A}_m \) for which

\[ \chi_0 = \chi_1^r, \quad \chi_1 \in \mathcal{A}_m. \]

We define a character \( \chi_2 \) on \( U'_m \) by setting

\[ \chi_2(\hat{a}) = \chi_1^r(\hat{a}), \quad \text{for } \hat{a} \in U'_m. \]

Since \( P_m \cap U'_m = (1) \), \( \chi_2 \) extends onto \( P_m \cdot U'_m \) by setting

\[ \chi_2(\hat{a}) = 1, \quad \text{for } \hat{a} \in P_m. \]

Now let \( \hat{a} \) be an element in \( P_m \cdot U'_m \cap I_m' \), and let \( \hat{a} = \hat{a}_\mathcal{L} \), where \( \hat{a} \in P_m \) and
We get $\alpha \in F_0$, because $\tilde{a}$, and thus in particular the $\mathfrak{p}$-component of $\tilde{a}$, is invariant under the action of the Galois group $G$. Let $(\tilde{a})$ denote the ideal of $F_m$ corresponding to the idèle $\tilde{a}$. Then we have $(\tilde{a}) = (\alpha)$, $\alpha \in F_0$. The principal ideal $(\alpha)$ is a norm of an ideal of $F_m$. Since there exists only one prime divisor which is ramified by $F_m/F_0$, it follows that $\alpha$ is a norm of an element in $F_m^{(2)}$. Thus $\tilde{a}$ is an element in $B^0(I_m) = I_m^\nu$. We get then $\tilde{b} \in I_m^\nu$, because $\tilde{a} = \tilde{a}^0 \in I_m^\nu$. From (11) it follows further that $\tilde{b} \in U_m^\nu$. Hence $P_m \cdot U_m^\nu \cap I_m^\nu = (P_m \cdot U_m^\nu)^\nu$. This enables us to extend $\chi_2$ on $P_m \cdot U_m^\nu \cdot I_m^\nu$ by setting

$$\chi_2(\tilde{a}^\nu) = 1, \quad \text{for } \tilde{a} \in I_m^\nu,$$

because $\chi_2$ previously defined on $P_m \cdot U_m^\nu$ is trivial on $(P_m \cdot U_m^\nu)^\nu$.

We next consider the continuity of $\chi_2$ defined on $P_m \cdot U_m^\nu \cdot I_m^\nu$ by (18), (19) and (20). For this purpose it suffices to consider $\chi_2$ only on $P_m \cdot U_m^\nu \cdot I_m^\nu \cap U_m^\nu$, which is the direct product of $D = P_m \cdot U_m^\nu \cdot I_m^\nu \cap U_m^\nu$ and $U_m^\nu$ (as topological group). That $\chi_2$ is continuous on $U_m^\nu$ is clear by (18). Thus we have only to consider $\chi_2$ on $D$. Let $\tilde{a} \in D$ and $\tilde{a} = \tilde{a} \tilde{b} \tilde{c}^\nu$, where $\tilde{a} \in P_m$, $\tilde{b} \in U_m^\nu$, $\tilde{c} \in I_m^\nu$. Then the ideal $(\alpha)$ corresponding to the principal idèle $\tilde{a}$ is an image by $\nu$ of an ideal of $F_m$. Then, by the remark at the beginning of this §3.3, there exists an element $\alpha'$ in $F_0$ and a unit $\varepsilon$ of $F_m$ for which we have $\alpha = \alpha' \cdot \varepsilon$. Then there exists $\beta \in F_m$ such that $\alpha' = \beta^{\nu 24}$. Thus we have $\tilde{a} = \tilde{a} \tilde{b} \tilde{c} \varepsilon \tilde{b} \tilde{c}^\nu$, where $\tilde{b} \in F_m^*$, $\varepsilon \in \mathcal{F}_m$, $\tilde{b} \in U_m^\nu$ and $\tilde{c} \in I_m^\nu$. Hence the $\mathfrak{p}$-component of $\tilde{a}$ is of the form $\varepsilon \cdot \alpha^\nu$, where $\varepsilon \in \mathcal{F}_m$ and $\alpha \in U_0$. Conversely an idèle of $F_m$ whose $\mathfrak{p}$-component is of the form $\varepsilon \cdot \alpha^\nu$ (\in $U_0$) and all other local components are 1 is clearly contained in $D$.

We now consider $U_0$ as contained in the multiplicative group $\Phi_m^*$ of the non-zero elements of $\Phi_m$. The structure of $U_0$ is as follows. Let $V$ denote the group of all the roots of unity contained in $\Phi_m$ whose orders are prime to $l$. Then the order $\nu$ of $V$ is equal to the absolute norm of $\mathfrak{p}$ minus 1. Let $H$ denote the subgroup of $U_0$ which consists of all the elements $\alpha$ in $U_0$ such that $\alpha \equiv 1 \pmod{\mathfrak{p}}$. As topological group, $U_0$ is the direct product of the subgroups $V$ and $H$. Now, by our assumption, $\Phi_m$ contains no primitive $l$-th root of unity. This immediately implies that $\Phi_m$ also contains no primitive $l$-th root of unity, because $[\Phi_m : \Phi_0] = l^m$ in our case. In such a case $H$ is, as topological group, isomorphic to the direct product of $[\Phi_m : \mathfrak{p}]$ groups all isomorphic to the additive group of $\mathbb{Z}_l$. In particular $H$ is torsion free. If we put $H^{(1)} = C^1(H)$ and $H^{(2)} = C^2(H)$, then we have $H^{(1)} \cap H^{(2)} = (1)$. Moreover it is easily observed by local class field theory that the direct product $H^{(1)} \cdot H^{(2)}$ is an open subgroup of finite index in $H$. Therefore, if $\alpha_1 \in H^{(1)}$, $\alpha_2 \in H^{(2)}$ and

\( \alpha_1 \cdot \alpha_2 \equiv 1 \pmod{\mathfrak{P}} \) for a sufficiently large integer \( c \), then there exists an integer \( d \) independent of \( c \) such that \( \alpha_1 \equiv \alpha_2 \equiv 1 \pmod{\mathfrak{P}^{c-d}}. \)

Since \( \chi_2 \) is of order a power of \( l \) on \( D \), \( \chi_2 \) is continuous on \( D \) if and only if \( \chi_2 \) is continuous on \( D' \). Then \( \chi_2 \) is continuous on \( D \) if \( \chi_2 \) is continuous on \( D' \), where \( D' = \{ s \in \mathbb{E}_m : H' \text{ and } \alpha' \in H' \} \). Since the assumptions in Theorem 4 implies that \( l \) is odd, we have \( \varepsilon' = 1 \) if \( \varepsilon \in H \): namely we have \( \tau_\varepsilon (\mathbb{E}_m) \cap H \). Thus, if \( \varepsilon \alpha' \equiv 1 \pmod{\mathfrak{P}} \) for a sufficiently large integer \( c \), where \( \varepsilon \in \mathbb{E}_m \cap H' \) and \( \alpha' \in H' \), then we have \( \varepsilon \equiv 1 \pmod{\mathfrak{P}^{c-d}}. \) On the other hand we have \( \chi_2 (s \alpha') = \chi_2 (s) \chi^{-1} (s) \) locally at \( \mathfrak{P} \). Since \( \chi_1 \) is continuous, \( \chi_2 \) is also continuous on \( D' \). Therefore, as noticed above, it follows that \( \chi_2 \) is continuous on \( P_m \cdot U_m \cdot \mu_m \).

Now, as a continuous character, \( \chi_2 \) extends uniquely onto the closure of \( P_m \cdot U_m \cdot \mu_m \). By this procedure the value group of \( \chi_2 \) remains unchanged, because the order of the original \( \chi_2 \) is finite. The closure of \( P_m \cdot U_m \cdot \mu_m \) is a closed subgroup of the locally compact abelian group \( I_m \), and therefore \( \chi_2 \) now extends onto the whole group \( I_m \). The restriction of \( \chi_2 \) thus defined on \( I_m \) to the unit idèle group \( U_m \) is of finite order. Hence \( \chi_2 \) thus extended onto \( I_m \) is of finite order, because \( I_m / P_m \cdot U_m \) is of finite order and \( \chi_2 (P_m) = 1 \). Then we can take \( \chi_2 \) extended on \( I_m \) so as to be of order a power of \( l \). \( \chi_2 \) is then an element of \( \mathcal{A}_m \), and moreover we have, by (18), \( \chi_1 \cdot \chi_2 \equiv \mathcal{A}'_m \). It follows finally from (17), (20) and (2) that \( (\chi_1 \cdot \chi_2)^t = \chi_1 = \chi_0 \). The existence of such \( \chi_1 \cdot \chi_2 \) in \( \mathcal{A}'_m \) completes the proof of Lemma 7.

References


[Remark added in proof 6 Feb. 1968 at the 'Goethe Institut' in Brannenburg] I have heard in Japan that A. Brumer has proved the $p$-adic analogue of Dirichlet's unit theorem (cf. § 1.4 and § 2.7) for absolutely abelian fields and that his paper will appear in a forthcoming issue of Mathematika, to which I am not yet accessible.