On the unit group of an absolutely cyclic number field of degree five

Dedicated to Professor Iyanaga on his 60th birthday

By Ryozo MORIKAWA

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1. Let $K$ be a Galois extension of odd degree $n$ over the rational number field $Q$. Then $K$ is totally real and the group of units of $K$ has $(n-1)$ generators mod $±1$. Let $H$ be the group of totally positive units of $K$. Then $H$ has also $(n-1)$ generators, and it is known that in case $n=3$ these generators can be taken to conjugate to each other (cf. Hasse [1]). We shall show in this paper that the same is true for $n=5$.

In the following let $K$ be a cyclic field of degree 5 over $Q$, $σ$ a generator of the Galois group $G(K/Q)$ and $H$ the group of totally positive units of $K$. For $ξ ∈ K$, $ξ^{(i)}$ means $σ^{-1}(ξ) ∈ K$ $(i=1, 2, 3, 4, 5)$. Then the points

$$P(ξ) = (\log ξ^{(1)}, \log ξ^{(2)}, \log ξ^{(3)}, \log ξ^{(4)}, \log ξ^{(5)}) ∈ R^5$$

for $ξ ∈ H$ form a lattice $L$ lying in the hyperplane $π : x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Obviously the five points $P(ξ^{(1)}), \ldots, P(ξ^{(5)})$ lie at the same distance from the origin $O$ of $R^5$.

Let $η(≠ 1)$ be a unit in $H$ such that $P(η) ∈ L$ lies nearest to $O$. Then our main result is that $H$ is generated by any four of $η^{(1)}, η^{(2)}, η^{(3)}, η^{(4)}, η^{(5)}$, or geometrically expressed, $L$ is generated by $P(η^{(1)}), \ldots, P(η^{(5)})$.

We shall namely prove the following theorem.

**Theorem.** Let $K$ be an absolutely cyclic field of degree 5, and $H$ the group of totally positive units of $K$. Then $H$ is generated by $η ∈ H$ and its conjugates, where $η$ is an element $(≠ 1)$ of $H$ such that

$$\sum_{i=1}^5 (\log η^{(i)})^2 ≤ \sum_{i=1}^5 (\log ξ^{(i)})^2$$

holds for any element $ξ ∈ H$ $(ξ ≠ 1)$.

2. We shall first prove the following general proposition. Let $M$ be an $n$-dimensional lattice in $R^n$, which is generated by $n$ vectors $\overrightarrow{OQ_1}, \overrightarrow{OQ_2}, \ldots, \overrightarrow{OQ_n}$. Let $d_i$ be the length of $\overrightarrow{OQ_i}$ $(i=1, 2, \ldots, n)$. 

(A) For any point \( X \in \mathbb{R}^n \), there exists a point \( Y \) of \( M \), such that the distance

\[
XY \leq \frac{1}{2} \left( \sum_{i=1}^{n} d_i^2 \right)^{1/2}.
\]

Here we can replace the sign \( \leq \) by \( < \) except the case: \( \overrightarrow{OQ_i} \perp \overrightarrow{OQ_j} \); for any \( i \neq j \).

**Proof.** We shall prove it by induction on the dimension \( n \).

1) If \( n = 1 \) the assertion is trivial.

2) For \( n \geq 2 \) let \( N \) be the sublattice of \( M \) generated by \( \overrightarrow{OQ_1}, \ldots, \overrightarrow{OQ_{n-1}} \). Then \( M = \mathbb{Z} \cdot \overrightarrow{OQ_n} + N \), and each \( i \overrightarrow{OQ_n} + N \) forms an \( n-1 \) dimensional lattice in the hyperplane \( \pi_i \), where \( \pi_i \parallel \pi_j \) for any \( i \neq j \). For any given point \( X \in \mathbb{R}^n \) we can choose a suitable \( i \) and a point \( Z \in \pi_i \) such that \( \overrightarrow{XZ} \perp \pi_i \) and \( \overrightarrow{XZ} \leq \frac{d_n}{2} \). We can replace \( \leq \) by \( < \), if \( \overrightarrow{OQ_n} \) is not orthogonal to \( \pi_i \). With respect to the point \( Z \in \pi_i \), and the lattice \( N \), we can apply the assumption of the induction. Hence there exists a point \( Y \) of \( N \) such that \( \overrightarrow{YZ} \perp \pi_i \) and \( \overrightarrow{YZ} \leq \frac{d_n}{2} \). We can replace \( \leq \) by \( < \), except \( \overrightarrow{OQ_i} \parallel \overrightarrow{OQ_j} \) for any \( i \neq j \). Q.E.D.

3. Now we proceed to the proof of the theorem. With the same notations as in the introduction, let \( \tilde{L} \) denote the lattice in \( \pi \) generated by \( P(\gamma^{(i)}), \ldots, P(\gamma^{(4)}) \). Our aim is to prove \( \tilde{L} = L \). Now it is known that, \( l \) being an odd prime, any cyclic field of degree \( l \) over \( \mathbb{Q} \) has the property that any \( l-1 \) among \( \xi^{(i)} \) \( i = 1, 2, \ldots, l \) forms a system of independent units in \( K \) for any non rational unit \( \xi \) in \( K \) (cf. Hilbert [2] § 55). This implies obviously \( \dim \tilde{L} = 4 \). Take \( \tilde{L} \) as the lattice \( M \) in Proposition (A). Then \( Q_i = P(\gamma^{(i)}) \) \( (i = 1, 2, 3, 4) \) generate \( \tilde{L} \) and \( d_1 = \cdots = d_4 = \left( \sum_{i=1}^{5} (\log \gamma^{(i)}) \right)^{1/2} \). Moreover, for some \( i \neq j \) \( \overrightarrow{OQ_i} \) is not orthogonal to \( \overrightarrow{OQ_j} \). Hence from Proposition (A) follows the proposition:

(B) For any point \( X \) of \( \pi \) there exists a point \( Y \) of \( \tilde{L} \) such that the distance \( \overrightarrow{XY} < d \), where \( d^2 = \sum_{i=1}^{5} (\log \gamma^{(i)})^2 \).

It is rather a routine reasoning to deduce our theorem from Proposition (B).

**Remark.** Whether the similar result holds for a prime \( p \) \((\neq 3, \neq 5)\) or not is an open problem.

Tokyo Metropolitan University
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References
