On the convergence of nonlinear semi-groups II

By Isao MIYADERA

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§ 1. Introduction.

Let $X$ be a Banach space and let $\{T(\xi) ; \xi \geq 0\}$ be a family of (nonlinear) operators from $X$ into itself satisfying the following conditions:

(i) $T(0) = I$ (the identity) and $T(\xi + \eta) = T(\xi)T(\eta)$ for $\xi, \eta \geq 0$.
(ii) For each $x \in X$, $T(\xi)x$ is strongly continuous in $\xi \geq 0$.
(iii) There is a constant $\omega \geq 0$ such that

$$
||T(\xi)x - T(\eta)y|| \leq \omega \xi ||x - y||
$$

for $x, y \in X$ and $\xi \geq 0$.

We call such a family $\{T(\xi) ; \xi \geq 0\}$ simply nonlinear semi-group of local type. In particular, if $\omega = 0$, it is called a nonlinear contraction semi-group.

We define the infinitesimal generator $A_0$ of $\{T(\xi) ; \xi \geq 0\}$ by

$$
A_0x = \lim_{\delta \to 0^+} \delta^{-1}(T(\delta) - I)x
$$

and the weak infinitesimal generator $A'$ by

$$
A'x = \lim_{\delta \to 0^+} \delta^{-1}(T(\delta) - I)x,
$$

where the notation "$\omega$-lim" means the weak limit in $X$.

Throughout this paper it is assumed that the dual $X^*$ of $X$ is uniformly convex. Our purpose is to prove the following theorem.

**Theorem 1.** Let $\{T(\xi_k) ; \xi_k \geq 0\}$ be a sequence of nonlinear semi-groups of local type satisfying the stability condition

$$
||T(\xi_k)x - T(\xi_k)y|| \leq \omega \xi_k ||x - y||
$$

for $\xi_k \geq 0$, $k \in \mathbb{N}$, and $x, y \in X$, where $\omega$ is a non-negative constant independent of $x, y, \xi_k$ and $k$. Let $A^{(k)}$ be the weak infinitesimal generator of $\{T^{(k)}(\xi) ; \xi \geq 0\}$ and assume $R(I - h_kA^{(k)}) = X$ for some $h_k \in (0, 1/\omega)$, and define $A x = \lim_k A^{(k)} x$.

Suppose that

(a) $D(A)$ (the domain of $A$) is dense in $X$,

(b) $R(I - h_kA) = X$ for some $h_k \in (0, 1/\omega)$,

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Then the strong closure $\overline{A}$ of $A$, which is not necessarily single-valued, generates a nonlinear semi-group $\{T(\xi) ; \xi \geq 0\}$ of local type (in sense of Theorem 2); and for each $x \in X$

\[(1.4) \quad T(\xi)x = \lim_{k \to \infty} T^{(k)}(\xi)x \quad \text{for} \quad \xi \geq 0, \]

and the convergence is uniform with respect to $\xi$ in every finite interval.

**Remarks.**

1°. A multi-valued operator $T$ is called to be the strong closure of $A$ if $G(T) = G(A)$, where the notation $G(\cdot)$ denotes the graph of operator; and we write $T = \overline{A}$.

2°. If $\{T^{(k)}(\xi) ; \xi \geq 0\} (k = 1, 2, 3, \ldots)$ are linear semi-groups (in this case, each $A^{(k)}$ becomes the infinitesimal generator and $R(I-h_{\xi}A^{(k)}) = X$ holds automatically), then $\overline{A}$ is single-valued; and the theorem is a special case of Trotter's theorem (see [9]).

3°. If we omit the condition (a), then $\overline{A}$ generates a nonlinear semi-group $\{T(\xi) ; \xi \geq 0\}$ of local type defined on $\overline{D(A)}$ and (1.4) holds on $\overline{D(A)}$.

4°. It is easy to see that

\[(A^{(k)}x - A^{(k)}y, F(x-y)) \leq \omega \|x-y\|^2 \quad \text{for} \quad x, y \in D(A^{(k)}),\]

where $F$ is the duality map from $X$ into $X^*$, i.e., $A^{(k)}\omega I$ is dissipative; and hence the condition $R(I-h_{\xi}A^{(k)}) = X$ shows that $A^{(k)}\omega I$ is $m$-dissipative. Conversely if $A^{(k)}\omega I$ is single-valued $m$-dissipative with dense domain, then $A^{(k)}$ is the weak infinitesimal generator of a nonlinear semi-group $\{T^{(k)}(\xi) ; \xi \geq 0\}$ of local type with (1.3) (see T. Kato [2] and S. Oharu [8]).

5°. In the previous paper [6] we discussed the case of $R(I-h_{\xi}A) = X$ under slightly different conditions.

We use the recent results on nonlinear semi-groups generated by multi-valued $m$-dissipative operators, obtained by Y. Kōmura [4, 5], T. Kato [3], and M.G. Crandall and A. Pazy [1]. In § 2 we shall explain a part of their results related to ours. The proof of Theorem 1 is given in § 3.

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**§ 2. Generation of nonlinear semi-groups.**

A multi-valued operator $A$ with domain $D(A)$ and range $R(A)$ in $X$ is said to be dissipative if

\[\Re (x'-y', F(x-y)) \leq 0 \quad \text{for any} \quad x' \in Ax, y' \in Ay,\]

where $F$ is the duality map from $X$ into $X^*$. If $A$ is dissipative and $R(I-\alpha \xi A)$

\[2) \quad \text{See (3.1).}\]
In this section we shall sketch a construction and some properties of nonlinear semi-groups generated by multi-valued \(m\)-dissipative operators ([1], [3], [4] and [5]).

Throughout this section let \(\omega \geq 0\) and let \(A-\omega I\) be \(m\)-dissipative. It is obtained that the set \(Ax\) is convex and weakly closed for each \(x \in D(A)\). According to Kato [3] we define the canonical restriction \(A^0\) of \(A\) by

\[
A^0 x = \{y'; y' \in Ax \text{ and } \|y'\| = \inf \{\|x'\|; x' \in Ax\}\}
\]

for \(x \in D(A)\). Since \(X\) is reflexive and \(Ax\) is weakly closed, \(A^0 x \neq \emptyset\) for \(x \in D(A)\); so that \(A^0\) is a multi-valued dissipative operator with \(D(A^0) = D(A)\). In particular if \(X\) is strictly convex, then \(A^0\) is single-valued.

From the dissipativity of \(A-\omega I\) we get

\[
\|x-y-\alpha(x'-y')\| \geq (1-\alpha \omega) \|x-y\|
\]

for \(x' \in Ax\), \(y' \in Ay\) and \(\alpha \in (0, 1/\omega)\); and hence for each \(\alpha \in (0, 1/\omega)\) \((I-\alpha A)^{-1}\) exists as a single-valued operator defined on \(X\) and

\[
\|(I-\alpha A)^{-1}x-(I-\alpha A)^{-1}y\| \leq (1-\alpha \omega)^{-1} \|x-y\|
\]

for \(x, y \in X\). If we put

\[
J_n = (I-n^{-1}A)^{-1} \quad \text{and} \quad A_n = n(J_n-I)
\]

for \(n > \omega\), then

\[
A_n x \in AJ_n x \quad \text{for} \quad x \in X,
\]

\[
\begin{align*}
\|A_n x - A_n y\| & \leq \frac{2n-\omega}{1-n^{-1}\omega} \|x-y\| \\
\Re(A_n x - A_n y, F(x-y)) & \leq \omega(1-n^{-1}\omega)^{-1} \|x-y\|
\end{align*}
\]

for \(x, y \in X\),

\[
\|A_n x\| \leq (1-n^{-1}\omega)^{-1} \|Ax\| \quad \text{for} \quad x \in D(A),
\]

where \(\|Ax\| = \inf \{\|x'\|; x' \in Ax\}\) (we note that \(\|x'\| = \|Ax\|\) for all \(x' \in A^0 x\)), and

\[
\lim_{n \to \infty} J_n x = x \quad \text{for} \quad x \in \overline{D(A)}.
\]

It follows from (2.4) that each \(A_n\) generates a nonlinear semi-group \(\{T_n(\xi)\}; \xi \geq 0\) of local type satisfying

\[
\|T_n(\xi)x - T_n(\xi)y\| \leq \exp \left(\frac{\omega \xi}{1-n^{-1}\omega}\right) \|x-y\|
\]

for \(x, y \in X\) and \(\xi \geq 0\), and

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3) It is known that \(R(I-\alpha A) = X\) implies \(R(I-\alpha A) = X\) for all \(\alpha > 0\), if \(A\) is dissipative.
for each \( x \in X \), \( T_n(\xi)x \in C^1([0, \infty) ; X) \)

\[(d/d\xi)T_n(\xi)x = A_n T_n(\xi)x \quad \text{for} \quad \xi \geq 0\]

(for example, see the proof of Theorem 4.1 in [6]).

Notice that

\[\|A_n T_n(\xi)x\| \leq e^{\omega \xi} \|A_n x\| \leq e^{\omega \xi} \|A x\| \]

for \( x \in D(A) \) and \( \xi \geq 0 \), where \( c_n = \omega (1 - n^{-1}\omega)^{-1} \) and \( d_n = (1 - n^{-1}\omega)^{-1} \).

Let \( x \in D(A) \) and let \( z_{mn}(\xi) = T_n(\xi)x - T_m(\xi)x \). We shall now estimate \( z_{mn}(\xi) \).

Note that \( c_n \leq 2\omega \) and \( d_n \leq 2 \) for \( n > 2\omega \). In the following let \( m \) and \( n \) be integers such that \( m, n > 2\omega \). From (2.9)

\[\|z_{mn}(\xi)\| \leq \int_{\xi}^{\infty} \|A_n T_n(\tau)x - A_m T_m(\tau)x\| d\tau \]

\[\leq 4e^{2\omega \xi} \|A x\| \xi, \]

\[(2.10)\]

\[\|z_{mn}(\xi) - u_{mn}(\eta)\| \leq n^{-1} \|A_n T_n(\eta)x\| + m^{-1} \|A_m T_m(\eta)x\| \]

\[\leq \left( \frac{1}{n - \omega} + \frac{1}{m - \omega} \right) e^{2\omega \eta} \|A x\| \]

for \( \eta \geq 0 \), where \( u_{mn}(\eta) = \int_{\eta}^{\infty} T_n(\tau)x - T_m(\tau)x \); and hence

\[\|u_{mn}(\eta)\| \leq \left( \frac{1}{n - \omega} + \frac{1}{m - \omega} \right) e^{2\omega \eta} \|A x\| + \|z_{mn}(\xi)\| \]

(2.11)

Since \( \text{Re} \ (A_n T_n(\eta)x - A_m T_m(\eta)x, F(u_{mn}(\eta))) \leq \omega \|u_{mn}(\eta)\|^2 \) by (2.3),

\[\text{Re} \ (A_n T_n(\eta)x - A_m T_m(\eta)x, F(z_{mn}(\xi))) \leq \text{Re} \ (A_n T_n(\eta)x - A_m T_m(\eta)x, F(z_{mn}(\xi))) - F(u_{mn}(\eta))) + \omega \|u_{mn}(\eta)\|^2 \]

\[\leq 4e^{2\omega \xi} \|A x\| \|F(z_{mn}(\xi)) - F(u_{mn}(\eta))\| + \omega \|u_{mn}(\eta)\|^2 \]

hence

\[
\left\{
\begin{array}{ll}
\|z_{mn}(\xi)\|^2 = \int_{\xi}^{\infty} (d/d\eta) \|z_{mn}(\eta)\|^2 d\eta \\
&= 2 \int_{\xi}^{\infty} \text{Re} \ (A_n T_n(\eta)x - A_m T_m(\eta)x, F(z_{mn}(\xi))) d\eta \\
&\leq 8e^{2\omega \xi} \|A x\| \int_{\xi}^{\infty} \|F(z_{mn}(\xi)) - F(u_{mn}(\eta))\| d\eta \\
&+ 2\omega \int_{\xi}^{\infty} \|u_{mn}(\eta)\|^2 d\eta.
\end{array}
\right.

(2.13)

It follows from (2.12) and (2.13) that

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4) \( C^1([0, \infty) ; X) \) denotes the set of all strongly continuously differentiable \( X \)-valued functions on \([0, \infty)\).

5) See T. Kato [2; Lemma 1.3].
\[ \|z_{mn}(\xi)\|^2 \leq 8e^{2\omega_2} \| A\xi \| \int_0^\xi \| F(x_{mn}(\eta)) - F(u_{mn}(\eta))\| d\eta \]

\[ +4\omega \left( \frac{1}{n - \omega} + \frac{1}{m - \omega} \right)^2 e^{2\omega_2} \| A\xi \|^2 + 4\omega \int_0^\xi \| z_{mn}(\eta)\|^2 d\eta . \]

Consequently for any fixed \( \beta > 0 \) we have

\[ \|z_{mn}(\xi)\|^2 \leq K_{mn}(\beta) + 4\omega \int_0^\xi \| z_{mn}(\eta)\|^2 d\eta \]

for \( \xi \in [0, \beta] \), where

\[ K_{mn}(\beta) = 8e^{2\omega_2} \| A\xi \| \int_0^\beta \| F(x_{mn}(\eta)) - F(u_{mn}(\eta))\| d\eta \]

\[ +4\omega \left( \frac{1}{n - \omega} + \frac{1}{m - \omega} \right)^2 e^{2\omega_2} \| A\xi \|^2 \beta . \]

From this integral inequality we get

(2.14) \[ \|z_{mn}(\xi)\| \leq \sqrt{K_{mn}(\beta)} e^{2\omega_2} \quad \text{for} \quad \xi \in [0, \beta] . \]

Since \( F \) is uniformly continuous on any bounded set of \( X \) (see [2; Lemma 1.2]), (2.10) and (2.11) show that \( K_{mn}(\beta) \to 0 \) as \( m, n \to \infty \). Therefore it follows from (2.14) that

(2.15) \[ \lim_{m, n} \| T_n(\xi) x - T_m(\xi) x \| = 0 \quad \text{uniformly in} \quad \xi \in [0, \beta] . \]

By (2.7), the above (2.15) holds good for each \( x \in D(A) \).

Now we define \( \{ T(\xi); \xi \geq 0 \} \) by

(2.16) \[ T(\xi) x = \lim_{n} T_n(\xi) x \quad \text{for} \quad \xi \geq 0 \quad \text{and} \quad x \in D(A) . \]

It is clear that \( \{ T(\xi); \xi \geq 0 \} \) is a nonlinear semi-group of local type defined on \( D(A) \) such that

\[ \| T(\xi) x - T(\xi) y \| \leq e^{\omega_2} \| x - y \| \]

for \( x, y \in D(A) \) and \( \xi \geq 0 \). The following results are due to T. Kato [3].

**Theorem 2.** (I) The above \( \{ T(\xi); \xi \geq 0 \} \) is a unique semi-group of local type satisfying the following conditions;

(a) for each \( x \in D(A) \), \( T(\xi) x \) is strongly absolutely continuous on every finite interval,

(b) for each \( x \in D(A) (= D(A^\theta)) \), \( T(\xi) x \in D(A) \) for all \( \xi \geq 0 \) and

\[ (d/d\xi) T(\xi) x \in A^\theta T(\xi) x (\subset AT(\xi) x) \quad \text{for} \quad a.e. \quad \xi , \]

where \( (d/d\xi) T(\xi) x \) denotes the strong derivative of \( T(\xi) x \).

(II) In particular if \( X \) is uniformly convex, then

(c) for each \( x \in D(A) \)

\[ D^* T(\xi) x = A^\theta T(\xi) x \quad \text{for all} \quad \xi \geq 0 \]
and $A^\alpha T(\xi)x$ is strongly right-hand continuous in $\xi \geq 0$, where $D^+ T(\xi)x$ denotes the strong right-hand derivative of $T(\xi)x$,

(d) for each $x \in D(A)$ the strong derivative $(d/d\xi) T(\xi)x = A^\alpha T(\xi)x$ exists and is strongly continuous except at a countable number of values $\xi$.

REMARKS. 1°. In case of $\omega = 0$ (i.e., $A$ is $m$-dissipative), the above results have been given by T. Kato [3] (in this case, of course, $\{T(\xi) ; \xi \geq 0\}$ is a nonlinear contraction semi-group). And his results can be extended to our case (i.e., $A - \omega I$ ($\omega \geq 0$) is $m$-dissipative).

2°. In (1), if $A$ is single-valued (so that $A^0 = A$), then it is known that $A$ is the weak infinitesimal generator of $\{T(\xi) ; \xi \geq 0\}$ and for each $x \in D(A)$ $AT(\xi)x$ is weakly continuous in $\xi \geq 0$ (see T. Kato [2] and S. Oharu [8]).

§ 3. Proof of Theorem 1.

For $x, y \in D(A^{(k)})$,

$$\text{Re } (A^{(k)}x - A^{(k)}y, F(x - y)) = \lim_{\xi \to 0} \left( \xi^{-1} [T^{(k)}(\xi)x - x] - \xi^{-1} [T^{(k)}(\xi)y - y], F(x - y) \right) \leq \omega \|x - y\|^2;$$

this shows that $A^{(k)} - \omega I$ are dissipative. Moreover it follows from the assumption $R(I - h_k A^{(k)}) = X$ that $R(I - \alpha_k (A^{(k)} - \omega I)) = X$ for each $k$, where $\alpha_k = h_k (1 - h_k \omega)^{-1}$. Thus we have

(3.1) $A^{(k)} - \omega I$ are $m$-dissipative.

Fix $k$. From the arguments in § 2, for each $n > \omega$

(3.2) $J_n^{(k)} = (I - n^{-1} A^{(k)})^{-1}$ exists and

$$\|J_n^{(k)} x - J_n^{(k)} y\| \leq (1 - n^{-1} \omega)^{-1} \|x - y\| \quad \text{for } x, y \in X,$$

and if we put

(3.3) $A_n^{(k)} = n (J_n^{(k)} - I) (= A^{(k)} J_n^{(k)}$, because $A^{(k)}$ is single-valued),

then

(3.4) $A_n^{(k)}$ is the infinitesimal generator of a nonlinear semi-group

$$\{T_n^{(k)}(\xi) ; \xi \geq 0\} \text{ of local type such that } \|T_n^{(k)}(\xi)x - T_n^{(k)}(\xi)y\| \leq \exp \left( \frac{\omega \xi}{1 - n^{-1} \omega} \right) \|x - y\| \text{ for } x, y \in X \text{ and } \xi \geq 0;$$

and for each $x \in D(A^{(k)})$

(3.5) $T_n^{(k)}(\xi)x = \lim_n T_n^{(k)}(\xi)x \quad \text{for } \xi \geq 0$.

Let $x \in D(A^{(k)})$ and put

$$z_{mn}^{(k)}(\eta) = T_m^{(k)}(\eta)x - T_m^{(k)}(\eta)x \quad \text{for } \eta \geq 0,$$

where $m$ and $n$ are integers such that $m, n > 2\omega$.

From (2.10), (2.12) and (2.14)
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\begin{equation}
\|z^{(k)}_{mn}(\eta)\| \leq 4e^{2\omega\nu}\|A^{(k)}x\| \eta \quad \text{for } \eta \geq 0,
\end{equation}

\begin{equation}
\|z^{(k)}_{mn}(\eta) - u^{(k)}_{mn}(\eta)\| \leq \left(\frac{1}{n-\omega} + \frac{1}{m-\omega}\right)e^{2\omega\nu}\|A^{(k)}x\|
\end{equation}

for \(\eta \geq 0\), where \(u^{(k)}_{mn}(\eta) = f^{(k)}_{n}T^{(k)}_{m}(\eta)x - f^{(k)}_{m}T^{(k)}_{m}(\eta)x\), and

\begin{equation}
\|z^{(k)}_{mn}(\xi)\| \leq \sqrt{K^{(k)}_{mn}(\beta)}e^{2\omega\xi} \quad \text{for } \xi \in [0, \beta],
\end{equation}

where

\begin{equation}
K^{(k)}_{mn}(\beta) = 8e^{2\omega\beta}\|A^{(k)}x\| \int_{0}^{\beta} \|F(z^{(k)}_{mn}(\eta)) - F(u^{(k)}_{mn}(\eta))\| \, d\eta
+ 4\omega\left(\frac{1}{n-\omega} + \frac{1}{m-\omega}\right)^{2} e^{2\omega\beta}\|A^{(k)}x\|^{2}\beta.
\end{equation}

(We note that \(\|A^{(k)}x\| = \|A^{(k)}x\|\) because \(A^{(k)}\) is single-valued.)

From the above estimations we have the following

**Lemma 1.** Let \(\beta > 0\). For each \(x \in D(A)\) the convergence (3.5) is uniform with respect to \(k\) and \(\xi \in [0, \beta]\).

**Proof.** Let \(x \in D(A)\). Since \(\lim_{k} A^{(k)}x = Ax\), there exist \(k_{0}\) and \(M > 0\) such that \(x \in D(A^{(k)})\) and \(\|A^{(k)}x\| \leq M\) for \(k \geq k_{0}\). It follows from (3.6) and (3.7) that the set

\[B = \{z^{(k)}_{mn}(\eta), u^{(k)}_{mn}(\eta); \eta \in [0, \beta], k \geq k_{0}\text{ and } m, n > 2\omega\}

is bounded. Since \(F\) is uniformly continuous on \(B\), for every \(\varepsilon > 0\) there is \(\delta = \delta_{0} > 0\) such that \(z, u \in B\) and \(\|z - u\| < \delta\) imply \(\|F(z) - F(u)\| < 2^{-1}Ke^{2}\), where \(K = (8e^{2\omega\beta}M\beta)^{-1}\). Choose an integer \(N (= N_{0})\) such that \(N > 2\omega\) and \(2(N - \omega)^{-1}e^{2\omega\beta}M \leq \min (\delta, \varepsilon/\sqrt{8\omega\beta})\).

Let \(m, n > N\). By (3.7)

\[\|z^{(k)}_{mn}(\eta) - u^{(k)}_{mn}(\eta)\| < 2(N - \omega)^{-1}e^{2\omega\beta}M \leq \delta\]

for \(\eta \in [0, \beta]\) and \(k \geq k_{0}\), so that

\[\|F(z^{(k)}_{mn}(\eta)) - F(u^{(k)}_{mn}(\eta))\| < 2^{-1}Ke^{2}\]

for \(\eta \in [0, \beta]\) and \(k \geq k_{0}\). Hence

\[8e^{2\omega\beta}\|A^{(k)}x\| \int_{0}^{\beta} \|F(z^{(k)}_{mn}(\eta)) - F(u^{(k)}_{mn}(\eta))\| \, d\eta
\leq 8e^{2\omega\beta}M\beta 2^{-1}Ke^{2} = e^{2}/2,
\]

and

\[4\omega\left(\frac{1}{n-\omega} + \frac{1}{m-\omega}\right)^{2} e^{2\omega\beta}\|A^{(k)}x\|^{2}\beta
\leq 4\omega\left(\frac{2}{N - \omega} e^{2\omega\beta}M\right)^{2}\beta \leq \varepsilon^{2}/2.
\]

Consequently \(K^{(k)}_{mn}(\beta) \leq \varepsilon^{2}\) for \(k \geq k_{0}\). Therefore it follows from (3.8) that
Since \( A^{(b)} - \omega I \) are dissipative (see (3.1)), the limit operator \( A - \omega I \) is also dissipative. Combining this and \( R(I - \varepsilon_0 A) = X \) (the assumption (b)) we have the following.

**Lemma 2.** For each \( n > \omega \)

\[(I - n^{-1}A)^{-1} \text{ has a unique extension } J_n \text{ defined on } X\]

such that \( \|J_n x - J_n y\| \leq (1-n^{-1}\omega)^{-1}\|x-y\| \) for \( x, y \in X \), and

\[(3.12) \quad A - \omega I \text{ is } m\text{-dissipative and } J_n = (I - n^{-1}A)^{-1}.\]

**Proof.** At first we remark that

\[(3.13) \quad R(I - n^{-1}A) = X \quad \text{for all } n > \omega \quad (\text{see S. Oharu [7; Lemma 4])}.\]

From the dissipativity of \( A - \omega I \), for each \( n > \omega \) \( (I - n^{-1}A)^{-1} \) exists and

\[\|(I - n^{-1}A)^{-1} x - (I - n^{-1}A)^{-1} y\| \leq (1-n^{-1}\omega)\|x-y\|\]

for \( x, y \in R(I - n^{-1}A) \). Thus (3.11) follows from (3.13).

We shall now prove (3.12). Let \( m, n > \omega \). For \( x \in R(I - m^{-1}A) \)

\[(I - n^{-1}A)(I - m^{-1}A)^{-1} x = (1-m/n)(I - m^{-1}A)^{-1} x + (m/n)x,\]

so that

\[(I - m^{-1}A)^{-1} x = (1-n^{-1}A)^{-1}((1-m/n)(I - m^{-1}A)^{-1} x + (m/n)x)\]

i.e.,

\[J_m x = J_n((1-m/n)J_m x + (m/n)x)\]

for \( x \in R(I - m^{-1}A) \). From \( R(I - m^{-1}A) = X \) we have

\[(3.14) \quad J_m x = J_n((1-m/n)J_m x + (m/n)x) \quad \text{for all } x \in X.\]

Consequently

\[(3.15) \quad R(J_n) = R(J_m),\]

\[(3.16) \quad n(x - J_n^{-1}x) = m(x - J_m^{-1}x) \quad \text{for } x \in D,\]

where \( D \) is the set \( R(J_n) \) independent of \( n > \omega \) and \( J_n^{-1} \) are multi-valued mappings defined by \( J_n^{-1} x = \{y : J_n y = x\} \) (see S. Oharu [7; Lemma 6]).

Define \( \tilde{A} \) by

\[(3.17) \quad \tilde{A} x = n(x - J_n^{-1}x) \quad \text{for } x \in D.\]

It is easy to see that \( \tilde{A} \supseteq A \) (i.e., \( D \supseteq D(A) \)) and \( \tilde{A} x \supseteq Ax \) for \( x \in D(A) \) and the graph \( G(\tilde{A}) \) of \( \tilde{A} \) is closed. Hence \( G(\tilde{A}) \supseteq G(A) \). Moreover \( G(\tilde{A}) \subseteq G(A) \). In fact, let \( y \in \tilde{A} x \). There is \( x' \in X \) such that \( x = J_n x' \) and \( y = n(x - x') \). Since
\( R(I^{-n-1}A) = X \), there exists a sequence \( \{x_k\} \) in \( D(A) \) such that \( (I^{-n-1}A)x_k \rightarrow x' \) as \( k \rightarrow \infty \). Hence

\[
x_k = J_n(I^{-n-1}A)x_k \rightarrow J_n x' = x, \quad \text{and} \quad Ax_k \rightarrow n(x-x') = y.
\]

Thus \( G(\tilde{A}) = \overline{G(A)} \) i.e., \( \tilde{A} = \overline{A} \) (the strong closure of \( A \)). And then we get \( J_n = (I^{-n-1}A)^{-1} \).

Finally we shall prove that \( \tilde{A} - \omega I \) is \( m \)-dissipative. For \( x' \in \mathcal{A}x \) and \( y' \in \mathcal{A}y \) there exist \( \{x_k\} \) and \( \{y_k\} \) in \( D(A) \) such that \( x_k \rightarrow x, \ Ax_k \rightarrow x' \) and \( y_k \rightarrow y, \ Ay_k \rightarrow y' \). Since \( \text{Re} \left( (A - \omega I)x_k - (A - \omega I)y_k, F(x_k - y_k) \right) \leq 0 \), it follows from the continuity of \( F \) that

\[
\text{Re} \left( (x'-\omega x)-(y'-\omega y), F(x-y) \right) \leq 0.
\]

This shows that \( \tilde{A} - \omega I \) is dissipative. From \( R(I^{-n-1}A) = X \) for \( n > \omega \) we have \( R(I^{-\alpha}(\tilde{A} - \omega I)) = X \) for \( \alpha > 0 \). Thus \( \tilde{A} - \omega I \) is \( m \)-dissipative. Q.E.D.

**Remark.** The above lemma is also true for multi-valued operators; i.e., if \( A - \omega I \) is multi-valued dissipative and if \( R(I^{-h_0}A) = X \) for some \( h_0 \in (0, 1/\omega) \), then \( R(I^{-h}A) = X \) for all \( h \in (0, 1/\omega) \) and (3.11) and (3.12) hold good.

By Lemma 2 and Theorem 2, \( \tilde{A} \) generates a nonlinear semi-group \( \{T(\xi); \ \xi \geq 0\} \) of local type; and for each \( x \in X \)

\[
(3.18) \quad T(\xi)x = \lim_{n \uparrow} T_n(\xi)x
\]

uniformly with respect to \( \xi \) in every finite interval, where \( \{T_n(\xi); \ \xi \geq 0\} \) is a nonlinear semi-group of local type generated by \( A_n = n(J_n-I) \) and

\[
(3.19) \quad \|T_n(\xi)x - T_n(\xi)y\| \leq \exp \left( \omega_n^\xi \right) \|x - y\|
\]

for \( x, y \in X \) and \( \xi \geq 0 \).

We shall show

\[
(3.20) \quad \lim_{k} J_n^{(k)}x = J_n x \quad \text{for} \quad x \in X \quad \text{and} \quad n.
\]

In fact, for \( y = (I^{-n-1}A)x \)

\[
\|J_n^{(k)}y - J_n y\| = \|J_n^{(k)}(I^{-n-1}A^{(k)})x\| \\
\leq (1-n^{-1})^{-1}\|y - (I^{-n-1}A^{(k)})x\| \\
= n^{-1}(1-n^{-1})^{-1}\|A^{(k)}x - Ax\| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.
\]

Then (3.20) follows from \( R(I^{-n-1}A) = X \) (see (3.13)). Hence

\[
(3.21) \quad \lim_{k} A_n^{(k)}x = A_n x \quad \text{for} \quad x \in X \quad \text{and} \quad n.
\]

6) From the assumption (a) \( (D(A) \) is dense in \( X \), \( \{T(\xi); \ \xi \geq 0\} \) is defined on \( X \).
Since each $A_n$ is Lipschitz continuous uniformly in $x \in X$ (see (2.4)), we have

$$(3.22) \quad R(I-hA_n) = X \quad \text{for sufficiently small } h > 0.$$  

(This is really true for $h \in (0, (1/\omega)/n')$.)

Consequently, by Theorem 2.3 in [6], for each $n$ we have

$$(3.23) \quad \sup_{0 < t_2 < \beta} \| T_{n}^{(k)}(\xi)x - T_{n}(\xi)x \| \to 0 \quad \text{(as } k \to \infty)$$

for any $\beta > 0$ and $x \in X$.

We can now prove the convergence (1.4). Let $\beta > 0$ be arbitrarily fixed, and let $x \in D(A)$. From Lemma 1 and (3.18), for each $\varepsilon > 0$ there is an integer $N (= N_\varepsilon)$ such that

$$\sup_{0 < t_2 < \beta} \| T_{n}^{(k)}(\xi)x - T_{n}(\xi)x \| < \varepsilon/2 \quad \text{for } n > N \text{ and } k,$$

$$\sup_{0 < t_2 < \beta} \| T_{n}(\xi)x - T(\xi)x \| < \varepsilon/2 \quad \text{for } n > N.$$  

Thus for $n > N$ and $k$

$$\sup_{0 < t_2 < \beta} \| T_{n}^{(k)}(\xi)x - T(\xi)x \| < \varepsilon + \sup_{0 < t_2 < \beta} \| T_{n}^{(k)}(\xi)x - T_{n}(\xi)x \|.$$  

Going $k \to \infty$, it follows from (3.23) that

$$(3.24) \quad \sup_{0 < t_2 < \beta} \| T_{n}^{(k)}(\xi)x - T(\xi)x \| \to 0 \quad \text{as } k \to \infty.$$  

Finally, by the stability condition (1.3) and $D(A) = X$, (3.24) holds good for every $x \in X$. This completes the proof of Theorem 1.

Georgetown University, Washington, D.C.
and
Waseda University, Tokyo

References