On the theory of commutative formal groups

By Taira Honda

(Received Nov. 10, 1969)

The theory of (commutative) formal groups was initiated by M. Lazard and J. Dieudonné around 1954. Lazard [11], [12] studied commutative formal groups over an arbitrary commutative ring by treating the coefficients of power series explicitly. Whereas Dieudonné investigated formal groups over a field of characteristic $p > 0$ exclusively. He reduced in [4] the study of commutative formal groups over a perfect field of characteristic $p > 0$ to that of modules over a certain non-commutative ring, so-called Dieudonné modules, and obtained in [5] a complete classification of isogeny classes of commutative formal groups over an algebraically closed field of characteristic $p > 0$. Later Manin [16] studied isomorphism classes of simple formal groups. The study of one-dimensional formal groups over $p$-adic integer rings was begun by Lubin [13] and a number of interesting results were obtained by him and Tate.

In this paper we first construct a certain general family of commutative formal groups of arbitrary dimension over a $p$-adic integer ring. Over the ring $W(k)$ of Witt vectors over a perfect field of characteristic $p > 0$, this exhausts all the commutative formal groups. These are attached to a certain type of matrices with elements in the ring $W(k)_{\sigma}[[T]]$ of non-commutative power series, where $\sigma$ is the Frobenius of $W(k)$, and homomorphisms of these formal groups are described in terms of matrices over $W(k)_{\sigma}[[T]]$. By reducing the coefficients of formal groups over $W(k)$ mod $pW(k)$ we get formal groups over $k$. It is shown that all the commutative formal groups over $k$ are obtained in this manner. Moreover homomorphisms of commutative formal groups over $k$ are also described in terms of $W(k)_{\sigma}[[T]]$-modules by lifting these homomorphisms to power series over $W(k)$. Thus we get the main results of Dieudonné [4] again by the method quite different from his. In [4] he used tools peculiar to characteristic $p > 0$ and his construction of formal groups was indirect, whereas in our method the relation between formal groups over $W(k)$ and those over $k$ is transparent and the construction of formal groups is explicit and elementary.

We now explain briefly how to construct commutative formal groups over $W(k)$ in case of dimension one. Take an element $u$ of $W(k)_{\sigma}[[T]]$ of the
form \( p + \sum_{v=1}^{\infty} c_v T^v \) (\( c_v \in W(k) \)) and put \( p u^{-1} = \sum_{v=0}^{\infty} b_v T^v \). The \( b_v \) are elements of the fraction field of \( W(k) \) and \( b_0 = 1 \). Form \( f(x) = \sum_{v=0}^{\infty} b_v x^{p^v} \) and \( F(x, y) = f^{-1}(f(x) + f(y)) \). Then \( F \) is a formal group over \( W(k) \). In some special case this fact can be proved by using the basic lemma of Lubin-Tate [14] (cf. [10]). In general case we have to adopt another idea. Any formal group over \( W(k) \) is isomorphic to one obtained in this manner. Let \( v \) be another element of \( W(k)[[T]] \) of the form mentioned above and let \( g(x) \) and \( G(x, y) \) be the corresponding power series and the formal group, respectively. It is known that any homomorphism of \( F \) to \( G \) is of the form \( g^{-1}(cf(x)) \) with \( c \in W(k) \). We assert that \( g^{-1}(cf(x)) \) is in reality a homomorphism over \( W(k) \), if and only if there is \( t \in W(k)[[T]] \) such that \( vc = tu \). All these results will be generalized and proved for an arbitrary dimension and for more general coefficient rings of characteristic 0 with discrete valuation.

Our results can be applied to construct and characterize formal groups over \( \mathbb{Z} \) corresponding to a certain type of Dirichlet series with matrix coefficients, thus generalizing the results of the last half of our previous paper [10]. In particular we get an interesting interpretation of the Dirichlet series obtained from a representation of Hecke operators in the space of cusp forms of dimension \(-2\) with respect to a congruence unit group \( \Gamma_N \) of a maximal order of an indefinite quaternion algebra over \( \mathbb{Q} \) (Shimura [19]). There is an intimate connection between this Dirichlet series and a formal completion of the Jacobian \( J_N \).

\section{Invariant differential forms on a formal group.}

1.1. Let \( S \) be a ring. We denote by \( S^m \) the module consisting of all the column vectors of dimension \( m \) with components in \( S \) and by \( M_m(S) \) the full matrix ring of order \( m \) with elements in \( S \). \( I_m \) denotes the identity matrix of order \( m \). For \( a = (a_1, \ldots, a_m) \in S^m \) we write \( a^\nu \) for \( (a_1^\nu, \ldots, a_m^\nu) \).

Let \( R \) be a commutative ring with the identity. Let \( x \) be the set of \( n \) variables \( x_1, \ldots, x_n \). We denote by \( R[[x]] \) the ring of formal power series on \( x_1, \ldots, x_n \). For basic properties of \( R[[x]] \) we refer to Bourbaki [3]. We shall often regard \( x \) as the column vector \( (x_1, \ldots, x_n) \) in \( R[[x]]^n \). Let \( f \) and \( g \) be power series in \( R[[x]] \). We shall say that \( f \) is congruent to \( g \) modulo degree \( r \), \( f \equiv g \mod \deg r \), if \( f \) and \( g \) differ only in terms of total degree \( \geq r \). Let \( I \) be a submodule of \( R \). \( f \) is said to be congruent to \( g \) modulo \( I \), \( f \equiv g \mod I \), if all the coefficients of \( f-g \) belong to \( I \). We shall write \( f \equiv g \mod \deg r \), \( \mod I \), if there are \( \varphi, \phi \in R[[x]] \) such that \( f-g = \varphi + \phi \), \( \varphi \equiv 0 \mod \deg r \) and \( \phi \equiv 0 \mod I \). These definitions extend to \( R[[x]]^m \). If \( f = (f_1, \ldots, f_m) \) and
The theory of commutative formal groups

Let $g = (g_1, \ldots, g_m)$ be elements of $R[[x]]^m$, $f \equiv g \mod \ast$ will mean $f_i \equiv g_i \mod \ast$ for $1 \leq i \leq n$. We write $R[[x]]^n = \{ f \in R[[x]]^m | f \equiv 0 \mod \deg 1 \}$.

Let $x' = (x'_1, \ldots, x'_m)$ be another set of variables. If $f(x') = (f_1(x'), \ldots, f_m(x'))$ $(f_i(x') = f_i(x'_1, \ldots, x'_m))$ is in $R[[x']]^m$ and $\varphi(x) = (\varphi_1(x), \ldots, \varphi_m(x))$ is in $R[[x]]^m$, the power series $f_i(\varphi(x)) = f_i(\varphi_1(x), \ldots, \varphi_m(x))$ is well-defined and $(f_i(\varphi(x)), \ldots, f_i(\varphi(x)))$ is an element of $R[[x]]^m$. We denote it by $f(\varphi(x))$ or simply by $f \circ \varphi$, if there is no fear of ambiguity. Define the identity function $i$ of $R[[x]]^m$ by $i(x) = x$. If $\varphi(x)$ is an element of $R[[x]]^m$ such that $\varphi(x) \equiv P x \mod \deg 2$ with an invertible matrix $P$ in $M_n(R)$, there is a unique element $\psi(x)$ in $R[[x]]^m$ satisfying $\varphi \circ \psi = \psi \circ \varphi = i$. We shall call this $\psi$ the inverse function of $\varphi$ and denote it by $\varphi^{-1}$.

We adopt the classical definition of formal group.

**Definition.** Let $x$ and $y$ be sets (or vectors) of $n$ variables. An $n$-dimensional formal group over $R$ is an element $F(x, y)$ of $R[[x, y]]$ satisfying:

i) $F(x, y) \equiv x + y \mod \deg 2$,

ii) $F(F(x, y), z) = F(x, F(y, z))$.

If $F$ satisfies $F(x, y) = F(y, x)$ moreover, $F$ is said to be commutative.

It follows from (i) that there is a unique $i_F(x) \in R[[x]]^m$ such that $F(x, i_F(x)) = F(i_F(x), x) = 0$. Part (ii) shows that $F(x, 0) = x$ and $F(0, y) = y$.

**Definition.** Let $F$ and $G$ be formal groups over $R$, of dimension $n$ and $m$, respectively. An element $\varphi$ of $R[[x]]^m$, where $x = (x_1, \ldots, x_n)$, is said to be a homomorphism of $F$ to $G$, if $\varphi$ satisfies $\varphi \circ F = G \circ \varphi$, where $(G \circ \varphi)(x, y)$ stands for $G(\varphi(x), \varphi(y))$. If $m = n$ and $\varphi$ is invertible, $\varphi^{-1}$ is also a homomorphism of $G$ to $F$. Such $\varphi$ is called an isomorphism and $G$ is said to be (weakly) isomorphic to $F$, $F \sim G$ over $R$. If there is an isomorphism $\varphi$ of $F$ to $G$ such that $\varphi(x) \equiv x \mod \deg 2$, we shall say that $G$ is strongly isomorphic to $F$ and write $\varphi : F \approx G$ over $R$.

If $G$ is commutative, the set $\text{Hom}_R(F, G)$ of all homomorphisms of $F$ to $G$ over $R$ forms a module by defining $(\varphi_1 + \varphi_2)(x) = G(\varphi_1(x), \varphi_2(x))$ for $\varphi_1, \varphi_2 \in \text{Hom}_R(F, G)$. In particular $\text{End}_R G = \text{Hom}_R (G, G)$ becomes a ring by defining the multiplication by composition of functions.

1.2. Let $A = R[[x]]$ be as in 1.1. We denote by $\mathfrak{D}(A; R)$ the space of derivations of $A$ over $R$. It is a free left $A$-module with a base $D_1, \ldots, D_n$, where $D_i = \partial / \partial x_i$ (cf. [3]). Denote by $\mathfrak{D}^{\ast}(A; R)$ the dual $A$-module of $\mathfrak{D}(A; R)$, the space of differentials of $A$ over $R$. For $f \in A$ we map $D \mapsto Df$ of $\mathfrak{D}(A; R)$ into $A$ defines a differential, which we denote by $df$. A differential of this form is called exact. It is well-known that $dx = (dx_1, \ldots, dx_n)$ is an $A$-base of $\mathfrak{D}^{\ast}(A; R)$ and $df = \sum_{i=1}^n (D_if)dx_i$ for any $f \in A$.
Let $B = R[[x']]$ be another ring of power series on $m$ variables and let
\[ \omega = \sum_{j=1}^{m} \phi_j(x')dx'_j \]
be a differential in $\mathfrak{D}^*(B; R)$. If $\varphi \in R[[x]]^m$, $\sum_{j=1}^{m} \phi_j(\varphi(x))d\varphi_j(x)$
is a differential in $\mathfrak{D}^*(A; R)$. We denote it by $\varphi^*(\omega)$. $\varphi^*$ is an $R$-homomorphism of
$\mathfrak{D}^*(B; R)$ into $\mathfrak{D}^*(A; R)$.

Let $F$ be an $n$-dimensional formal group over $R$. Introducing a new set
$t = (t_1, \ldots, t_n)$ of variables we may consider that $F$ is also defined over $R_t = R[t]$.

**Definition.** The right translation $T_t$ on $F$ is an element of $R_t[[x]]^n$
defined by $T_t(x) = F(x, t)$. A differential $\omega$ in $\mathfrak{D}^*(A; R)$ is said to be a right
invariant differential on $F$ if $T_t^*(\omega) = \omega$.

We denote by $\mathfrak{D}^*(F; R)$ the space consisting of all right invariant differen-
tials on $F$. As in the case of a Lie group or an algebraic group, we
have:

**Proposition 1.1.** If $F$ is an $n$-dimensional formal group over $R$, $\mathfrak{D}^*(F; R)$
is a free $R$-module of rank $n$. More precisely, $(\psi_{ij}(z))$ denoting the inverse
matrix of $(\partial/\partial x_j)F_i(0, z)$, we have $\psi_{ij}(0) = \delta_{ij}$ and
$\omega_i = \sum_{j=1}^{n} \psi_{ij}(a)dx_j$ ($1 \leq i \leq n$)
form an $R$-basis of $\mathfrak{D}^*(F; R)$. Moreover the base \{\omega_1, \ldots, \omega_n\} is characterized
by these two properties.

**Proof.** Differentiating $F_i(u, F(v, w)) = F_i(F(u, v), w)$ relative to $u_j$, we get

\[ \frac{\partial}{\partial x_j}F_i(u, F(v, w)) = \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_k}F_i(F(u, v), w) \right) \left( \frac{\partial}{\partial x_j}F_k(u, v) \right), \]

so that

\[ \frac{\partial}{\partial x_j}F_i(0, F(v, w)) = \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_k}F_i(v, w) \right) \left( \frac{\partial}{\partial x_j}F_k(0, v) \right) \]

or by matrix notation

\[ (\partial/\partial x_j)F_i(0, F(v, w)) = (\partial/\partial x_j)F_i(F(v, w))(\partial/\partial x_k)F_k(0, v)). \]

Since \( (\partial/\partial x_j)F_i(0, z) \equiv \delta_{ij} \mod \deg 1 \), the matrix \((\partial/\partial x_j)F_i(0, z)\) is invertible,
\( \phi_{ij}(z) \in R[[z]] \) and $\phi_{ij}(0) = \delta_{ij}$. Hence (1.1) is equivalent to

\[ (T_t \psi_{ij}(z))(\partial/\partial x_j)F_i(z, t) = (\psi_{ij}(z)). \]

Now a differential $\omega = \sum_{i=1}^{n} \phi_i(x)dx_i$ in $\mathfrak{D}^*(A; R)$ is right invariant on $F$, if and
only if

\[ \phi_j(x) = \sum_{k=1}^{n} \phi_k(F(x, t))(\partial/\partial x_j)F_k(x, t). \]

This shows $\omega_1, \ldots, \omega_n \in \mathfrak{D}^*(F; R)$ by (1.2). On the other hand we get from

(1.3) \[ \phi_j(0) = \sum_{k=1}^{n} \phi_k(t)(\partial/\partial x_j)F_k(0, t), \]

which implies that, if \( \omega \in \mathfrak{X}^*(F; R) \), \( \omega = 0 \Leftrightarrow \phi_i(0) = 0 \) for \( 1 \leq i \leq n \). Therefore the map \( \Phi: \omega \mapsto (\phi_1(0), \cdots, \phi_n(0)) \) defines an \( R \)-isomorphism of \( \mathfrak{X}^*(F; R) \) into \( R^n \). Since the \( \Phi(\omega_i) \) (\( 1 \leq i \leq n \)) are the unit vectors of \( R^n \), the map \( \Phi \) is surjective and \( \{ \omega_1, \cdots, \omega_n \} \) is a base of \( \mathfrak{X}^*(F; R) \).

We shall call this \( \{ \omega_1, \cdots, \omega_n \} \) the canonical base of \( \mathfrak{X}^*(F; R) \).

**Proposition 1.2.** Let \( F, G \) be formal groups over \( R \) and \( \varphi \in \text{Hom}_R(F, G) \). If \( \eta \in \mathfrak{X}^*(G; R) \), then \( \varphi^*(\eta) \in \mathfrak{X}^*(F; R) \).

**Proof.** Write \( \eta = \sum_{i=1}^m \psi_i(x')dx_i \) where \( m \) is the dimension of \( G \). Then

\[
T_i(\varphi^*(\eta)) = T_i\left( \sum_{i=1}^m \psi_i(\varphi(x))dx_i(x) \right)
= \sum_{i=1}^m \psi_i(\varphi(F(x, t)))d\varphi_i(F(x, t))
= \sum_{i=1}^m \psi_i(G(\varphi(x), \varphi(t)))dG(\varphi_i(x), \varphi_i(t))
= \sum_{i=1}^m \psi_i(\varphi(x))d\varphi_i(x)
= \varphi^*(\eta).
\]

1.3. We now study invariant differential forms on a commutative formal group.

**Proposition 1.3.** Let \( F \) be a commutative formal group over \( R \). Then every differential in \( \mathfrak{X}(F; R) \) is closed.

**Proof.** Let \( \omega_i = \sum_{j=1}^n \psi_{ij}(x)dx_j \) (\( 1 \leq i \leq n \)) be the canonical base of \( \mathfrak{X}^*(F; R) \). We shall prove \( d\omega_i = 0 \) for \( 1 \leq i \leq n \). First \( d\omega_i \) is a right invariant 2-form, since

\[
T_i^*(d\omega_i) = T_i^*\left( \sum_{j=1}^n d\psi_{ij}(x) \wedge dx_j \right)
= \sum_j d\psi_{ij}(F(x, t)) \wedge dF_j(x, t)
= d(T_i^*(\omega_i))
= d\omega_i.
\]

Now differentiating

\[
\sum_{k=1}^n \frac{\partial}{\partial z_k} F_i(0, z) \psi_{kj}(z) = \delta_{ij}
\]
relative to \( z_i \) and putting \( z = 0 \), we get

\[
\sum_k (\frac{\partial^2}{\partial x_k \partial y_i}) F_i(0, 0) \psi_{kj}(0) + \sum_k (\frac{\partial}{\partial x_k}) F_i(0, 0) (\frac{\partial}{\partial x_i}) \psi_{kj}(0) = 0,
\]
which is reduced to
\[
(\partial^2/\partial x_j \partial y_j)F_i(0, 0) + (\partial/\partial x_j)\phi_{ij}(0) = 0 ,
\]
since
\[
\phi_{kj}(0) = \delta_{kj} \quad \text{and} \quad (\partial/\partial x_k)F_i(0, 0) = \delta_{ik} .
\]
Hence, by the commutativity of \( F \) we get
\[
(\partial/\partial x_j)\phi_{ij}(0) = -(\partial^2/\partial x_j \partial y_j)F_i(0, 0)
= -(\partial^2/\partial x_j \partial y_j)F_i(0, 0)
= (\partial/\partial x_j)\phi_{ij}(0) .
\]
Since
\[
d\omega_i = \sum_{j \neq i} (\partial/\partial x_j)\phi_{ij}(x)dx_i \wedge dx_j
= \sum_{j < i} ((\partial/\partial x_i)\phi_{ij}(x)-(\partial/\partial x_j)\phi_{ij}(x))dx_i \wedge dx_j,
\]
the coefficients of \( dx_i \wedge dx_j \) in \( d\omega_i \) have no constant term. So we have only to prove that, if \( \eta = \sum_{i < j} \lambda_{ij}(x)dx_i \wedge dx_j \) is right invariant on \( F \) and \( \lambda_{ij}(0) = 0 \) for all \( 1 \leq i < j \leq n \), \( \eta \) must be equal to 0. An easy computation shows that \( T^*_\eta(\eta) = \eta \) is equivalent to
\[
\lambda_{kl}(x) = \sum_{i < j} \lambda_{ij}(F(x, t)) \left| \begin{array}{cc}
(\partial/\partial x_k)F_i(x, t) & (\partial/\partial x_l)F_i(x, t) \\
(\partial/\partial x_l)F_j(x, t) & (\partial/\partial x_k)F_j(x, t)
\end{array} \right|,
\]
which implies
\[
\lambda_{kl}(0) = \sum_{i < j} \lambda_{ij}(t) \left| \begin{array}{cc}
(\partial/\partial x_k)F_i(0, t) & (\partial/\partial x_l)F_i(0, t) \\
(\partial/\partial x_l)F_j(0, t) & (\partial/\partial x_k)F_j(0, t)
\end{array} \right|,
\]
for \( 1 \leq k < l \leq n \). Since the matrix \( (\partial/\partial x_j)F_i(0, t) \) is regular, this shows in fact \( \lambda_{ij}(0) = 0 \) for all \( i < j \Rightarrow \lambda_{ij}(t) = 0 \) for all \( i < j \).

We now consider the case where \( R \) is a \( \mathbb{Q} \)-algebra. In this case every power series in \( R[[x]] \) is termwise integrable with respect to \( x_i \). The following lemma is essentially well-known in elementary analysis and the proof is easy.

**Lemma 1.4.** If \( R \) is a \( \mathbb{Q} \)-algebra, a closed differential in \( \mathfrak{D}^*(A ; R) \) is exact.

The following theorem, mentioned in [10], was also proved in [7] in a slightly different manner.

**Theorem 1.** Let \( F \) be an \( n \)-dimensional commutative formal group over a \( \mathbb{Q} \)-algebra \( R \) and let \( \omega = \langle \omega_1, \cdots, \omega_n \rangle \) be the canonical base of \( \mathfrak{D}^*(F; R) \). Then there exists a unique element \( f \) of \( R[[x]] \) such that \( \omega = df \). This \( f \) satisfies
\[
f(x) \equiv x \mod \deg 2
\]
and
\[
F(x, y) = f^{-1}(f(x)+f(y)) .
\]
In particular \( F(x, y) = x+y \) over \( R \).
The theory of commutative formal groups

219

PROOF. The existence of \( f \) follows from Proposition 1.3 and Lemma 1.4. The uniqueness follows from the fact that \( d\varphi = 0 \) for \( \varphi \in R[[x]] \), if and only if \( \varphi \) is a constant. Since \( \varphi_t(0) = \delta_t \), we have \( f(x) \equiv x \mod \deg 2 \). Now, \( df(x) \) being right invariant, we have

\[
df(F(x, t)) = df(x),
\]

which implies

\[
f(F(x, t)) - f(x) \in R[[t]].
\]

Writing \( g(t) = f(F(x, t)) - f(x) \) and putting \( x = 0 \) we get

\[
g(t) = f(t).
\]

Thus we have

\[
f(F(x, t)) = f(x) + f(t)
\]

or

\[
F(x, t) = f^{-1}(f(x) + f(t)).
\]

This completes the proof of our theorem.

1.4. Let \( R \) be an integral domain of characteristic 0 and \( K \) its fraction field.

LEMMA 1.5. Let \( x = t(x_1, \ldots, x_n) \) and \( y = t(y_1, \ldots, y_n) \) be sets of \( n \) variables. If \( \varphi \in K[[x]]^m \) satisfies

\[
\varphi(x + y) = \varphi(x) + \varphi(y),
\]

\( \varphi \) must be linear, i.e. there is an \( m \times n \) matrix \( C \) over \( K \) such that \( \varphi(x) = Cx \).

PROOF. We have only to consider the case where \( m = 1 \) and \( \varphi \) is a homogeneous polynomial. Then our assertion is verified by a simple computation. (See the proof of Lemma 3.2)

Let \( F \) be a commutative formal group over \( R \), of dimension \( n \). By Theorem 1 there is \( f(x) \in K[[x]]^n \) such that \( f \equiv i \mod \deg 2 \) and \( F(x, y) = f^{-1}(f(x) + f(y)) \).

If there is another element \( h \) of \( K[[x]]^n \) satisfying \( h \equiv i \mod \deg 2 \) and \( F(x, y) = h^{-1}(h(x) + h(y)) \), we have

\[
(f \circ h^{-1})(x + y) = (f \circ h^{-1})(x) + (f \circ h^{-1})(y).
\]

Hence we get \( f \circ h^{-1} = i \) or \( f = h \) by Lemma 1.5.

DEFINITION. Let \( R \) and \( K \) be as above; let \( F \) be an \( n \)-dimensional commutative formal group over \( R \). The unique element \( f \) of \( K[[x]]^n \), such that \( f \equiv i \mod \deg 2 \) and \( F(x, y) = f^{-1}(f(x) + f(y)) \), is called the transformer of \( F \).

Let \( G \) be another commutative formal group over \( R \), of dimension \( m \) and with the transformer \( g \). If \( \varphi \in \text{Hom}_R(F, G) \), we have

\[
\varphi(f^{-1}(f(x) + f(y))) = g^{-1}(g(\varphi(x) + g(\varphi(y)))).
\]

Substituting \( x, y \) by \( f^{-1}(x), f^{-1}(y) \), respectively, we get

\[
(g \circ \varphi \circ f^{-1})(x + y) = (g \circ \varphi \circ f^{-1})(x) + (g \circ \varphi \circ f^{-1})(y).
\]
Hence by Lemma 1.5 there is an \( m \times n \) matrix \( C \) over \( K \) such that \((g \circ \varphi \circ f^{-1})(x) = Cx\). This implies \( \varphi(x) = g^{-1}(Cf(x)) \). As \( \varphi(x) \equiv Cx \pmod{\deg 2} \), \( C \) is a matrix with elements in \( R \).

**Proposition 1.6.** Let \( F, f, G, g \) be as above. Every element \( \varphi \) of \( \operatorname{Hom}_R(F, G) \) has the form \( g^{-1} \circ (Cf) \), where \( C \) is an \( m \times n \) matrix over \( R \). Conversely, \( C \) being an \( m \times n \) matrix over \( R \), \( g^{-1} \circ (Cf) \in \operatorname{Hom}_R(F, G) \), if and only if \( g^{-1} \circ (Cf) \) has coefficients in \( R \). The map \( \varphi \rightarrow C \) yields an isomorphism of \( \operatorname{Hom}_R(F, G) \) into the module of \( m \times n \) matrices over \( R \). If \( F = G \) in particular, this map is a ring isomorphism of \( \operatorname{End}_R F \) into \( M_n(R) \).

**Proof.** The first assertion has already been proved. The second follows from
\[
(g^{-1} \circ (Cf)) \circ F = G \circ (g^{-1} \circ (Cf)).
\]
The rests follow from the definitions.

§ 2. Formal groups over a \( p \)-adic integer ring.

Throughout the rest of this paper we exclusively deal with commutative formal groups. By a formal group we always mean a commutative one.

Let \( K \) be a discrete valuation field of characteristic 0 and let \( o \) and \( \mathfrak{p} \) be the ring of integers in \( K \) and the maximal ideal of \( o \), respectively. We assume that the residue class field \( k = o / \mathfrak{p} \) is of characteristic \( p > 0 \). Consider the following condition on \( K \):

\[ (F) \text{ There are an endomorphism } \sigma \text{ of } K \text{ and a power } q \text{ of } p \text{ such that } \sigma^q \equiv \sigma^2 \pmod{\mathfrak{p}} \text{ for any } \sigma \in o. \]

We note \( \sigma^p = \sigma \), since \( \sigma \) sends a unit of \( o \) to \( o \) and \( p^p = p^\sigma = p \). In this section we study formal groups over \( o \), when \( K \) satisfies \((F)\). We do not assume the completeness of \( K \).

Let \( K_o \) be a finite extension of the \( p \)-adic number field \( Q_p \) and let \( q \) be the cardinal of its residue field. Then it is well-known that an unramified extension of \( K_o \) (of finite or infinite degree) or its completion satisfies \((F)\) with a Frobenius \( \sigma \).

2.1. Let \( K_o[[T]] \) be the non-commutative power series ring on \( T \) with the multiplication rule: \( Ta = \alpha^T \) for \( \alpha \in K \). We denote by \( \mathfrak{B}_{m,n} \) (resp. \( \mathfrak{A}_{m,n} \)) the module consisting of all \( m \times n \) matrices over \( K_o[[T]] \) (resp. \( o[[T]] \)).

Let \( x = (x_1, \ldots, x_n) \) be a set of \( n \) variables. For \( f \in K[[x]] \) and \( u = \sum_{\nu=0}^\infty C_\nu T^\nu \in \mathfrak{B}_{1,n} \) (where the \( C_\nu \) are matrices over \( K \)), we define an element \( \nu f \) of \( K[[x]] \) by
\[
(u \ast f)(x) = \sum_{\nu=0}^\infty C_\nu f^{\nu}(x^{q^\nu}).
\]
This is well-defined, since \( f(x) \) has no constant term. If \( v = \sum_{\nu=0}^{\infty} D_{\nu}T^\nu \) is in \( \mathfrak{B}_{r,t} \), we have

\[
(vu) * f = v * (u * f),
\]

since

\[
(v * (u * f))(x) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} C_{\nu}^{\nu}f_{\mu+\nu}(x^{\mu+\nu})
\]

\[
= \sum_{x=0}^{\infty} \sum_{\mu=0}^{\infty} D_{\nu}C_{\nu}^{\nu}f_{\mu}(x^{\mu})
\]

\[
= ((vu) * f)(x).
\]

From now on we fix a prime element \( \pi \) of \( \mathfrak{a} \).

**Lemma 2.1.** For any rational integers \( \nu \geq 0, a \geq 1 \) and \( m \geq 1 \) we have

\[
\pi^{-\nu}(X+\pi Y)^{mp^a} \equiv \pi^{-\nu}X^{mp^a} \mod \mathfrak{p}.
\]

In particular we have

\[
m^{-1}(X+pY)^m \equiv m^{-1}X^m \mod \mathfrak{p}.
\]

for \( m \geq 1 \).

This is Lemma 4 of [10]. As the proof is elementary and easy, we omit it here.

We write \( \mathfrak{A}_n \) (resp. \( \mathfrak{B}_n \)) for \( \mathfrak{A}_{n,n} \) (resp. \( \mathfrak{B}_{n,n} \)).

**Definition.** An element \( u \) of \( \mathfrak{A}_n \) is said to be special, if \( u \equiv \pi I_n \mod \deg 1 \). Let \( P \) be an invertible matrix in \( M_n(\mathfrak{a}) \) and let \( u \) be a special element of \( \mathfrak{A}_n \).

An element \( f \) of \( K[[x]]_{\mathfrak{a}} \) is said to be of type \((P; u)\), if \( f \) satisfies the following two conditions:

i) \( f(x) \equiv Px \mod \deg 2 \),

ii) \( (u * f)(x) \equiv 0 \mod \mathfrak{p} \).

If \( f \) is of type \((I_n; u)\), we shall simply say that \( f \) is of type \( u \).

Let \( u \in \mathfrak{A}_n \) be special and put \( w = u^{-1}\pi(\in \mathfrak{B}_n) \). Then, \( i \) being the identity function,

\[
(u * (w * i))(x) = ((uw)*i)(x) = \pi x \equiv 0 \mod \mathfrak{p}.
\]

This implies that \((u^{-1}\pi)*i\) is of type \( u \).

**Lemma 2.2.** Let \( u \in \mathfrak{A}_n \) be special and put \( u^{-1}\pi = I_n + \sum_{\nu=1}^{\infty} B_{\nu}T^\nu \). Then we have \( \pi^*B_\nu \in M_n(\mathfrak{a}) \) for \( \nu \geq 0 \).

**Proof.** Write \( u = \pi I_n + \sum_{\nu=1}^{\infty} C_{\nu}T^\nu \) and replace \( T \) by \( \pi T \) in the equality

\[
(\pi I_n + \sum_{\nu=1}^{\infty} C_{\nu}T^\nu)(I_n + \sum_{\nu=1}^{\infty} B_{\nu}T^\nu) = \pi I_n.
\]

Then we get
\[
\left( I_n + \sum_{\nu=1}^{\infty} \pi^{\nu-\nu-1} C_{\nu} T^{\nu} \right) \left( I_n + \sum_{\nu=1}^{\infty} \pi^{1+\nu-\nu-1} B_{\nu} T^{\nu} \right) = I_n.
\]

This implies \( \pi^n B_{\nu} \in M_\nu(0) \), since \( \pi^n \) is also a prime element of \( o \).

2.2. The following two lemmas play crucial roles in our further investigation and will be used repeatedly.

**Lemma 2.3.** Let \( f \in K[[x]] \) be of type \((P; u)\) and let \( v \) be an element of \( A_m, n \). Let \( \phi \) be an element of \( K[[x']] \), \( x' \) being a finite set of variables. If the coefficients (of components) of \( \phi \), of terms of (total) degree \( \leq r-1 \), belong to \( o \) for some \( r \geq 2 \), we have

\[
v*(f \circ \phi) \equiv (v*f) \circ \phi \mod \deg (r+1), \mod p.
\]

If \( \phi \in o[[x']] \) in particular, we have

\[
v*(f \circ \phi) \equiv (v*f) \circ \phi \mod p.
\]

**Lemma 2.4.** If \( f \) (resp. \( g \)) \( \in K[[x]] \) is of type \((P; u)\) (resp. of type \((Q; u)\)), then \( g^{-1} o f \in o[[x]] \).

Put \( h = (u^{-1} \pi) * i \). First we will prove the first assertion of Lemma 2.3 for \( f = h \). Write

\[
u^{-1} \pi = I_n + \sum_{\nu=1}^{\infty} B_{\nu} T^{\nu}, \quad v = \sum_{\nu=0}^{\infty} A_{\nu} T^{\nu}.
\]

We have

\[
((v*h) \circ \phi)(x') = (((v u^{-1} \pi) * i) \circ \phi)(x') = \sum_{\nu} A_{\nu} B_{\nu} \phi(x')^{q^\nu}.
\]

Now

\[
B_{\mu} \phi(x')^{q^\nu} = \pi^n B_{\mu} \pi^{-\nu} \phi(x')^{q^\nu}
\]

and \( \pi^n B_{\mu} \subset M_\nu(0) \) by Lemma 2.2. We will prove

\[
\pi^{-\nu} \phi(x')^{q^\nu} \equiv \pi^{-\nu} (\phi^{q^\nu}(x' q^\nu))^{q^n} \mod \deg (r+1), \mod p.
\]

If \( \mu = \nu = 0 \), (2.4) is trivial. If \( \mu = 0 \) and \( \nu \geq 1 \), we have

\[
\phi(x')^{q^\nu} \equiv \phi^{q^\nu}(x') \mod \deg (r+1), \mod p,
\]

since terms of \( \phi \) of degree \( \geq r \) do not affect this congruence. (Note \( \phi(0) = 0 \).) Assume \( \mu \geq 1 \). Because

\[
\phi(x')^{q^\nu} \equiv \phi^{q^\nu}(x' q^\nu) \mod \deg r, \mod p,
\]

we get (2.4) by Lemma 2.1 and by the fact \( \phi(0) = 0 \). This completes the proof of (2.4). Thus we get from (2.2), (2.3) and (2.4)

\[
((v*h) \circ \phi)(x') \equiv \sum_{\mu, \nu} A_{\mu} B_{\nu} (\phi^{q^\nu}(x' q^\nu))^{q^n} \mod \deg (r+1), \mod p
\]

\[
= (v*(h \circ \phi))(x')
\]
PROOF OF LEMMA 2.4. Since \( g^{-1} \circ f = (g^{-1} \circ h) \circ (h^{-1} \circ f) \) and \( (h^{-1} \circ g)(x) \equiv Qx \mod \text{deg} 2 \), we have only to prove \( h^{-1} \circ f \equiv 0 \mod \text{deg} 2 \). Assume that the coefficients of \( \varphi \), of (total) degree \( \leq r-1 \), are integers for some \( r \geq 2 \). By Lemma 2.3 for \( f = h \) we have

\[
\pi \varphi = (u \ast h) \circ \varphi \equiv u \ast (h \circ \varphi) \mod \text{deg} (r+1), \mod p.
\]

This implies that the \( r \)-th degree coefficients of \( \varphi \) are also integers. This completes our proof by induction.

PROOF OF LEMMA 2.3. We have only to prove the first assertion. Notations being as above,

\[
v \ast (f \circ \varphi) = v \ast ((h \circ \varphi) \circ \varphi) = v \ast (h \circ (\varphi \circ \varphi))
\]

\[
\equiv (v \ast h) \circ (\varphi \circ \varphi) \mod \text{deg} (r+1), \mod p.
\]

Since \( \varphi(x) \equiv Px \mod \text{deg} 2 \), we have

\[
((v \ast h) \circ \varphi)(x) \equiv A_0Px \equiv (v \ast (h \circ \varphi))(x) \mod \text{deg} 2.
\]

Put \( \lambda_1(x) = ((v \ast h) \circ \varphi)(x) - A_0Px \) and \( \lambda_2(x) = (v \ast (h \circ \varphi))(x) - A_0Px \). Then \( \lambda_1 \equiv \lambda_2 \equiv 0 \mod \text{deg} 2 \) and \( \lambda_1 \equiv \lambda_2 \mod p \) by what we have proved. It follows from this

\[
\lambda_1 \circ \varphi \equiv \lambda_2 \circ \varphi \mod \text{deg} (r+1), \mod p,
\]

since the terms of \( \varphi \) of degree \( r \) do not affect this congruence. Hence we get

\[
v \ast (f \circ \varphi) \equiv ((v \ast h) \circ \varphi) \circ \varphi \mod \text{deg} (r+1), \mod p
\]

\[
= A_0P\varphi + \lambda_1 \circ \varphi
\]

\[
= A_0P\varphi + \lambda_2 \circ \varphi \mod \text{deg} (r+1), \mod p
\]

\[
= (v \ast (h \circ \varphi)) \circ \varphi
\]

\[
= (v \ast f) \circ \varphi.
\]

This completes the proof of our lemma.

2.3. The results of 2.2 first allow us to construct certain formal groups over \( \mathfrak{a} \).

THEOREM 2. Assume \( K \) satisfies \( (F) \). Let \( P \) be an invertible matrix in \( M_n(\mathfrak{a}) \) and let \( u \) be a special element of \( \mathfrak{a}_n \). If \( f \in K[[x]] \) is of type \( (P; u) \), \( F(x, y) = f^{-1}(f(x)+f(y)) \) is a formal group over \( \mathfrak{a} \). Let \( g \in K[[x]] \) be of type \( (Q; u) \) for an invertible matrix \( Q \) and put \( G(x, y) = g^{-1}(g(x)+g(y)) \). Then we have \( G \cong F \) over \( \mathfrak{a} \). If \( P = Q \) in particular, we have \( G \cong F \) over \( \mathfrak{a} \).

PROOF. Form \( h = (u^{-1} \pi) \ast i \) and \( H(x, y) = h^{-1}(h(x)+h(y)) \). It is clear that
Assume that the coefficients of $H$, of terms of degree $\leq r-1$, are integers for some $r \geq 2$. By Lemma 2.3 we have
\[
\pi H(x, y) = (u \ast h) \circ H(x, y)
\]
\[
\equiv (u \ast (h \circ H))(x, y) \mod \deg (r+1), \mod p
\]
\[
= (u \ast h)(x) + (u \ast h)(y)
\]
\[
= \pi x + \pi y \equiv 0 \mod p.
\]
This implies that the $r$-th degree coefficients of $H$ are also integers. This proves $H(x, y) \in \mathcal{O}[x, y]$ by induction. All the assertions of our theorem follow from this and from Lemma 2.4, because $F = \varphi^{-1} \circ H \circ \varphi$ if $f = h \circ \varphi$.

As for examples, see § 5.

**Proposition 2.5.** Let $P$ be an invertible matrix in $M_n(\mathcal{O})$ and let $u$ be a special element of $\mathcal{O}$. Then $f \in K[[x]]$ of type $(P; u)$, if and only if $f$ is of the form $(u^{-1} \pi) \ast i \circ \varphi$ with $\varphi \in \mathcal{O}[x]$ such that $\varphi(x) = Px \mod \deg 2$.

**Proof.** “Only if” part is Lemma 2.4. Conversely, if $\varphi \in \mathcal{O}[x]$ and $\varphi(x) \equiv Px \mod \deg 2$, we have, writing $h = (u^{-1} \pi) \ast i$,
\[
(h \circ \varphi)(x) \equiv Px \mod \deg 2
\]
and by Lemma 2.3
\[
u \ast (h \circ \varphi) \equiv (u \ast h) \circ \varphi = \pi \varphi \equiv 0 \mod p.
\]
This completes our proof.

Dually to Proposition 2.5 we have

**Proposition 2.6.** Let $f \in K[[x]]$ be of type $(P; u)$ for an invertible matrix $P$ of $M_n(\mathcal{O})$ and a special element $u$ of $\mathcal{O}$; Let $v$ be a matrix in $\mathcal{U}_{m,n}$. Then
\[
v \ast f \equiv 0 \mod p,
\]
if and only if there exists $t \in \mathcal{U}_{m,n}$ such that $v = tu$.

**Proof.** If $v = tu$ with $t \in \mathcal{U}_{m,n}$, then
\[
v \ast f = t \ast (u \ast f) \equiv 0 \mod p.
\]
Conversely, assume $v \ast f \equiv 0 \mod p$ for $v \in \mathcal{U}_{m,n}$. Put $h = (u^{-1} \pi) \ast i$ and $\varphi = h^{-1} \circ f$.

Since $\varphi$ is an invertible element of $\mathcal{O}[x]$ by Lemma 2.4, we have
\[
(v \ast h) \circ \varphi \equiv v \ast (h \circ \varphi) = v \ast f \equiv 0 \mod p
\]
by Lemma 2.3, so that
\[
v \ast h = ((v \ast h) \circ \varphi) \circ \varphi^{-1} \equiv 0 \mod p.
\]
Put $vu^{-1} \pi = \sum_{v=0}^{\infty} A_v T^v$. Since
\[
v \ast h = v \ast ((u^{-1} \pi) \ast i) = (vu^{-1} \pi) \ast i,
\]
we have from (2.5)
\[ \sum_{\nu=0}^{\infty} A_{\nu} x^{q^\nu} \equiv 0 \mod p, \]
which implies \( vu^{-1} = (vu^{-1} \pi) \pi^{-1} \in \kappa_{m,n} \). This completes our proof.

2.4. We now study homomorphisms of formal groups constructed in Theorem 2. \( M_{m,n}(\kappa) \) denotes the module of all the \( m \times n \) matrices with elements in \( \kappa \).

**Theorem 3.** Assume \( K \) satisfies (F). Let \( u \in \kappa_n \) and \( v \in \kappa_m \) be special and let \( f \in K[[x]]^{m} \) (resp. \( g \in K[[x]]^{n} \)) be of type \( u \) (resp. of type \( v \)). Form \( F(x, y) = f^{-1}(f(x)+f(y)) \) and \( G(x, y) = g^{-1}(g(x)+g(y)) \). Then \( g^{-1} \circ (Cf) \in \text{Hom}_{\kappa}(F, G) \) for \( C \in M_{m,n}(\kappa) \), if and only if there exists \( t \in \kappa_{m,n} \) such that \( vC = tu \).

**Proof.** Put \( \varphi = g^{-1} \circ (Cf) \). By Proposition 1.6 \( \varphi \in \text{Hom}_{\kappa}(F, G) \) if and only if \( \varphi \in \kappa[[x]]^{m} \). In view of Lemma 2.4 we may assume \( f = (u^{-1} \pi) \cdot i \) and \( g = (v^{-1} \pi) \cdot i \). If \( \varphi \in \kappa[[x]]^{m} \), we have by Lemma 2.3
\[
(vC)^{*} f = v*(Cf) = v*(g \circ \varphi) \\
\equiv (v*g) \circ \varphi = \pi \varphi \equiv 0 \mod p.
\]
Hence, by Proposition 2.6, there exists \( t \in \kappa_{m,n} \) such that \( vC = tu \). Conversely, suppose that there is \( t \in \kappa_{m,n} \) such that \( vC = tu \). As \( \varphi(x) \equiv Cx \mod \deg 2 \), the first-degree coefficients of \( \varphi \) are integral. Assume that \( i \)-th degree coefficients of \( \varphi \) are integral for \( i \leq r-1 \) (\( r \geq 2 \)). By Lemma 2.3 we have then
\[
\pi \varphi = (v*g) \circ \varphi \\
\equiv v*(g \circ \varphi) \mod \deg (r+1), \mod p \\
= v*(Cf) = (vC)^{*} f \\
= (tu)^{*} f = t*(u*f) \\
\equiv 0 \mod p.
\]
This shows that the \( r \)-th degree coefficients of \( \varphi \) are integral. Hence we get \( \varphi \in \kappa[[x]]^{m} \) by induction.

**Corollary.** Let \( F, G \) be as in Theorem 3. The module \( \text{Hom}_{\kappa}(F, G) \) is canonically isomorphic to \( M_{m,n}(\kappa) \cap v^{-1}\kappa_{m,n}u \).

By Theorem 3 \( g^{-1} \circ (Cf) \in \text{Hom}_{\kappa}(F, G) \) for \( C \in M_{m,n}(\kappa) \), if and only if \( C \in v^{-1}\kappa_{m,n}u \). Our assertion follows from this and from Proposition 1.6.

§ 3. The non-ramified case.

Let \( K, \kappa, p \) and \( k \) be as in § 2. In § 3 we assume moreover that:
(F₁) The valuation of \( K \) is unramified and (F) is satisfied with \( q = p \).
The ring \( W(k') \) of Witt vectors over a perfect field \( k' \) of characteristic
$p > 0$ satisfies $(F_1)$ (cf. [22]). Under $(F_1)$ we can take $p$ as the fixed prime element of $o$.

3.1. Let $x$ be the set of $n$ variables as usual. Let $N$ be the set of all the non-negative rational integers. For $\alpha = (\alpha_1, \ldots , \alpha_n) \in N^n$ we write $x^\alpha$ for \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). Then \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) is the degree of \( x^\alpha \). For \( 1 \leq i \leq n \), let \( \varepsilon_i \) denote the vector of \( N^n \) whose \( j \)-th component is \( \delta_{ij} \) \((1 \leq j \leq n)\). Then \( x^\varepsilon_i = x^i \) for \( r \in N \). Every element of \( K[[x]] \) is written in the form \( \sum_{\alpha \in N^n} a_\alpha x^\alpha \) \((a_\alpha \in K)\).

**Lemma 3.1.** For \( r \geq 2 \) define the form \( A_r(X, Y) \) in \( \mathbb{Z}[X, Y] \) as follows: If \( r \) is not a power of a prime number, we put \( A_r(X, Y) = (X+Y)^r - X^r - Y^r \). If \( r \) is a power of a prime number \( p \), we put \( A_r(X, Y) = l^{-1}((X+Y)^p - X^p - Y^p) \). Then \( A_r \) is a primitive polynomial in \( \mathbb{Z}[X, Y] \).

**Proof.** Easy. See also [11], III.

For any commutative ring $R$, $A_r$ is considered a polynomial in $R[X, Y]$.

**Lemma 3.2.** Let \( \lambda(x) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha x^\alpha \) \((a_\alpha \in K)\) be a form of degree \( r \) satisfying

\[
(3.1) \quad \lambda(x+y) = \lambda(x) + \lambda(y) \quad \text{mod } p.
\]

Then, if \( r \) is not a power of \( p \), \( a_\alpha \in \mathbb{Z} \) for all \( \alpha \). If \( r \) is a power of \( p \), \( a_\alpha \in \mathbb{Z} \) for all \( \alpha \) and \( a_\alpha \in \mathbb{Z} \) for all \( \alpha \neq r \varepsilon_i \) \((1 \leq i \leq n)\).

**Proof.** Take \( \alpha \in N^n \) such that \( |\alpha| = r \). If two of \( \alpha_1, \ldots , \alpha_n \), say \( \alpha_1 \) and \( \alpha_2 \), are not equal to 0, the coefficient of \( x_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \) on the left side of \( (3.1) \) is \( a_\alpha \) and no term of this form appears on the right. Hence we have \( a_\alpha \in \mathbb{Z} \) for such \( \alpha \). If \( \alpha = r \varepsilon_i \), we have

\[
a_\alpha \{ (x_i+y_i)^p - x_i^p - y_i^p \} \equiv 0 \quad \text{mod } p
\]

from \( (3.1) \). Then our assertion is a direct consequence of Lemma 3.1.

**Proposition 3.3.** Let \( F \) be an \( n \)-dimensional formal group over \( o \) and let \( f \) be its transformer. Then there exists a special element \( u \) of \( \mathbb{Z}_n \) such that \( f \) is of type \( u \).

**Proof.** As \( f(x) \equiv x \mod \deg 2 \), we have \( pf(x) \equiv 0 \mod \deg 2, \mod p \). Suppose that for \( \mu \geq 0 \) there are matrices \( C_1, \ldots , C_\mu \) in \( M_n(o) \) satisfying

\[
(3.2) \quad pf(x) + \sum_{\nu=1}^\mu C_\nu f^{\nu}(x^\nu) \equiv 0 \mod (p^{\nu+1}), \mod p.
\]

Write \( f_\nu(x) = \sum_{\alpha} a_{\alpha, \nu} x^\alpha \) for \( 1 \leq i \leq n \). Since \( df_\nu(x) \in \mathbb{D}^*(F; o) \) by the results of §1, the \( \left( \partial / \partial x_i \right) f_\nu(x) \) have integral coefficients. In particular we have \( a_{\nu, \alpha, \nu} \in \mathbb{Z} \) for \( 1 \leq j \leq n \). Hence by Lemma 2.1 we get

\[
a_{\alpha, \nu}(x+py)^\nu = a_{\nu, \alpha} a_{\nu, \nu}^{-1}(x_1+py_1)^{\alpha_1} \prod_{j=2}^n (x_j+py_j)^{\alpha_j} \equiv a_{\nu, \alpha} a_{\nu, \nu}^{-1} x_1^{\alpha_1} \prod_{j=2}^n (x_j+py_j)^{\alpha_j} \mod p.
\]
The theory of commutative formal groups

\[ = x_1^{p \alpha_1} a_{\alpha_1} \prod_{j=2}^{n} (x_j + py_j)^{\alpha_j}. \]

By repeating the same argument we have

\[ a_{\alpha_i} (x + py)^{\alpha_i} \equiv a_{\alpha_i} x^{\alpha_i} \mod p. \]

Put now

\[ p f(x) + \sum_{\nu=1}^{\mu} f^{\alpha_{\nu}}(x^{p^\nu}) \equiv \sum_{|\beta| \equiv p+1} b_{\beta} x^{\beta} \mod p \quad (b_{\beta} \equiv K^n). \]

Substituting \( x \) by \( F(x, y) \) in (3.4) we get

\[ p f(F(x, y)) + \sum_{\nu=1}^{\mu} f^{\alpha_{\nu}}(F(x, y)^{p^\nu}) \equiv \sum_{|\beta| \equiv p+1} b_{\beta} F(x, y)^{\beta} \mod p. \]

By (3.3) the left side of (3.5) is congruent mod \( p \) to

\[ p f(F(x, y)) + \sum_{\nu=1}^{\mu} C_{\nu} f^{\alpha_{\nu}}(F^{p^\nu}(x^{p^\nu}, y^{p^\nu})) \]

\[ \equiv \sum_{|\beta| \equiv p+1} b_{\beta} (x^{\beta} + y^{\beta}). \]

Thus, denoting by \( b_{\beta,i} \) the \( i \)-th component of \( b_{\beta} \), we get

\[ \sum_{|\beta| \equiv p+1} b_{\beta,i} \{ F(x, y)^{\beta} - x^{\beta} - y^{\beta} \} \equiv 0 \mod p \]

for \( 1 \leq i \leq n \). Let \( r \) be the minimum value of \( |\beta| \) such that \( b_{\beta,i} \equiv p \) for some \( i \). Then (3.6) implies

\[ \sum_{|\beta| = r} b_{\beta,i} \{ (x+y)^{\beta} - x^{\beta} - y^{\beta} \} \equiv 0 \mod p. \]

Applying Lemma 3.2 to this we see \( r \geq p^{r+1} \). At any rate we have

\[ \sum_{|\beta| \equiv p^{r+1}} b_{\beta,i} \{ (x+y)^{\beta} - x^{\beta} - y^{\beta} \} \equiv 0 \mod p. \]

Hence, by Lemma 3.2, \( b_{\beta,i} \equiv 0 \) for \( \beta = p^{r+1} \epsilon_j \) \((1 \leq j \leq n)\) and \( b_{\beta,i} \equiv p \) for other \( \beta \) such that \( |\beta| = p^{r+1} \). Therefore we can find a matrix \( C_{p+1} \) in \( M_n(\mathbb{O}) \) satisfying

\[ p f(x) + \sum_{\nu=1}^{\mu} C_{\nu} f^{\alpha_{\nu}}(x^{p^\nu}) \equiv -C_{p+1} x^{p^{r+1}} \mod \deg (p^{r+1}+1), \mod p, \]

from which follows

\[ p f(x) + \sum_{\nu=1}^{\mu+1} C_{\nu} f^{\alpha_{\nu}}(x^{p^\nu}) \equiv 0 \mod \deg (p^{r+1}+1), \mod p. \]
Thus we have been able to replace $\mu$ by $\mu + 1$ in (3.2). This implies the existence of $C_1, C_2, \ldots, C_v, \ldots \in M_n(\mathfrak{o})$ satisfying

$$\varphi f(x) + \sum_{v=1}^{\infty} C_v f^{(v)}(x^p^v) \equiv 0 \mod p.$$  

This means that $f$ is of type $u$, where $u = \varphi I_n + \sum_{v=1}^{\infty} C_v T^v$.

**3.2.** By Theorem 2 and Proposition 3.3 every $n$-dimensional formal group over $\mathfrak{o}$ is obtained from a special element of $\mathfrak{A}_n$. Let $F$ and $G$ be $n$-dimensional formal groups over $\mathfrak{o}$, with the transformers $f$ and $g$. By Proposition 3.3 there exist special elements $u, v$ of $\mathfrak{A}_n$ such that $f$ (resp. $g$) is of type $u$ (resp. of type $v$). By the uniqueness of transformer $F \approx G$ over $\mathfrak{o}$ if and only if $g^{-1} f \in \mathfrak{o}[\![x]\!]$. By Theorem 3 this happens if and only if there is $t \in \mathfrak{A}_n$ such that $v = tu$. It is clear that such $t$ is a unit in $\mathfrak{A}_n$. Let $u'$ and $v'$ be elements of $\mathfrak{A}_n$. We shall say that $v'$ is left associate with $u'$, if there is a unit $t'$ in $\mathfrak{A}_n$ such that $v' = t'u'$. We have proved the following theorem:

**THEOREM 4.** Assume $K$ satisfies $(F_1)$. Then every $n$-dimensional formal group over $\mathfrak{o}$ is obtained from a special element of $\mathfrak{A}_n$ by the method of Theorem 2. The strong isomorphism classes of $n$-dimensional groups over $\mathfrak{o}$ correspond bijectively to the left associate classes of special elements of $\mathfrak{A}_n$.

**COROLLARY.** Let $M$ be a complete system of representatives of $\mathfrak{o} \mod \mathfrak{p}$. Then the strong isomorphism classes of $n$-dimensional formal groups over $\mathfrak{o}$ correspond bijectively to the special elements of $\mathfrak{A}_n$ whose coefficient matrices have elements in $M$.

**PROOF.** Let $u = \varphi I_n + \sum_{v=1}^{\infty} C_v T^v$ be a fixed special element of $\mathfrak{A}_n$ and let $t = I_n + \sum_{v=1}^{\infty} A_v T^v$ be a unit in $\mathfrak{A}_n$. Then we have

$$tu = \varphi I_n + \sum_{v=1}^{\infty} \left( \varphi A_v + \sum_{\mu < v} A_{\mu} C_{\mu}^{(v)} \right) T^v.$$  

Therefore we can choose $A_1, A_2, \ldots$ successively and uniquely so that the coefficients of the $T^v$ in $tu$ have all their elements in $M$. Our assertion follows from this and from Theorem 4.

**3.3.** As for the classification of (strong) isomorphism classes of $n$-dimensional groups over $\mathfrak{o}$, it is preferable to construct a module space over $\mathfrak{o}$. In the following we will perform it in case $n = 1$ and $\mathfrak{o}$ is complete.

The following lemma is a slight modification of Lemma 2.1 of [16].

**LEMMA 3.4.** In addition to the condition $(F_2)$, suppose that $\mathfrak{o}$ is complete. Let $u = \varphi + \sum_{v=1}^{\infty} c_v T^v$ (as $\mathfrak{o} \in \mathfrak{o}$) be a special element of $\mathfrak{o}[\![T]\!]$. If all the $c_v$ are in $\mathfrak{p}$, there is a unit $t$ in $\mathfrak{o}[\![T]\!]$ such that $tu = \varphi$. If $c_{h}, \ldots, c_{h-1} \in \mathfrak{p}$ but $c_h \in \mathfrak{p}$, then there is a unit $t$ in $\mathfrak{o}[\![T]\!]$ such that $tu$ is of the form $\varphi + \sum_{v=1}^{h} b_v T^v$ where
PROOF. If all the \( c_i \) are in \( \mathfrak{p} \), it suffices to put \( t = p u^{-1} \). Assume \( c_1, \ldots, c_{h-1} \in \mathfrak{p} \) but \( c_h \not\in \mathfrak{p} \). We will show that for every \( i \geq 1 \) we can choose \( b_{h-1}', \ldots, b_1' \in \mathfrak{o} \) and a unit \( t_i \) of \( \mathfrak{o}[[T]] \) satisfying
\[
\begin{align*}
& b_{h-1}' \equiv b_{h-1} \mod \mathfrak{p}^i, \\
& b_1' \equiv c_1 \mod \mathfrak{p} \\
& (1 \leq i \leq h), \\
& t_i \equiv 1 \mod \deg 1, \\
& t_{i+1} \equiv t_i \mod \mathfrak{p}^i, \\
& t_i u \equiv p + \sum_{v=1}^{h} b_v T^v \mod \mathfrak{p}^i.
\end{align*}
\]
(3.9)

First put \( b_{h-1}' = \cdots = b_2' = 0, \ b_1' = c_h \) and \( t_1 = c_h \left( \sum_{v=h}^{\infty} c_v T^{v-h} \right)^{-1} \). As \( c_h \) is a unit, \( t_i \in \mathfrak{o}[[T]] \). Since
\[
t_i u \equiv c_h T^h \mod \mathfrak{p},
\]
(3.9) is satisfied by \( \{b_v'; t_1\} \) with \( i = 1 \). Suppose that we have already found \( \{b_v'; t_i\} \) for \( 1 \leq j \leq i \) satisfying (3.9). We try to determine \( b_{v+1}' = b_v' + p^j c_v' \) \( (1 \leq v \leq h) \) and \( t_{i+1} = t_i + p^j v_i \) so that
\[
(t_i + p^j v_i) u \equiv p + \sum_{v=1}^{h} (b_v' + p^j c_v') T^v \mod \mathfrak{p}^{i+1}.
\]
(3.10)

Put \( w_i = p^{-i} \left( t_i u - \left( p + \sum_{v=1}^{h} b_v' T^v \right) \right) \) \( (\in \mathfrak{o}[[T]]) \). Since \( p^j u \equiv p \left( \sum_{v=h}^{\infty} c_v T^v \right) \mod \mathfrak{p}^{i+1}, \) (3.10) is reduced to
\[
v_i \sum_{v=h}^{\infty} c_v T^v \equiv \sum_{v=1}^{h} d_v T^v - w_i \mod \mathfrak{p}.
\]
(3.11)

As \( w_i \) has no constant term, we can choose \( d_1', \ldots, d_h' \in \mathfrak{o} \) so that the right hand side of (3.11) has no term of degree \( \leq h \). Hence we can find a series \( v_i \in \mathfrak{o}[[T]] \), without constant term and satisfying (3.11). By induction this proves the existence of \( \{b_v'; t_i\} \) for all \( i \). Put \( t = \lim_{i \to \infty} t_i \) and \( b_v = \lim_{i \to \infty} b_v' \) for \( 1 \leq v \leq h \). Then \( \{b_v; t\} \) satisfy the requirement of our lemma.

Let \( F \) be a 1-dimensional formal group over \( \mathfrak{o} \). We shall say that \( F \) is of height \( h \) if the reduction of \( F \) modulo \( \mathfrak{p} \) is of height \( h \) (cf. [11]).

**Proposition 3.5.** Let \( K \) be a complete discrete valuation field satisfying (F1). The strong isomorphism classes of 1-dimensional formal groups over \( \mathfrak{o} \) of height \( h \) \( (1 \leq h < \infty) \), correspond bijectively to the special elements of the form \( u = p + \sum_{v=1}^{h} b_v T^v \) where \( b_1, \ldots, b_{h-1} \in \mathfrak{p} \) but \( b_h \) is a unit of \( \mathfrak{o} \). Let \( v = p + \sum_{v=1}^{h} c_v T^v \) be another special element of this form. Then the formal group obtained from \( u \) is weakly isomorphic to the one obtained from \( v \), if and only if there exists a unit \( c \) of \( \mathfrak{o} \) such that \( c_v = c_1 \sigma c_v \) for \( 1 \leq v \leq h \).

**Proof.** Let \( F \) be a 1-dimensional formal group over \( \mathfrak{o} \). Then its transformer \( f \) is of type \( u' \) for a special element \( u' \). If all the coefficients of \( u' \)
are in \( p \), then \( F(x, y) = x + y \) by Lemma 3.4 and Theorem 2. If not, \( f \) is also of type \( u \), where \( u \) is a special element of the form \( p + \sum_{i=1}^{h} b_i T^i \) (\( b_1, \ldots, b_{n-1} \in p, b_n \in p \)). We will prove that \( F \) is of height \( h \). Since

\[
(1 + p^{-1} \sum_{i=1}^{h} b_i T^i)^{-1} u = p + b_n T^n + \cdots,
\]

it suffices to prove that a formal group obtained from a special element \( u^* \) of the form \( p + b_n T^n + \cdots \) (\( b_n \in p \)) is of height \( h \). Put \( (pu^{*i}) * i = h \). Then

\[
h(x) = x - p^{-1} b_n x^{p^n} + \cdots
\]

and so

\[
h^{-1}(ph(x)) = px - b_n x^{p^n} + \cdots + p^{-1} b_n (px - \cdots)^{p^n} + \cdots
\]

\[
\equiv -b_n x^{p^n} + \cdots \mod p,
\]

which prove that \( h^{-1}(h(x) + h(y)) \) is of height \( h \).

Now suppose that there exist a unit \( c \) in \( o \) and a unit \( t = ay T^v \) in \( o[[T]] \) such that \( vc = tu \). Comparing the \((v+h)\)-th degree coefficients of both members of

\[
\left( \sum_{v=0}^{\infty} a_v T^v \right) \left( p + \sum_{v=1}^{h} b_v T^v \right) = \left( p + \sum_{v=1}^{h} c_v T^v \right) c
\]

for \( v > 0 \), we get

\[
(3.12) \quad a_v b_h^v + \sum_{u+\mu=n} a_{u+\mu} b_{h-\mu}^n + p a_{v+h} = 0.
\]

Since \( b_h \) is a unit, it follows from (3.12) that \( a_v \in p \) for \( v \geq 1 \). Hence we get \( a_v \in p^i \) for \( v \geq 1 \) again by (3.12). Repeating the same argument we see \( a_v \in p^i \) for every \( v \geq 1 \) and for every \( i \geq 1 \). This implies \( a_v = 0 \) for \( v \geq 1 \), and \( t = a_o = c \).

Our proposition follows from this, from Theorem 3 and from Theorem 4.

In the above proof we proved that \( vc = tu \) implied \( t = c \). Thereby we did not use the fact that \( c \) (resp. \( t \)) is a unit. Therefore we get by Theorem 3;

**Proposition 3.6.** Let \( u, v \) be as in Proposition 3.5 and let \( F, G \) be formal groups attached to them. Then the module \( \text{Hom}_o(F, G) \) is canonically isomorphic to \( \{ c \in o \mid vc = cu \} \).

### § 4. Formal groups over a field of characteristic \( p \geq 0 \).

Let \( K \) be a discrete valuation field satisfying (F) of § 2. For a power series \( f \in o[[x]]^m \), \( f^* \) denotes the power series in \( k[[x]]^m \) obtained by reducing the coefficients of \( f \) modulo \( p \). In § 4 we will study the reductions of formal groups over \( o \) and their homomorphisms.

**4.1.** Our first task is to prove two lemmas.
LEMMA 4.1. Let \( f \in K[[x]]^n \) be of type \((P; u)\) and let \( \phi(x') \in O[[x']]^m \) where \( x' \) is a finite set of variables. Then we have
\[
f^{-1}(\pi \phi(x')) \equiv 0 \mod p.
\]

PROOF. Put \( h = (u^{-1} \pi) \ast i \). By Lemma 2.4 it suffices to prove
\[
h^{-1}(\pi x) \equiv 0 \mod p.
\]
Write \( h(x) = \sum \pi^v B_v x^v \) and \( h^{-1}(\pi x) = l(x) \). Since \( l(x) \equiv \pi x \mod \text{deg } 2 \), the first-degree coefficients of \( l \) are in \( p \). Assume for \( r \geq 2 \) that the \( i \)-th degree coefficients of \( l \) are in \( p \) for all \( i \leq r-1 \). Write \( l(x) = \pi^r x + \Delta^r(x) \) where \( \Delta^r(x) \in O[[x]]^m \) and \( \Delta^r(x) \equiv 0 \mod \text{deg } r \). Then it follows from \( h(l(x)) = \pi x \)
\[
(4.1) \quad l(x) + \sum_{v=1}^{r-1} \pi^v B_v \pi^r x^v \equiv \pi x \mod \text{deg } (r+1).
\]
Since \( \pi^v B_v \in \pi M_n(\alpha) \) for \( v \geq 1 \) by Lemma 2.2, it follows from (4.1)
\[
l(x) \equiv 0 \mod \text{deg } (r+1), \mod p.
\]
Hence the \( r \)-th degree coefficients of \( l \) are also in \( p \). Thus we get \( l \equiv 0 \mod p \) by induction.

LEMMA 4.2. Let \( u \in \mathfrak{A}_n \) be special and let \( f \in K[[x]]^n \) be of type \( u \). Let \( \phi_1 \in K[[x']]^n \) and \( \phi_2 \in O[[x']]^m \). Then \( f \circ \phi_1 \equiv f \circ \phi_2 \mod p \), if and only if \( \phi_1 \equiv \phi_2 \mod p \).

PROOF. Suppose \( \phi_1 \equiv \phi_2 \mod p \). Then we have clearly \( \phi_1 \in O[[x]]^n \). Put \( h = (u^{-1} \pi) \ast i \) and \( h^{-1} \circ f = \phi \). Since \( \phi \in O[[x]]^n \) by Lemma 2.4 and \( \phi \circ \phi_1 \equiv \phi \circ \phi_2 \mod p \), we obtain by Lemma 2.1 and 2.2
\[
h \circ (\phi \circ \phi_1) \equiv h \circ (\phi \circ \phi_2) \mod p
\]
i.e. \( f \circ \phi_1 \equiv f \circ \phi_2 \mod p \). Conversely assume \( f \circ \phi_1 \equiv f \circ \phi_2 \mod p \) and put \( \pi \lambda = f^{-1}(f \circ \phi_1 - f \circ \phi_2) \). Then \( \lambda \in O[[x]]^n \) by Lemma 4.1. Since \( F(x, y) = f^{-1}(f(x) + f(y)) \) has coefficients in \( o \), it follows from
\[
f \circ \phi_1 = f \circ \phi_2 + f \circ (\pi \lambda)
\]
i.e. \( \phi_1 = F(\phi_2, \pi \lambda) \) that \( \phi_1 \equiv \phi_2 \mod p \).

4.2. We now study a certain type of homomorphisms of \( F^* \) to \( G^* \) for formal groups \( F, G \) over \( o \).

THEOREM 5. Suppose \( K \) satisfies \((F)\). Let \( F \) and \( G \) be formal groups over \( o \), of dimension \( n \) and \( m \) and with transformers \( f \) and \( g \), respectively. Suppose that \( f \) (resp. \( g \)) is of type \( u \) (resp. of type \( v \)) for special elements \( u \in \mathfrak{A}_n \) and \( v \in \mathfrak{A}_m \).

(i) Put \( \varphi = \varphi_w = g^{-1} \circ (w \ast f) \) for \( w \in \mathfrak{A}_{m,n} \). Then \( \varphi(x) \in O[[x]]^n \) if and only if there exists \( t \in \mathfrak{A}_{m,n} \) such that \( wt = tu \).

(ii) If \( \varphi_w \in O[[x]]^n \), then \( \varphi_w \in \text{Hom}_o(F^*, G^*) \).
(iii) Let \( h \) be of type \( v' \) for a special element \( v' \in \mathfrak{H} \). If \( \varphi_{w'} = h^{-1} \circ (w' \ast g) \) has integral coefficients for \( w' \in \mathfrak{H}_{m,n} \), then \( \varphi_{w'}^{-1} \circ \varphi_{w'} = \varphi_{w'}^{*} \).

**Proof.** In order to prove (i) we may assume \( g = (v^{-1} \ast i) \). Suppose there is \( t \in \mathfrak{H}_{m,n} \) such that \( vw = tu \). Clearly the first-degree coefficients of \( \varphi \) are integers. Assume for \( r \geq 2 \) that the \( i \)-th degree coefficients of \( \varphi \) are integers for \( i \leq r-1 \). By Lemma 2.3 we have

\[
\pi \varphi = (v \ast g) \circ \varphi \equiv v \ast (g \circ \varphi) \mod \deg (r+1), \mod p
\]

\[
= v \ast (w \ast f) = (vw) \ast f = (tu) \ast f
\]

\[
= t \ast (u \ast f) \equiv 0 \mod p.
\]

This implies that the \( r \)-th degree coefficients of \( \varphi \) are also integers. This shows \( \varphi(x) \in \mathfrak{O}[[x]]^m \) by induction. Conversely, suppose \( \varphi = \varphi_w \in \mathfrak{O}[[x]]^m \). By Lemma 2.3 we get

\[
(vw) \ast f = v \ast (w \ast f) = v \ast (g \circ \varphi)
\]

\[
\equiv (v \ast g) \circ \varphi = \pi \varphi \equiv 0 \mod p.
\]

Hence, by Proposition 2.6 we can find \( t \in \mathfrak{H}_{m,n} \) such that \( vw = tu \). This proves (i). Now we have

\[
g \circ (\varphi \circ F) = (g \circ \varphi) \circ F = (w \ast f) \circ F
\]

and by Lemma 2.3

\[
((w \ast f) \circ F)(x, y) \equiv (w \ast (f \circ F))(x, y) \mod p
\]

\[
= (w \ast f)(x) + (w \ast f)(y)
\]

\[
= (g \circ \varphi)(x) + (g \circ \varphi)(y)
\]

\[
= g(G(\varphi(x), \varphi(y))).
\]

Thus we get \( g \circ (\varphi \circ F) = g \circ (G \circ \varphi) \mod p \). By Lemma 4.2 it follows from this that \( \varphi \circ F = G \circ \varphi \mod p \). This implies \( \varphi^* \in \text{Hom}_k(F^*, G^*) \). Let us prove (iii). By Lemma 2.3 we have

\[
h \circ (\varphi_{w'} \circ \varphi_w) = (h \circ \varphi_{w'}) \circ \varphi_w = (w' \ast g) \circ \varphi_w
\]

\[
\equiv w' \ast (g \circ \varphi_w) \mod p
\]

\[
= w' \ast (w \ast f) = (w' \ast f).
\]

By (i) there is \( t' \in \mathfrak{H}_{m,n} \) such that \( v'w' = t'v \). Since \( v'w'w = t'vw = t'tu \), \( \varphi_{w'}w \)

\[
= h^{-1} \circ ((w'w) \ast f) \text{ has integral coefficients by (i). Since}
\]

\[
h \circ (\varphi_{w'} \circ \varphi_w) \equiv h \circ \varphi_{w'} \mod p
\]

as we have shown, it follows from Lemma 4.2 that

\[
\varphi_{w'} \circ \varphi_w \equiv \varphi_{w'} \mod p.
\]

This proves (iii).
The theory of commutative formal groups

**COROLLARY.** Put \( E = e[[T]] \). The submodule of \( \text{Hom}_k(F^*, G^*) \), consisting of homomorphisms of the form \( \varphi_{w} \) (\( w \in \mathfrak{A}_{m,n} \)), is canonically isomorphic to the module of all right \( E \)-homomorphisms of \( E^n/uE^n \) into \( E^m/vE^m \). In particular the subring of \( \text{End}_k F^* \), consisting of homomorphisms of the form \( (f^{-1} \circ (w*f))^* \) (\( w \in \mathfrak{A}_n \)), is canonically isomorphic to the right \( E \)-endomorphism ring of \( E^n/uE^n \).

**PROOF.** If \( tu = vw \), then

\[
t(uE^n) = vwE^n \subset vE^m.
\]

Thus \( t \) induces a right \( E \)-homomorphism \( \Phi_t \) of \( E^n/uE^n \) into \( E^m/vE^m \). Conversely, as is easily verified, every right \( E \)-homomorphism of \( E^n/uE^n \) into \( E^m/vE^m \) is of the form \( \Phi_t \) with \( t \in \mathfrak{A}_{m,n} \) such that \( tu \in v\mathfrak{A}_{m,n} \). We will show that \( \varphi^* = 0 \) if and only if \( \Phi_t = 0 \).

**4.3.** If \( K \) satisfies (\( F_i \)), every element of \( \text{Hom}_k(F^*, G^*) \) is of the form \( \varphi_{w} \) with \( w \in \mathfrak{A}_{m,n} \). To prove it we need the following lemma.

**LEMMA 4.3.** Suppose \( K \) satisfies (\( F_i \)). Let \( F \) be an \( n \)-dimensional formal group over \( o \) and let \( f \) be its transformer. Put \( M = \{ \phi \in K[[x]] \mid (\phi \circ F)(x, y) = \phi(x) + \phi(y) \mod p \} \). Then \( M \) is topologically generated by \( p[[x]] \) and by \( \{ f^i(x^p) \mid 1 \leq i \leq n, \nu \geq 0 \} \) as \( o \)-module. (We define the topology of \( K[[x]] \) by taking \( I_\nu = \{ f \in K[[x]] \mid f \equiv 0 \mod \deg (\nu+1) \} \) as a base of neighborhoods of 0.)

**PROOF.** It is clear that \( p[[x]] \subset M \). By Lemma 2.3 and by Proposition 3.3 we have

\[
f^{\alpha}(F(x, y)^{p^\nu}) = ((T^*f) \circ F)(x, y)
\]

\[
= (T^*f \circ F)(x, y) \mod p
\]

\[
= (f^*f)(x) + (T^*f)(y)
\]

\[
= f^{\alpha}(x^{p^\nu}) + f^{\alpha}(y^{p^\nu}).
\]

This implies \( f^{i\alpha}(x^{p^\nu}) \in M \) for \( 1 \leq i \leq n, \nu \geq 0 \). Let \( \phi \) be any element of \( M \) and let \( r \) be the lowest degree such that \( \phi \neq 0 \mod \deg (r+1), \mod p \). Then \( \phi \in M \) implies that the \( r \)-th degree homogeneous part \( \phi^{r\alpha} \) of \( \phi \) satisfies

\[
\phi^{r\alpha}(x+y) = \phi^{r\alpha}(x) + \phi^{r\alpha}(y) \mod \deg (r+1), \mod p .
\]

By Lemma 3.2 (\( 4.2 \)) implies that \( r \) is a power of \( p \), say \( p^b \) (if \( r < \infty \)) and that there exist \( c_1, \ldots, c_n \in o \) satisfying

\[
\phi(x) - \sum_{i=1}^{n} c_i x^{p^b} \equiv 0 \mod \deg (r+1), \mod p .
\]

Hence we get
Applying the same argument to the left side of (4.3) in place of $\phi$ and repeating this procedure we see in fact that $\mathfrak{g}[\mathfrak{x}]$ and the $f^\nu(x^\nu)$ ($1 \leq i \leq n$, $\nu \geq 0$) generate a dense $\mathfrak{o}$-submodule of $M$.

THEOREM 6. Suppose $K$ satisfies $(F_2)$. The map $\Phi: \varphi \rightarrow \varphi^*_\mathfrak{o}$, defined in Theorem 5, is a bijection of $\text{Hom}_E(E^n/uE^n, E^m/vE^m)$ onto $\text{Hom}_\mathfrak{o}(F^*, G^*)$. In particular $\text{End}_E F^*$ is canonically isomorphic to $\text{End}_\mathfrak{o}(E^n/uE^n)$.

PROOF. It suffices to prove the surjectivity. We may assume $f = (u^{-1}\pi)^*121$ and $g = (v^{-1}\pi)^*121$. For $\varphi^* \in \text{Hom}_E(F^*, G^*)$, take $\varphi \in \mathfrak{o}[\mathfrak{x}, \mathfrak{y}]$ such that $\varphi^* = \varphi^*$. Since $\varphi \circ F \equiv G \circ \varphi \mod \mathfrak{p}$, we get by Lemma 4.2

\begin{equation}
\phi^* = (u^{-1}\pi)^* \circ (v^{-1}\pi)^* \circ \varphi \mod \mathfrak{p}.
\end{equation}

Put $\phi = \phi \circ \varphi$. Then (4.4) implies

\begin{equation}
\phi(F(x, y)) = \phi(x) + \phi(y) \mod \mathfrak{p}.
\end{equation}

By Lemma 4.3 it follows from (4.5) that there exists $w \in \mathfrak{m}_{m,n}$ satisfying

$\phi \equiv w * f \mod \mathfrak{p},$

or

$\phi \equiv w * f \mod \mathfrak{p}.$

By Lemma 4.2 this implies that $g^{-1} \circ (w * f) \in \mathfrak{o}[\mathfrak{x}, \mathfrak{y}]$ and $\varphi \equiv g^{-1} \circ (w * f) \mod \mathfrak{p}$. Thus we have $\varphi^* = \varphi_*^*$, which was to be proved.

4.4. Now we will show that, if $K$ satisfies $(F_2)$, any formal group over $k$ is obtained by reducing a formal group over $\mathfrak{o}$.

The following lemma is due to [12].

LEMMA 4.4. Let $R$ be a commutative ring and let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be systems of $n$ variables. Suppose that a form $\Delta(X, Y)$ of degree $r$ in $R[X, Y]$ is a commutative 2-cocycle, i.e.

$\Delta(X, Y) = \Delta(Y, X),$

(4.6)

$\Delta(Y, Z) - \Delta(X + Y, Z) + \Delta(X, Y + Z) - \Delta(X, Y) = 0.$

Then, if $r$ is not a power of a prime number, $\Delta$ is a 2-coboundary, i.e. there is a form $\Gamma(X)$ of degree $r$ such that

$\Delta(X, Y) = \Gamma(X) - \Gamma(X + Y) + \Gamma(Y).$

If $r$ is a power of a prime, $\Delta$ is cohomologous to a linear combination of $\Delta_i(X_i, Y_i)$ ($1 \leq i \leq n$) with coefficients in $R$.

PROOF. In case $n = 1$ this is Lemma 3 of [11]. (For the proof of this case see also [7], p. 62.) In general we can reduce the case $n = m$ to the case $n = m - 1$ by making use of the result of Lyndon [15] on normal co-
homology groups. (See also [12]). For the convenience of the reader we will perform this reduction in the following. We first note \( \mathcal{A}(X, 0) = 0 = \mathcal{A}(0, X) \). (Put \( Y = Z = 0 \) in (4.6)). Let us write \( X' = (X_1, \ldots, X_{m-1}), \ Y' = (Y_1, \ldots, Y_{m-1}) \), i.e. \( X = (X', X_m), \ Y = (Y', Y_m) \) and \( \mathcal{A}(X, Y) = \mathcal{A}(X', X_m, Y', Y_m) \). Define \( \mathcal{A}_1 \) by

\[
\mathcal{A}_1(X, Y) = \mathcal{A}(X, Y) - \left\{ \mathcal{A}(0, X_m, X', 0) \mathcal{A}(0, X_m+Y_m, X'+Y', 0) + \mathcal{A}(0, Y_m, Y', 0) \right\}.
\]

Then \( \mathcal{A}_1 \) is also a commutative 2-cocycle cohomologous to \( \mathcal{A} \). Putting \( X' = 0, Y_m = 0 \) in (4.7) we get

\[
\mathcal{A}_1(0, X_m, Y', 0) = 0
\]

and by commutativity

\[
\mathcal{A}_1(Y', 0, 0, Y_m) = 0.
\]

Now putting \( X' = 0, Y_m = Z_m = 0 \) in (4.6) for \( \mathcal{A} = \mathcal{A}_1 \) we get

\[
\mathcal{A}_1(Y', 0, Z', 0) - \mathcal{A}_1(Y', X_m, Z', 0) + \mathcal{A}_1(0, X_m, Y'+Z', 0) - \mathcal{A}_1(0, X_m, Y', 0) = 0.
\]

By (4.8) this implies

\[
\mathcal{A}_1(Y', X_m, Z', 0) = \mathcal{A}_1(Y', 0, Z', 0).
\]

In the same way we obtain

\[
\mathcal{A}_1(X', Y_m, 0, Z_m) = \mathcal{A}_1(0, Y_m, 0, Z_m).
\]

Putting \( Y' = Z_m = 0 \) in (4.6) for \( \mathcal{A}_1 = \mathcal{A} \) we get

\[
\mathcal{A}_1(0, Y_m, Z', 0) - \mathcal{A}_1(X', X_m+Y_m, Z', 0) + \mathcal{A}_1(X', X_m, Z', Y_m) - \mathcal{A}_1(X', X_m, 0, Y_m) = 0.
\]

By (4.8), (4.9) and (4.10) this implies

\[
\mathcal{A}_1(X', X_m, Z', Y_m) = \mathcal{A}_1(X', 0, Z', 0) + \mathcal{A}_1(0, X_m, 0, Y_m),
\]

which completes the reduction: the case \( n = m \) \( \Rightarrow \) the case \( n = m-1 \).

**Theorem 7.** Suppose \( K \) satisfies (F1) of § 3. For any formal group \( F_* \) over \( k \) there exists a formal group \( F \) over \( o \) such that \( F_* = F_* \).

**Proof.** Let \( n \) be the dimension of \( F_* \). Take \( \phi(x) \in o[[x]]^n \) such that \( \phi(x) \equiv x \mod \deg 2 \) and \( u(T) = pI_n + \sum_{v=1} c_v T^v \in \mathfrak{a}_n \) and form \( f = (p u^{-1} \ast i) \circ \phi \). Then \( F(x, y) = f^{-1}(f(x)+f(y)) \) is a formal group over \( o \). We will prove that we can choose the coefficients of \( \phi \) and \( C_1, C_2, \ldots \) successively so that \( F_* = F_* \). Suppose that we have already chosen the \( i \)-th degree coefficients of \( \phi \) for \( i \leq r-1 \) and the \( C_v \) for \( p^r < r \) so that

\[
F_* \equiv F_* \mod \deg r.
\]
Letting the other coefficients of \( \varphi \) be equal to 0 and the \( C_i \) for \( p^r \geq r \) be equal to 0-matrix for example, form \( g = (p u^{-1}) \circ \varphi \) and \( G(x, y) = g^{-1}(g(x) + g(y)) \). Then \( G \) is a formal group over \( o \) and we have

\[
G^* \equiv F^* \mod \deg r.
\]

It follows from (4.12) and from the associative law of formal group that the \( r \)-th degree homogeneous part \( J \) of \( G^* - F^* \) is a commutative 2-cocycle in \( k[[x]] \) (cf. [11], [12]). If \( r \) is not a power of \( p \), we can find by Lemma 4.4 \( \phi \in o[[x]]^n \) whose components are forms of degree \( r \) and satisfy

\[
G^*(x, y) - F^*(x, y) \equiv \phi^*(x) - \phi^*(x+y) + \phi^*(y) \mod \deg (r+1).
\]

Let \( h \) be the element of \( o[[x]]^n \), obtained by replacing \( \varphi \) by \( \varphi - \phi \) in the definition of \( g \) and put \( H(x, y) = h^{-1}(h(x) + h(y)) \). Since \( h \equiv g - \phi \mod \deg (r+1) \), we get

\[
H(x, y) = h^{-1}(h(x) + h(y)) \equiv g^{-1}(g(x) + g(y)) - \{ \phi(x) + \phi(y) - \phi(x+y) \} \mod \deg (r+1).
\]

This implies

\[
H^*(x, y) \equiv G^*(x, y) - \{ \phi^*(x) + \phi^*(y) - \phi^*(x+y) \} \mod \deg (r+1)
\]

\[
\equiv F^*(x, y) \mod \deg (r+1).
\]

Thus we have been able to replace \( r \) by \( r+1 \) in (4.11). If \( r \) is a power of \( p \), say \( r = p^s \), we can find by Lemma 4.4 \( \phi \in o[[x]]^n \) whose components are forms of degree \( r \) and \( D \in M_n(o) \) such that

\[
G^*(x, y) - F^*(x, y) \equiv \phi^*(x) - \phi^*(x+y) + \phi^*(y) - D^* A_r(x, y) \mod \deg (r+1),
\]

where we have written \( A_r(x, y) = (A_r(x_1, y_1), \ldots, A_r(x_n, y_n)) \). Replacing \( \varphi - \phi \) and \( u \) by \( u + DT^h \) in the definition of \( g \), we get an element \( h \) of \( o[[x]]^n \). Since

\[
\varphi(p I_n + \sum_{v=1}^{h} C_v T_v + DT^h) = p(p I_n + \sum_{v=1}^{h} C_v T_v)^{-1} - p^{-1} DT^h \mod \deg (h+1),
\]

we have

\[
h(x) \equiv g(x) - \phi(x) - p^{-1} Dx^r \mod \deg (r+1).
\]

Put \( H(x, y) = h^{-1}(h(x) + h(y)) \). Then we get from (4.15)

\[
H(x, y) \equiv G(x, y) - \{ \phi(x) + \phi(y) - \phi(x+y) \} + D^* A_r(x, y) \mod \deg (r+1).
\]

It follows from (4.14) and (4.16) that

\[
H^*(x, y) \equiv G^*(x, y) - \{ \phi^*(x) + \phi^*(y) - \phi^*(x+y) \} + D^* A_r(x, y) \equiv F^*(x, y) \mod \deg (r+1).
\]
The theory of commutative formal groups 237

Thus we have been able to replace \( r \) by \( r+1 \) in (4.11) in this case too. This proves the existence of \( u \) and \( \varphi \) satisfying \( F^* = F_* \).

When \( K \) satisfies \((F_1)\), all the formal groups over \( k \) are obtained from special elements by Theorem 7 and homomorphisms of these groups are described in Theorem 6 and its corollary. In case where \( o \) is the ring of Witt vectors over a perfect field \( k' \) of characteristic \( p > 0 \), these results are nothing other than the main results of Dieudonné \([4]\). Using these results Dieudonné \([5]\) gave a complete classification of isogeny classes of formal groups over \( k' \) when \( k' \) is algebraically closed. For this see also \([2]\), \([8]\) and \([16]\).

§ 5. Examples and applications.

5.1. The group of Witt vectors of length \( n \).

Let \( k \) be a perfect field of characteristic \( p > 0 \) and let \( o = W(k) \) be the ring of Witt vectors over \( k \). Put \( u = p I_n - C_1T \) where \( C_1 = \begin{pmatrix} 0 & 1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \in M_n(o) \). Then it is easily verified that the reduction of the formal group with the transformer \((pu^{-1})*i\) is the group of Witt vectors of length \( n \) (cf. \([5]\), p. 120).

5.2. The group \( G_{n,m} \) for \( n \geq 2, m \geq 1 \).

Let \( k, o \) and \( C_1 \) be as in 5.1. Put \( u = p I_n - C_1T - C_{m+1}T^{m+1} \) with \( C_{m+1} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \) and form \( h = (pu^{-1})*i \) and \( H(x, y) = h^{-1}(h(x)+h(y)) \).

Then, as is seen from \([5]\), \( H^* \) is the group \( G_{n,m} (= G_{n,\varphi, m} \) by the notation of \([5]\)). Suppose that \( o \) contains a primitive \((pm^-n-1)\)-th root \( w \) of unity. Put \( W = \begin{pmatrix} w^{p^{n-1}} & 0 \\ 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & w^p \end{pmatrix} \). Then as \( w^a = w^p \), we have \( WC_1 = C_1W^a \) and \( WC_{m+1} = C_{m+1}W^{\varphi m+1} \), so that \( WU = uW \). By Theorem 3 this implies \( h^{-1}(Wh(x)) \in \text{End}_k H \). On the other hand \((T*i)(x) = x^p \in \text{End}_k H^* \), since \( H \) is defined over \( Z_p \). Let \( E \) be the \( Z_p \)-subalgebra of \( \text{End}_k H^* \) generated by \((h^{-1}\circ (Wh))^* \) and \( T*i \). The coefficients of components of \( h^{-1}\circ (Wh) \) are polynomials in \( Q_p[w] \). Since \( h^{-1}\circ (Wh) \in \mathfrak{c}([x])^w \), these polynomials belong to \( Z_p[w] \), the ring of integers in \( Q_p(w) \). Therefore we have

\[
(T*i)\circ (h^{-1}\circ (Wh))^* = (h^{-1}\circ (W^a h))^* \circ (T*i) .
\]

If \( (m, n) = 1 \) and \( k \) is algebraically closed, \( \text{End}_k H^* \) is isomorphic to the (unique) maximal order in the central division algebra of rank \((m+n)^2 \) over \( Q_p \), and
invariant $n/(m+n)$ ([5], p. 129-130). Since $Q_p(w)$ is the unramified extension of degree $m+n$ of $Q_p$ and $T\ast i$ is clearly a prime element in $\text{End}_k H^*$, (5.1) implies $E = \text{End}_k H^*$ when $(m, n) = 1$.

5.3. The Lubin-Tate group $(n = 1)$. Suppose $K$ satisfies $(F)$ of § 2. For $\alpha \in \mathfrak{o}$, $\alpha \neq 0$, $u_\alpha = \pi - \alpha^{a-1}T$ is a special element. Put $f_\alpha = ((u_\alpha^{-1}\pi) \ast i)$. An easy computation shows

$$f_\alpha(x) = \sum_{v=0}^{\infty} \pi^{(1+\alpha^{a-1}v)} \alpha^{a-1}x^v.$$ 

By Theorem 2, $F_\alpha(x, y) = f_\alpha^{-1}(f_\alpha(x) + f_\alpha(y))$ is a formal group over $\mathfrak{o}$. Since $\alpha u_\alpha = u_\alpha^{-1} \pi, f_\alpha^{-1}(\alpha f_\alpha(x))$ has integral coefficients by Theorem 3. When $\pi^a = \pi$ and $\alpha = 1$, $F_\alpha$ coincides with the group constructed in [10], Theorem 2. (Theorem 2 of [10] can be reduced to the case $a = 1$ by replacing $K$ by its unramified extension of degree $a$.)

5.4. Interpretation of the Artin-Hasse function. Suppose $K$ satisfies $(F_1)$ of § 3. Put $g(x) = -\log(1-x) = \sum_{m=1}^{\infty} m^{-1} x^m$. It is easily verified that $g$ is of type $p-T$. Put now

$$L(\alpha, x) = \sum_{v=0}^{\infty} p^v \alpha^{av} x^v$$

for $\alpha \in \mathfrak{o}$. Then $g^{-1}(L(\alpha, x))$ has integral coefficients by the result of 5.3. This is a homomorphism of $F_\alpha$ to $g^{-1}(g(x) + g(y)) = x + y - xy$. Since $g^{-1}(x) = 1 - \exp(-x)$, $\exp(-L(\alpha, x))$ has coefficients in $\mathfrak{o}$. This is nothing other than the Artin-Hasse exponential function ([1]).

5.5. The characteristic equation for the Frobenius endomorphism. Suppose $K$ satisfies $(F)$. Assume $\pi^a = \pi$ and let $u$ be a special element of $\mathfrak{g}$ such that $uT = Tu$. This implies that all coefficients of $u$ are $\sigma$-invariant. Since the elements of $u$ and $T$ generate a commutative subring of $\mathfrak{o}[\lbrack T\rbrack]$, we can consider the cofactor matrix $w$ of $u$:

$$uw = wu = (\det u)I_n.$$ 

Form $f = (u^{-1}\pi) \ast i$ and $F(x, y) = f^{-1}(f(x) + f(y))$. By (5.3) and by Theorem 5, (i) $(f^{-1} \circ (w \ast f))^* \in \text{End}_k F^*$. Then by Theorem 5, (iii) and by Lemma 4.1, (5.4)

$$f^{-1} \circ ((\det u) \ast f) \equiv (f^{-1} \circ (u \ast f)) \circ (f^{-1} \circ (w \ast f)) \mod p$$

$$\equiv 0.$$ 

Write $\det u = \pi^n + \sum_{v=1}^{\infty} c_v T^v, c_v \in \mathfrak{o}$. Since $c_v = c_v, f^{-1} \circ (c_v f) \in \text{End}_k F$ for $v \geq 1$ by Theorem 3. Put $[c_v]^* = (f^{-1} \circ (c_v f))^*$ and $\xi(x) = x^q$. Since $f^q = f$, (5.4) implies that $\xi$ satisfies the equation

$$[\pi^n]^* + \sum_{v=1}^{\infty} [c_v]^* \xi^v = 0$$

in $\text{End}_k F^*$. 
§ 6. Formal groups over \(\mathbb{Z}\). Applications to zeta functions.

6.1. Suppose that for every prime number \(p\) and for every \(\nu \geq 1\) there is given a matrix \(C_{p,\nu}\) in \(M_n(\mathbb{Z})\) and that \(C_{p,\nu}\) commutes with \(C_{l,\mu}\) if \(p\) and \(l\) are distinct primes. Let \(s\) be a complex variable and consider the (formal) Dirichlet series

\[
(I_n + C_{p,\nu}p^{-s} + \cdots + C_{p,\nu}p^{-s+1-v\nu} + \cdots)^{-1} = \sum_{\nu=0}^{\infty} A_{p,\nu}p^{-\nu s}.
\]

Since \(A_{p,\nu}\) is expressed by \(C_{p,\nu}, \ldots, C_{p,\nu}\nu\) with coefficients in \(\mathbb{Z}\), \(A_{p,\nu}\) commutes with \(A_{l,\mu}\) if \(p \neq l\). Hence we can consider the global Dirichlet series

\[
\prod_p (I_n + C_{p,\nu}p^{-s} + \cdots + C_{p,\nu}p^{-s+1-v\nu} + \cdots)^{-1} = \sum_{m=1}^{\infty} A_m m^{-s},
\]

where \(A_{mm'} = A_mA_{m'} = A_{m'}A_m\) if \((m, m') = 1\).

**Theorem 8.** Let \(\{C_{p,\nu}\}\) and \(\{A_m\}\) be as above and form \(f(x) = \sum_{m=1}^{\infty} m^{-1}A_m x^m \in \mathbb{Q}[[x]]\). Then

\[
\prod_p (I_n + C_{p,\nu}p^{-s} + \cdots + C_{p,\nu}p^{-s+1-v\nu} + \cdots)^{-1} = \sum_{m=1}^{\infty} B_m m^{-s},
\]

for every \(p\) and \(F(x, y) = f^{-1}(f(x) + f(y))\) is a formal group over \(\mathbb{Z}\).

**Proof.** Put

\[
p f(x) + \sum_{\nu=1}^{\infty} C_{p,\nu} f(x^{p^\nu}) \equiv 0 \mod p \mathbb{Z}_p
\]

for every \(p\) and \(F(x, y) = f^{-1}(f(x) + f(y))\) is a formal group over \(\mathbb{Z}\).

**Corollary 1.** Any 1-dimensional formal group over \(\mathbb{Z}\) is strongly iso-
morphic to one obtained in Theorem 8. The strong isomorphism classes correspond bijectively to Dirichlet series of the form (6.1) with $n=1$ such that $0 \leq C_{p^v} < p$.

**Proof.** Let $F$ be a 1-dimensional formal group over $\mathbb{Z}$ and let $f$ be its transformer. By Theorem 4 we can find $C_p, C_{p^2}, \ldots \in \mathbb{Z}$ for every $p$ satisfying

$$pf(x) + \sum_{v=1}^{\infty} C_v f(x^{p^v}) \equiv 0 \mod p\mathbb{Z}.$$

Let $G$ be the formal group over $\mathbb{Z}$ obtained from the Dirichlet series $\prod_p \left(1 + \sum_{v=1}^{\infty} C_{p^v} p^{v-1} x^v \right)^{-1}$. By Theorem 8 and Theorem 2 $F \approx G$ over $\mathbb{Z}_p$ for every $p$. Since the strong isomorphism of $F$ to $G$ is unique, this implies $F \approx G$ over $\mathbb{Z}$. The second assertion is a consequence of the Corollary of Theorem 4.

**Corollary 2.** Notations and assumptions being as in Theorem 8, assume moreover that the $C_{p^v}$ commute with each other for a fixed prime $p$. Put $[C_{p^v}] = f^{-1} \circ (C_{p^v} f)$ and $\xi(x) = x^p$. Then $[C_{p^v}] \in \text{End}_\mathbb{Z} F$ for $v \geq 1$ and $\xi$ satisfies the equation

$$(6.5) \quad [pI_n]^* + \sum_{v=1}^{\infty} [C_{p^v}]^* \xi^v = 0$$

in $\text{End}_k F^*$, where $k = \mathbb{Z}/p\mathbb{Z}$.

**Proof.** Since $C_{p^v}$ commutes with $\sum_{n=0}^{\infty} C_v T^n$ for any $l$, $[C_{p^v}]$ is $l$-integral by Theorem 3. Hence $[C_{p^v}] \in \text{End}_\mathbb{Z} F$ by Proposition 1.6. The equation (6.5) is a direct consequence of (6.2) and of Lemma 4.1.

6.2. The results of 6.1 can be applied to zeta functions of the following types:

(a) Dirichlet $L$-functions.

(b) Zeta functions of elliptic curves over $\mathbb{Q}$.

(c) Dirichlet series obtained from a rational representation of Hecke operators in the space of cusp forms of dimension $-2$ with respect to a congruence unit group of an indefinite quaternion algebra over $\mathbb{Q}$ (cf. [19]).

We have already studied (a) and (b) in [10]. We note that we can remove the assumption on $S$ in [10], Theorem 5:

**Theorem 9.** Let $C$ be a 1-dimensional abelian variety over $\mathbb{Q}$ and let $F$ be a formal minimal model for $C$ over $\mathbb{Z}$ (cf. [10]). Let $L_p(s)$ be the $p$-factor of the $L$ function of $C$ and put $L_S(s) = \prod_{p \in S} L_p(s)$ for any set $S$ of prime numbers. Then the formal group obtained from $L_S(s)$ is strongly isomorphic to $F$ over $\bigcap_{p \in S} (\mathbb{Z}_p \cap \mathbb{Q})$.

**Proof.** Let $G$ be the formal group obtained from $L_S(s)$. Since $L_p(s) = 1/(1 \pm p^{-s})^{-1}$ or of the form $(1 - a_p p^{-s} + p^{1-2s})^{-1}$, $G$ is a formal group over $\mathbb{Z}$ by Theorem 8. As a strong isomorphism of $G$ to $F$ is unique if it exists, it
The theory of commutative formal groups

suffices to prove \( F \cong G \) over \( \mathbb{Z}_p \) for every \( p \in S \). Let \( C_p \) be the reduction of \( C \) modulo \( p \). The cases where \( C_p \) has a singular point were treated in \([10]\). Suppose that \( C_p \) is an abelian variety with \( L_p(s) = (1-a_p p^{-s} + p^{1-2s})^{-1} \). Since the Frobenius \( \xi \) of \( C_p \) satisfies

\[
\xi^2 - a_p \xi + p = 0,
\]

the transformer \( f \) of \( F \) satisfies

\[
f^{-1}(pf(x) - a_p f(x^p) + f(x^{p^2})) \equiv 0 \mod p \mathbb{Z}_p.
\]

By Lemma 4.2 it follows from (6.6)

\[
pf(x) - a_p f(x^p) + f(x^{p^2}) \equiv 0 \mod p \mathbb{Z}_p.
\]

The fact \( F \cong G \) over \( \mathbb{Z}_p \) follows from (6.7), Theorem 8 and Theorem 2. This completes the proof of our theorem.

Notations being as above, put \( L_0(s) = \prod_p L_p(s) \) and let \( G \) be the formal group attached to it. Then there is \( \varphi(x) \in \mathbb{Z}[[x]] \) such that \( \varphi(x) \equiv x \mod \deg 2 \) and \( \varphi \circ g = \varphi \circ G \). If the conjecture of Weil \([21]\) on \( L_0(s) \) is true, the power series \( \varphi \) would be the “q-expansion” of a suitable automorphic function with respect to \( \Gamma_0(N) \) where \( N \) is the conductor of \( C \).

It would be interesting to see that our results yield a simple proof of a special case of the main result of Eichler \([6]\) and Shimura \([18]\). Let \( j(z) \) be the elliptic modular function and put \( L = \mathbb{Q}(j(z), j(Nz)) \) for \( N \geq 2 \). Then \( L \) is a field of algebraic function over \( \mathbb{Q} \) and \( LC \) is the field of automorphic functions with respect to the subgroup \( \Gamma_0(N) \) of \( SL(2, \mathbb{Z}) \). We shall consider the case where the genus of \( L \) is equal to 1. Let \( C \) be a complete non-singular model for \( L \) over \( \mathbb{Q} \). Since \( j(z) \) has q-expansion

\[
j(z) = q^{-1} + 744 + \ldots
\]

with coefficients in \( \mathbb{Z} \) where \( q = \exp(2\pi \sqrt{-1} z) \), the infinite point \( z = i \infty \) corresponds to a rational point \( \mathfrak{q} \) on \( C \) and \( C \) can be considered an abelian variety over \( \mathbb{Q} \), with the origin \( \mathfrak{q} \). Expanding the group law of \( C \) by means of the local parameter \( j(z)^{-1} \) at \( \mathfrak{q} \), we get a formal group \( F \) over \( \mathbb{Q} \). By the theory of reduction there exists a finite set \( S' \) of prime numbers such that for \( p \in S' \) the reduction \( C_p \) of \( C \) mod \( p \) is non-singular and \( j(z)^{-1} \) is a local parameter at the origin of \( C_p \). Then, for \( p \in S' \) \( F \) has \( p \)-integral coefficients and the \( p \)-th power endomorphism of the reduction \( F_p \) of \( F \) mod \( p \) satisfies the same characteristic equation as that of \( C_p \). Let \( f \) be the transformer of \( F \). Then \( df(x) \) is the canonical invariant differential on \( F \), i.e. the \( j(z)^{-1} \)-expansion of a differential of the first kind on \( C \). Let \( \varphi(q) \) be the q-expansion of \( j(z)^{-1} \). Then \( \varphi(x) \in \mathbb{Z}[[x]] \) and \( \varphi(x) \equiv x \mod 2 \) by (6.8). Put
Then, as is well-known, $\sum_{m=1}^{\infty} a_m q^m$ is the $q$-expansion of a cusp form of dimension $-2$ with respect to $\Gamma_0(N)$ and by Hecke [9] the Dirichlet series $\sum_{m=1}^{\infty} a_m m^{-s}$ has an Euler product of the form

$$\prod_{p \mid N} (1 - a_p p^{-s})^{-1} \prod_{p \nmid N} (1 - a_p p^{-s} + p^{1-2s})^{-1}, \quad a_p \in \mathbb{Z}. $$

Form $G(x, y) = g^{-1}(g(x) + g(y))$ with $g = f \circ \phi$. By Theorem 8 $G$ is a formal group over $\mathbb{Z}$, so that $F$ is also a formal group over $\mathbb{Z}$. Let $p$ be a prime number such that $p \nmid S$ and $p \nmid N$. Then, by Corollary 2 of Theorem 8 the Frobenius of $G_p$ is a root of the equation

$$p - a_p X + X^2 = 0. $$

Since $F \simeq G$ over $\mathbb{Z}$, (6.9) is also the characteristic equation for the Frobenius of $F_p$, and then of $C_p$. Therefore $(1 - a_p p^{-s} + p^{1-2s})^{-1}$ coincides with the $L$ function of $C_p$. This proves the principal theorem of [18] in this case.

**Remark.** By considering Néron's minimal model for $L$, we can prove that the $p$-factor of the Hecke Dirichlet series coincides with that of the zeta function of $L$, assuming only that $j(z)^{-1}$ is a local parameter at the origin of $C_p$. See [10] as for the case $C_p$ is singular. In view of the conjecture of Weil it is plausible that $F$ is a formal minimal model for $C$.

6.3. We now deal with (c). We use the terminology, notations and results of Shimura [19]. Let $\Phi$ be an indefinite quaternion algebra over $\mathbb{Q}$ and let $\mathfrak{o}$ be a maximal order in $\Phi$. For a natural number $N$ prime to the discriminant of $\Phi$, $\Gamma_N$ denotes the group consisting of units $\gamma$ in $\mathfrak{o}$ such that $N(\gamma) = 1$ and $\gamma \equiv 1 \mod N_0$. $\Gamma_N$ is an automorphic functions relative to $\Gamma_N$ and let $n$ be its genus. Take $\mathcal{L}_N, \mathcal{C}_N$ and $J_N$ as in [19]. $\mathcal{L}_N$ is a function field over $\mathbb{Q}$ such that $\mathcal{L}_N C = \mathfrak{R}_N, \mathcal{C}_N$ is its complete non-singular model and $J_N$ is a Jacobian of $\mathcal{C}_N$, each defined over $\mathbb{Q}$. Let $\mathcal{D}_N(\mathcal{C}_N)$ and $\mathcal{D}_N(\mathfrak{J}_N)$ be the spaces of differentials of the first kind on $\mathcal{C}_N$ and $J_N$, respectively. For $f, g \in \mathcal{L}_N$, $gdf \in \mathcal{D}_N(\mathcal{C}_N)$ if and only if $g f' \in \mathcal{S}_N(\Gamma_N)$. Let $\omega = \{a_1, \ldots, a_n\}$ be a base of $\mathcal{D}_N(\mathcal{C}_N)$, defined over $\mathbb{Q}$. Fixing a canonical map $\mathcal{C}_N \rightarrow J_N$ (which may not be defined over $\mathbb{Q}$), let $w$ and $\eta$ be the corresponding bases of $\mathcal{S}_N(\Gamma_N)$ and $\mathcal{D}_N(\mathfrak{J}_N)$, respectively. For $\alpha \in \mathfrak{o}$ such that $N\alpha > 0$, $(N, \alpha) = 1$, $\Gamma_N \alpha \Gamma_N$ operates on $\mathcal{S}_N(\Gamma_N)$ on the one hand. Let $\mathcal{E}_N(\Gamma_N \alpha \Gamma_N)$ denote its representation matrix relative to $w$. On the other hand $\Gamma_N \alpha \Gamma_N$ yields a correspondence $X_{\xi}$ of $\mathcal{C}_N$ over $\mathbb{Q}$ where $\xi = \alpha \omega$ and then induces an endomorphism $\xi$ of $J_N$. This $\xi$ is defined over $\mathbb{Q}$ ([19], p. 325). Denoting by $M^d(\xi)$ the representation matrix of $\xi$ with respect to $\eta$, we have...
The theory of commutative formal groups 243

we have

\[(6.10) M^d(\xi) = \mathfrak{X}_d(\Gamma_N \alpha \Gamma_N)\]

([19], p. 327), where \(M^d(\xi) \in M_n(\mathbb{Q})\). By [19] the \(\mathfrak{X}_d(\Gamma_N \alpha \Gamma_N)\) are semi-simple and commute with each other, and their eigenvalues are algebraic integers. Hence there is a regular matrix \(P\) in \(M_n(\mathbb{Q})\) such that the \(P^{-1}\mathfrak{X}_d(\Gamma_N \alpha \Gamma_N)P\) are all in \(M_n(\mathbb{Z})\). By changing the bases if necessary, we may assume that the \(\mathfrak{X}_d(\Gamma_N \alpha \Gamma_N)\) are already in \(M_n(\mathbb{Z})\).

Let \(S_1\) be the set of prime numbers which fail to satisfy at least one of P. 1)-10) in [19]. Then \(S_1\) is a finite set. Let \(S_2\) be the set of prime divisors of \(d(\Phi)\). By Theorem 4 of [19] we have for \(p \in S_1 \cup S_2\)

\[(6.11) \tilde{X}_q = \xi + T\tilde{Y}_p,
\]

where \(q\) is an integral left \(\alpha\)-ideal such that \(N(q) = p\), \(T\) is the Frobenius of \(\mathfrak{X}_N\) and \(Y_p\) is defined on p. 315 of [19]. Correspondingly we have

\[(6.12) \tilde{\xi}_p = \pi + \pi' \circ \tilde{\eta}_p.\]

Now let \(t = \{t_1, \ldots, t_n\}\) be a system of local parameters \((\in \mathbb{Q}(J_N))\) at the origin of \(J_N\). Expanding the group law of \(J_N\) into power series relative to \(t\), we get an \(n\)-dimensional formal group \(F\) over \(\mathbb{Q}\). We shall call this formal group a formal model for \(J_N\). (A formal model is also obtained from the \(t\)-expansion of a base of \(\mathfrak{D}_o(J_N)\), defined over \(\mathbb{Q}\)). By the theory of reduction ([20], Chapter III) there is a finite set \(S_3\) of prime numbers such that for \(p \in S_3\):

(i) \(t\) is a system of local parameters at the origin of \(\tilde{J}_N = \text{the reduction of } J_N \mod p\).

(ii) The differentials \(\eta_1, \ldots, \eta_n\) have good reductions mod \(p\) and yield a base of \(\mathfrak{D}_o(\tilde{J}_N)\).

Assume \(p \in S_1 \cup S_2 \cup S_3\). Then \(F\) has coefficients in \(\mathbb{Z}_p\) and an endomorphism of \(\tilde{\xi}\) of \(J_N\), corresponding to some \(\Gamma_N \alpha \Gamma_N\), induces an endomorphism of \(F\) over \(\mathbb{Z}_p\). Let \(f\) be the transformer of \(F\) and let \(f^{-1} \circ (C(\xi)f)\) \((C(\xi) \in M_n(\mathbb{Z}_p))\) denote this endomorphism of \(F\). Since \(\tilde{\xi}'\) is also defined over \(\mathbb{Q}\), it induces the endomorphism \(f^{-1} \circ (C(\tilde{\xi}')f)\) of \(F\) over \(\mathbb{Z}_p\). Now it follows from (6.12) that

\[\tilde{\xi}_p' = \pi' + \tilde{\eta}_p \circ \pi\]

and then

\[(6.13) p - \tilde{\xi}_p' \circ \pi + \tilde{\eta}_p \circ \pi^2 = 0.\]

This implies

\[f^{-1}(pf(x) - C(\xi_p)f(x^p) + C(\eta_p)f(x^p)) \equiv 0 \mod p\mathbb{Z}_p,\]

or by Lemma 4.2
\[(6.14) \quad pf(x) - C(\xi_p) f(x^p) + C(\eta_p) f(x^{p^2}) \equiv 0 \mod p \mathbb{Z}_p.\]

Let \( E \) be the subring of \( \text{End}_Q J_N \) generated by endomorphisms corresponding to \( \{ I'N a I'N a o, N(a) > 0, (a, N) = 1 \} \). Then, as \( E \otimes \mathbb{Q} \) is a commutative semi-simple algebra over \( \mathbb{Q} \), the map \( \xi \mapsto \xi' \) yields an isomorphism of \( E \) into \( \text{End}_Q J_N \). Now \( J_N \) is self-dual and \( M^d(\xi) \) is the transposed matrix of \( M^d(\xi) \), since \( M^d(\xi) \in M_n(\mathbb{Q}) \). (For example see [20], p. 25). As \( M^d(\xi) \) is conjugate with \( M^d(\xi') \), \( M^d(\xi) \) and \( M^d(\xi') \) have the same trace. Therefore there is an invertible matrix \( P_1 \in M_n(\mathbb{Q}) \) such that
\[(6.15) \quad M^d(\xi) = P_1^{-1} M^d(\xi) P_1 \quad \text{for all} \quad \xi \in E.\]

Now since the \( t \)-expansion of \( \gamma \) is a base of \( \mathcal{D}^*(F; \mathbb{Q}) \) and \( C(\xi') \) \( (\xi \in E) \) is the representation matrix of \( \xi' \) relative to the canonical base \( df(x) \) of \( \mathcal{D}^*(F; \mathbb{Q}) \), we can find an invertible matrix \( P_2 \in M_n(\mathbb{Q}) \) such that
\[(6.16) \quad C(\xi') = P_2^{-1} M^d(\xi') P_2 \quad \text{for all} \quad \xi \in E.\]

Putting \( P_3 = P_1 P_2 \), we get from (6.15), (6.16)
\[(6.17) \quad C(\xi') = P_3^{-1} M^d(\xi') P_3 \quad \text{for all} \quad \xi \in E.\]

Let \( S_4 \) be the set of prime numbers \( p \) such that \( P_3 \) or \( P_3^{-1} \) is not \( p \)-integral, and put \( S = \bigcup_{i=1}^4 S_i \). \( S \) is a finite set. For \( p \in S \) we get from (6.14) and (6.17)
\[(6.18) \quad pf(x) - M^d(\xi_p) f(x^p) + M^d(\eta_p) P_2 f(x^{p^2}) \equiv 0 \mod p \mathbb{Z}_p.\]

Now replacing the parameters \( t = (t_1, \ldots, t_n) \) by \( u = P_3 t \), we obtain the formal model \( H(x, y) = P_3 F(P_3^{-1} x, P_3^{-1} y) \) of \( J_N \), with the transformer \( h(x) = P_3 f(P_3^{-1} x) \). For \( p \in S \) we have
\[ (P_3^{-1} x)^{p^v} \equiv P_3^{-1} x^{p^v} \mod p \mathbb{Z}_p \]
and then by Lemma 4.2
\[(6.19) \quad f((P_3^{-1} x)^{p^v}) \equiv f(P_3^{-1} x^{p^v}) \mod p \mathbb{Z}_p.\]

By (6.18) and (6.19) we get finally
\[(6.20) \quad ph(x) - M^d(\xi_p) h(x^p) + M^d(\eta_p) h(x^{p^2}) \equiv 0 \mod p \mathbb{Z}_p\]
for \( p \in S \).

Now we have
\[(6.21) \quad M^d(\xi_p) = I_2(p; No) \quad \text{and} \quad M^d(\eta_p) = R_2(p; No)\]
([19], p. 327). Let \( M \) be the product of all primes in \( S \) and put \( Z_S = \bigcap_{p \in S} (Z_p \cap \mathbb{Q}) \). The Dirichlet series
\[\prod_{p \in M \setminus S} \left( I_n - I_2(p; No) p^{-s} + R_2(p; No) p^{1-s} \right) = \sum_{(m, MN) = 1} \mathcal{I}_4(m; No) m^{-s}\]
is a main part of the one defined in [19]. Let $G$ be the formal group over $\mathbb{Z}$ corresponding to it by Theorem 8. By Theorem 2 it follows from (6.20) and (6.21) that $G \cong H$ over $\mathbb{Z}_p$ for every $p \not\in S$. Hence $G \cong H$ over $\mathbb{Z}_S$ by the uniqueness of strong isomorphism. We have proved the following theorem:

**Theorem 10.** Let notations be as in [19] and let $\mathcal{X}_S$ be an integral representation as above. Then there is a finite set $S$ of prime numbers such that the formal group obtained from the Dirichlet series $\sum_{(m, N)_1=1} \mathcal{X}_S(m; N)m^{-s}$ is strongly isomorphic over $\mathbb{Z}_S$ to a formal model for $J_N$.

Thus the matrix Dirichlet series $\sum \mathcal{X}_S(m; N)m^{-s}$ itself (not only its determinant) has important significance for $J_N$. What kind of curve over $\mathbb{Q}$ has a Jacobian whose formal completion is isomorphic to a formal group corresponding to a matrix Dirichlet series with Euler product?

6.4. All zeta functions, which we studied in 6.2 and 6.3, are of the form $\prod_p (1 + C_p p^{-s} + C_p^2 p^{-2s})^{-1}$. Do there exist number-theoretic Dirichlet series of the form (6.1) such that not all $C_p^2$ are equal to 0 for $\nu \geq 3$? If such ones exist, formal groups over $\mathbb{Z}$ obtained from them would be non-algebroid. Their transformers would be obtained from analytic functions, perhaps satisfying suitable kinds of differential equations.

Osaka University

**References**


France, 83 (1955), 251-274.


