Semigroups in an ordered Banach space

By Bruce D. CALVERT

(Received May 6, 1970)

This note gives the basic theory of linear and nonlinear semigroups in the class of Banach spaces $X$ with a cone (which includes Banach lattices and Banach spaces). A Hille-Yosida theorem and a perturbation theorem are given for normal cones. Spaces with two cones are considered, obtaining convergence of some integral curves $x(t)$, where $\frac{dx}{dt}(t) = Ax(t)$ and $A$ is a nonlinear dispersive operator. We refrain from generalizing all the Banach lattice results; also some of the theory extends readily to study $\frac{dx}{dt}(t) \in A_t x(t)$, where $A_t$ depends on $t$, is multivalued, and $X$ is an ordered l.c.s.

The author wishes to thank K. Gustafson for helpful comments and F. Browder, R. Rockafeller, J. Dorroh and K. Sato for kindly making preprints available.

Operators in ordered spaces.

Suppose $X$ a Banach space over the real numbers $R$ or the complex numbers $C$. Suppose $K$ a closed convex subset invariant under multiplication by positive numbers. $K$ is called a cone, and defines an ordering, $x \preceq y$ if $y - x$ is in $K$. Let $K^* = \{ f \in X^*: \text{Re} \langle f, x \rangle \geq 0 \text{ for all } x \in K \}$, $B^* = \{ f \in X^*: |f| \leq 1 \}$ and for $x$ in $X$ define $|x|_K = \sup \{ \text{Re} \langle f, x \rangle: f \in B^* \cap K^* \}$ the support functional of $B^* \cap K^*$. Rockafeller [13] page 39 points out that such a functional has subgradient $J_K(x)$ consisting of elements $f$ of $B^* \cap K^*$ where $\text{Re} \langle f, x \rangle = |x|_K$. We recall that $f$ is in the subgradient of $|x|_K$ means that for all $y$ in $X$

$$|y|_K \geq |x|_K + \text{Re} \langle f, y-x \rangle.$$

Lem. 1. Suppose $x(t), y(t)$ are strongly continuous, once weakly differentiable on the left,

$$\frac{dx}{dt}(t) = -Ax(t), \quad \frac{dy}{dt}(t) = -Ay(t),$$

and for each $x(t), y(t)$, there exists $w(t)$ in $J_K(x(t)-y(t))$ with $\text{Re} \langle w(t), Ax(t)-Ay(t) \rangle \geq 0$. Then $|x(t)-y(t)|_K$ is nonincreasing.

Proof.
\[ |x(t-h) - y(t-h)|_K - |x(t) - y(t)|_K \]
\[ \geq \text{Re} \left( w(t), (x(t-h) - x(t)) - (y(t-h) - y(t)) \right) , \]
with \( h > 0, w(t) \) as above, since \( w(t) \) is in the subgradient of \( |x(t) - y(t)|_K \).

Consequently
\[ \lim_{h \to 0} h^{-1} \left( |x(t-h) - y(t-h)|_K - |x(t) - y(t)|_K \right) \]
\[ \geq \text{Re} \left( w(t), Ax(t) - Ay(t) \right) \]
\[ \geq 0 , \]
giving the result, since \( |x(t) - y(t)|_K \) is continuous. Q. E. D.

**Lemma 2.** Suppose \( x(t), y(t) \) are once weakly differentiable on the right,
\[ \frac{dx}{dt}(t) = -Ax(t) , \quad \frac{dy}{dt}(t) = -Ay(t) , \]
and \( |x(t) - y(t)|_K \) is nonincreasing. Then for any \( w(t) \) in \( J_K(x(t) - y(t)) \),
\[ \text{Re} \left( w(t), Ax(t) - Ay(t) \right) \geq 0 . \]

**Proof.** Take any \( w(t) \) as above, \( h > 0, \)
\[ |x(t+h) - y(t+h)|_K \geq |x(t) - y(t)|_K \]
\[ + \text{Re} \left( w(t), (x(t+h) - x(t)) - (y(t+h) - y(t)) \right) . \]
Dividing by \( h \) and letting \( h \to 0 \) give the result. Q. E. D.

**Remark.** If \( X \) is a Banach lattice, then \( |x|_K = \|x^+\| \) and \( J_K(x^+) \subseteq J_Kx \subset J_{\text{t}0}(x^+) \) where \( J_{\text{t}0} \) is the subgradient of the support functional of \( B^* \) and \( J_K(x) \) is any positive duality map. We recall a positive duality map is a function \( J_1 \) from \( X \) to \( X^* \) with \( \|J_1x\| = 1, (J_1x, x) = \|x\|, (J_1x, y) = 0 \) if \( x \perp y \), and \( (J_1x, y) \geq 0 \) if \( x \geq 0 \) and \( y \geq 0 \). The first assertion follows from \[2\] Proposition 1.1, and the second from \[2\] Proposition 1.2, noting that if \( w(x) \) is in \( J_K(x) \) then \( w(x) \geq 0 \).

We recall (Krasnoselskii \[9\]) that \( K \) is normal if and only if there is an equivalent monotonic norm. The norm is monotonic means that \( 0 \leq x \leq y \) implies \( \|x\| \leq \|y\| \).

**Theorem 3.** \( K \) is normal if and only if \( |u|_K + |u|_K^+ \) is an equivalent norm.

**Proof.** In any case, \( |u|_K \leq \|u\| \), giving \( |u|_K + |u|_K^+ \leq 2\|u\| \). Suppose \( K \) is normal. As a consequence of Theorem 3.3, page 219 of Schaefer \[18\], there exists \( n \) in \( Z^+ \) such that for \( f \) in \( B^* \) there exist \( f_1, f_2 \) in \( K^* \), \( f = f_1 - f_2, \|f_1\| \leq n, \|f_2\| \leq n \). For \( u \) in \( X \), \( f \) in \( B^* \), \[ |(f, u)| = |(f_1, u) - (f_2, u)| \leq n |u|_K + n |u|_K^+. \]
Consequently, \( \|u\| = \sup \{|(f, u)| : f \text{ in } B^* \} \leq n(\|u\|_K + \|u\|_K^+) \).

Conversely, suppose \( K \) not normal. Then, by Theorem 1 of Krasnoselskii \[9\], there exist sequences \( x_n, y_n \) in \( K, \|x_n\| = \|y_n\| = 1, \) with \( \|x_n + y_n\| \to 0 \). Therefore \( |x_n + y_n|_K \to 0 \), giving \( |x_n|_K \to 0 \). Since \( |x_n|_K + |x_n|_K^+ \to 0 \), this is not an equivalent norm. Q. E. D.
We say $U : D(U) \to X$, $D(U) \subseteq X$, is $K$ nonexpansive if for $x, y$ in $D(U)$, 
$\| U(x) - U(y) \|_K \leq \| x - y \|_K$. We say $A : D(A) \to X$, $D(A) \subseteq X$, is $K$ accretive if for $x, y$ in $D(A)$, there exists $w$ in $J_K(x-y)$ with $\text{Re}(w, Ax - Ay) \geq 0$. Clearly $U$ is $K$ nonexpansive if and only if $-K$ nonexpansive. We say $A$ is $K$ dispersive if $-A$ is $K$ accretive. We say $U$ is nonexpansive if $K = \{0\}$, and $A$ is $g$ accretive or $-A$ is $g$ dissipative if $K = \{0\}$. See Lumer and Phillips [10].

**Lemma 4.** $A$ is $K$ accretive if and only if for all $d > 0$ $(I + dA)^{-1}$ is $K$ nonexpansive.

**Proof.** Similar to Theorem 9.1 of Browder [1], using the fact that $x \to |x|_K J_K x$ is upper semicontinuous from the strong topology of $X$ to subsets of $X^*$ with the weak* topology (c.f. Cudia [4], Theorem 4.3.).

The functionals 

$$\varphi^* (f, g) = \lim_{d \to +\infty} d^{-1} (|f + dg|_K - |f|_K),$$

$$\psi^* (f, g) = -\varphi^* (f, -g)$$

were introduced by Sato [15] as the maximum and minimum (0) gauge functionals on a Banach lattice.

**Theorem 5.**

$$\varphi^* (f, g) = \sup \{ \text{Re} (h, g) : h \in J_K(f) \},$$

$$\psi^* (f, g) = \inf \{ \text{Re} (h, g) : h \in J_K(f) \}.$$

**Proof.** We prove the first assertion; the second is similar. By definition of subgradient for $d > 0$ and $h$ in $J_K(f)$,

$$|f + dg|_K \geq |f|_K + \text{Re} (h, dg),$$

giving $\varphi^* (f, g) \geq \text{Re} (h, g)$. Fixing $f, g$ $\varphi(h) = \varphi^* (f, h)$ defined on the space spanned by $f, g$ has $\varphi(ah) = a \varphi(h)$, $a \geq 0$, and $\varphi(h) \leq |h|_K$.

By Theorem 5.4 of Schaefer [18], we can extend $\varphi$ to $\varphi$ in $B^* \cap K^*$, i.e. a positive linear functional with norm $\leq 1$. Since $\varphi(f) = |f|_K$, $\varphi$ is in $J_K(f)$.

For a comparison of various functionals as in Sato [14], Phillips [12], Hasegawa [7], Dorroh [5] we refer to Sato [15].

**Corollary 6.** $A$ is $K$ accretive if and only if $\varphi^* (f-g, Af-Ag) \geq 0$ for $f, g$ in $D(A)$. If $-A$ generates a $K$ nonexpansive semigroup then $\varphi^* (f-g, Af-Ag) \leq 0$ for $f, g$ in $D(A)$.

The first assertion follows from Theorem 5, the second from Lemma 2.

**Theorem 7.** Suppose $X$ a Banach space with cone $K$.

(a) Suppose $U(t)$ a continuous bounded semigroup of positive linear operators on $X$. Then the generator $A$ is $K$ dispersive in an equivalent norm, and $R(I-A) = X.$
(b) Suppose $K$ is normal, $A$ is a densely defined $K$ accretive linear operator with $R(I + A) = X$.

Then $-A$ generates a continuous bounded semigroup of positive linear operators on $X$.

**Proof.** (a) Renorm $X$ by $\|x\| = \sup \{ \|U(t)x\| : t \geq 0 \}$, so $\|U(t)\| \leq 1$. Given $f \in B^* \cap K^*$, and $t \geq 0$, then $U(t)^* f$ is in $B^* \cap K^*$. For $x \in X$, $t \geq 0$,

$$|U(t)x|_X = \sup \{ \text{Re}(f, U(t)x) : f \in B^* \cap K^* \}$$

$$= \sup \{ \text{Re}(U(t)^* f, x) : f \in B^* \cap K^* \}$$

$$\leq \|x\|_X.$$

By Lemma 2, $A$ is $K$ dispersive.

(b) It is enough to show $A$ is $g$ accretive in an equivalent norm. For $d > 0$, $(I + dA)^{-1}$ is $K$ nonexpansive, and consequently nonexpansive in $|x|_X + |x|_{-K}$, giving $A$ $g$ accretive in this norm, by Theorem 3. Q. E. D.

**Corollary.** That a densely defined linear operator $A$ is the generator of a continuous semigroup of positive linear operators on $X$ implies that there exists $m$ in $\mathbb{R}$ with $A - mI$ $K$ dispersive in an equivalent norm, $R(A - nI) = X$ for $n > m$, and conversely if $K$ is normal.

**Proof.** Supposing $A$ generates a positive semigroup $T_t$ of class $C^0$, then $\|T_t\| \leq Me^{mt}$ with constants $m, M$, by Hille's theorem (Page 232 of Yosida [19]). Then $U_t = e^{-mt}T_t$ is positive, and bounded, so the generator $A - mI$ is $K$ dispersive in an equivalent norm by (a) of Theorem 7.

Conversely, by (b), $U_t$ generated by $A - mI$ is positive, continuous and bounded, so $e^{mt}U_t = T_t$ is positive and continuous. Its generator is $A$. Q. E. D.

This answers a question of Sato [15]. Next we look at pseudo-resolvents and see that some results hold without their being a resolvent in the usual sense.

**Theorem 8.** Suppose $X$ a Banach space and $\{J_\lambda : \lambda \geq \lambda_0\}$ a pseudo-resolvent, i.e. a family of bounded operators in $X$ with

$$J_\lambda - J_\mu = (\mu - \lambda)J_\mu J_\lambda.$$

Suppose $\|J_\lambda\| \leq M$ for all $\lambda$.

(a) Then the multivalued operator $A = \lambda I - J_\lambda$ is well defined.

(b) If $X$ is reflexive, there is a bounded semigroup $U(t)$ on the closure of $D(A)$ with infinitesimal generator $A_1$ having $D(A_1) = D(A)$, and for $x \in D(A)$, $A_1(x)$ is in $A'(x) = \{ y \in A(x) : \|y\| = d(0, Ax) \}$.

(c) If $X$ is ordered and $J_\lambda$ are all positive then so is $U(t)$.

**Proof.** (a) By Lemma 1', page 217 of Yosida [19], $N(J) \cap cl(R(J)) = \{0\}$, where $R(J)$ is the common range and $N(J)$ the common nullspace of the family $\{J_\lambda\}$. As in Yosida we have for any $\lambda, \mu$
Semigroups in Banach space

\[ J_t J_\mu (\lambda I - J_t^\mu - \mu I + J_\mu^\mu) = (\lambda - \mu) J_t J_\mu - (J_\mu - J_t) \]

\[ = 0. \]

It follows that for \( x \) in \( X \), \( J_t (\lambda I - J_t^\mu - \mu I + J_\mu^\mu) x \) is in \( N(J) \), and hence in \( N(J) \cap cl(R(J)) = \{0\} \). Consequently \( (\lambda I - J_t^\mu) x = (\mu I - J_\mu^\mu) x \).

(b) This may be proved as on page 246 of Yosida [19] using a little of the technique of Theorem 9.23 of Browder [1]. Reflexivity is needed because \( D(A) \) will not be dense unless \( A \) is single valued.

(c) As in (a) of Theorem 7 there is an equivalent norm in which \( \{\lambda J_t\} \) are nonexpansive, hence \( K \) nonexpansive, giving \( A K \) dispersive, by Lemma 4, and \( U(t) \) \( K \) nonexpansive by Lemma 1. Q. E. D.

REMARK. There should be a nonlinear extension of this. We note the condition for the pseudo-resolvent to admit a potential operator \( V \) can be stated in terms of the multivalued \( A \) above, giving \( V = -A^{-1} \), (see Sato [16]).

Phillips [12], Gustafson and Sato [17] ask if their (linear) dispersive operators \( A \) are always dissipative. It is understood that \( A \) should be a densely defined operator; counterexamples defined on a one dimensional subspace are known. If Bohnenblusts' property \( P \) holds, which says, in the setting of cones, \( |x|_K = |y|_K \) and \( |x|_{-K} = |y|_{-K} \) imply \( \|x\| = \|y\| \), then as in Calvert [3], \( K \) nonexpansive implies nonexpansive, so that by Lemma 4, \( K \) accretive implies \( g \) accretive. This result is contained in Theorem 5.1 of Sato [15] for linear operators in a lattice.

To study perturbation theory, as in Sato and Gustafson [17], a related question is whether a \( K \) accretive operator is \( g \) accretive in some equivalent norm. By Theorem 3, the answer is yes if \( K \) is normal. The multiplicative perturbation results of [17] then follow from the results of Gustafson in the \( g \) accretive case.

THEOREM 9. Suppose \( A \) is \( K \) accretive, \((I + A) \) surjective, \( D(B) \supset D(A) \), and there exists \( a < 1 \), \( b \) in \( R^+ \), with \( \|Bx\| \leq a\|Ax\| + b\|x\| \) for \( x \) in \( D(A) \). Suppose \( K \) normal. Suppose

(a) \( B \) is \( K \) accretive, \((I + B) \) surjective, \( J_K \) uniformly continuous on bounded sets, and \( X \) reflexive, or

(b) \( A \) and \( B \) are linear, and \( A + B \) is \( K \) accretive.

Then \( -(A + B) \) generates a \( K \) nonexpansive semigroup on \( cl(D(A)) \).

PROOF. (a) Similar to Theorem 9.22 of Browder [1]. (Cases where \( J_K \) is uniformly continuous on bounded sets include \( X^* \) uniformly convex and either \( K = \{0\} \) or \( X \) is a Banach lattice.)

(b) The result follows from the accretivity in an equivalent norm \( \|\cdot\|_e \) for \( a < a_0 \), some small \( a_0 \), for then \( \|Bx\|_e \leq a\|Ax\|_e + b\|x\|_e \) with \( a < 1 \). The result
follows by Lemma 5.1 of Gustafson [6]; there exist \( c_j > 0 \) (\( j = 1, \ldots, n \)), \( a^1 < a, b^1 < \infty \), with \( \sum c_j = 1 \) and for \( k = 1, 2, \ldots, n, x \) in \( D(A) \),

\[
\|c_k \delta(x)\| \leq a \left( (A + \sum_{j=1}^{n} c_j \delta) x + b \|x\| \right).
\]

Q. E. D.

**Lemma 10.** Suppose \( x(t) \) strongly continuous, once weakly differentiable on the left, \( \frac{dx}{dt}(t) = -Ax(t) \), and for \( t \geq s \) there is \( w \) in \( J_K(x(t) - x(s)) \) with \( \text{Re} \langle w, Ax(t) - Ax(s) \rangle \geq 0 \). Then \( |Ax(t)|_K \) is nonincreasing.

**Proof.** Let \( y(t) = x(t-h) \), \( h > 0 \), then \( \frac{dy}{dt}(t) = -Ay(t) \), and by Lemma 1 \( |x(t) - y(t)|_K \) is nonincreasing. Dividing by \( h \) and letting \( h \to 0 \) give the result. Q. E. D.

Lemmas 1, 2, 10 compare with Propositions 1.10, 1.11, 1.12 of Calvert [2].

**Theorem 11.** Suppose \( X \) a Banach space with cones \( K \) and \( H \), \( H \subset K \). Suppose \( x(t) \) is once weakly differentiable, strongly continuous, \( \frac{dx}{dt}(t) = -Ax(t) \), and \( x(t) \) is increasing with respect to \( K \). Suppose the norm is monotonic with respect to \( K \), and \( A \) is \( K \) accretive or \( g \) accretive. Then \( |Ax(t)|_H \) is nonincreasing.

**Proof.** By Lemma 10, it suffices to show that for \( t \geq s \) there is \( w \) in \( J_K(x(t) - x(s)) \) with \( \text{Re} \langle w, Ax(t) - Ax(s) \rangle \geq 0 \). By Lemma 2, if \( A \) is \( K \) accretive, any \( w \) in \( J_K(x(t) - x(s)) \) gives \( \text{Re} \langle w, Ax(t) - Ax(s) \rangle \geq 0 \), and if \( A \) is \( g \) accretive, any \( w \) in \( J_g(x(t) - x(s)) \) gives \( \text{Re} \langle w, Ax(t) - Ax(s) \rangle \geq 0 \). Putting \( x = x(t) - x(s) \), it suffices to show that \( x \) in \( K \) implies \( J_K(x) \cap J_H(x) \) is nonempty and \( J_H(x) \) is nonempty. Since the norm is monotonic with respect to \( K \), Proposition 1.1 of Calvert [2] tells us there is \( f \) in \( K^* \cap B^* \) with \( \text{Re} \langle f, x \rangle = \|x\| \). Since \( K^* \subset H^* \), this \( f \) is in \( J_K(x) \cap J_H(x) \) and \( J_H(x) \).

**Corollary.** If in addition \( Ax(0) \) is in \( -H \) then \( Ax(t) \) is in \( -H \) as long as \( x(t) \) is defined, and \( x(t) \) is increasing with respect to \( H \).

**Remark.** A broad sufficient condition for \( x(t) \) to be increasing if \( \frac{dx}{dt}(t) = -Ax(t) \) and \( Ax(0) \) is in the cone \( -K \) is: for \( z \) in \( D(A) \) there exists \( k \) in \( R^+ \) and \( N \) a neighborhood of \( z \) in \( X \), such that \( A + kI \) restricted to \( N \cap D(A) \) is \( K \) accretive. The proof is similar to Proposition 1.12 of Calvert [2]. Applications to the problems of [8] can be made: if the resistance function is differentiable and monotonic, then it satisfies the above by Proposition 1.3 of Calvert [3]. Integral \( x(t) \) curves increasing with \( t \) were considered by Olubumo [11].

**Theorem 12.** Suppose \( X \) a Banach space with monotonic norm with respect to a cone \( K \). Suppose \( A \) is \( K \) accretive, demiclosed, and \( x(t), t \in [0, \infty) \), is a strongly continuous once weakly differentiable curve with \( \frac{dx}{dt}(t) = -Ax(t) \), and
Suppose $A^{-1}$ bounded, then $x(t)$ converges to a zero of $A$.

**Proof.** Suppose $e \in K$, and for $x \in K$ put $\sup x = \inf \{p \in R : x \leq pe\}$ (possibly $\infty$) and $\inf x = \sup \{p \in R : pe \leq x\}$. For $b \in (0, 1]$ let $K(b) = \{x \in K : \inf x \geq b \sup x\}$ c.f. Krasnoselskii [9] page 27. $K(b)$ is a cone which allows plastering (i.e. there exists $f \in X^*$, $k$ in $R$, with $Re(f, x) \geq k\|x\|$ for $x \in K(b)$), and consequently is fully regular (i.e. any bounded set directed under $\preceq$ is convergent).

By the Corollary, $x(t)$ is increasing in $K$. Let $H = K(b)$, with any $b$ in $(0, 1]$ and $e = -Ax(0)$. By the Corollary, since $Ax(0)$ is in $-H$, $x(t)$ is increasing with respect to $H$. By Theorem 11, $|Ax(t)|_K$ is decreasing as $t$ increases. By Theorem 3, $\{Ax(t) : t \geq 0\}$ is bounded. Since $A^{-1}$ is bounded, $x(t)$ is bounded. Since $H$ is fully regular, $x(t)$ converges as $t \to \infty$ to a point $z$. For $f \in -K^*$, $(f, Ax(t))$ must converge to 0. Since $K^*$ is reproducing, i.e. $X^* = K^* \cap K^*$, $Ax(t)$ converges weakly to zero. Since $A$ is demiclosed, $x$ is in $D(A)$ and $A(x) = 0$. Q. E. D.

**Example 13.** Suppose $X$ a Banach space with monotonic norm and $U(t)$ a bounded positive continuous linear semigroup, with generator $-A$, $A^{-1}$ bounded. Then for $x_0$ in $D(A)$, $g \in X$, $g \geq A(x)$, there is a solution $x(t)$ to

$$\frac{dx}{dt}(t) = g - Ax(t), \text{ with } x(0) = x_0,$$

and $x(t)$ converges to $x$ with $A(x) = g$.

**Proof.** By Theorem 7, $A$ is $K$ accretive. The operator $A_g$ taking $x$ to $A(x) - g$ is $K$ accretive. For $x_0$ in $D(A) = D(A_g)$ there is a strong solution to

$$\frac{dx}{dt}(t) = -A(x(t)) + g, \quad x(0) = x_0.$$

$A_g$ is demiclosed and $A_g^{-1}$ is bounded since $A$ has these properties. $A_g(x_0) \preceq 0$, so that the conclusion follows from Theorem 12. Q. E. D.

**Definition.** Suppose $X$ a Banach space with fully regular normal cone $K$. Suppose $A$ is $K$ accretive, $R(I+A) = X$. We say an element $x$ of $X$ is harmonic if $Ax = 0$, subharmonic if $Ax \leq 0$, (as in Yosida [19] page 411).

**Theorem 14.** Suppose there exists $z$ with $\{(I+dA)^{-1}z : d > 0\}$ bounded. Then any subharmonic element $x$ has a least harmonic majorant $x_h = \lim_{d \to \infty} (I+dA)^{-1}x$.

**Proof.** Given $d > 0$, $(I+dA)x \leq x$. Since $(I+dA)^{-1}$ is $K$ nonexpansive, $x_d = (I+dA)^{-1}x \geq x$. Moreover if $d > e > 0$, $(I+eA)x_d = ed^{-1} x + (1-ed^{-1})x_d$, which is $\geq x$. Consequently $x_d \geq (I+eA)^{-1}x = x_e$. Now $\{x_d : d > 0\}$ is bounded, since $K$ normal implies $\{(I+dA)^{-1} : d > 0\}$ is equi-Lipschitz. Since $K$ is fully regular, and $x_d$ increases as $d$ increases, there exists $x_h = \lim_{d \to \infty} x_d$. Now $(I+A)x_d = d^{-1}x + (1-d^{-1})x_d = y_d$ converges to $x_h$. $(1+A)^{-1}y_d = x_d$ gives $(1+A)^{-1}x_h = x_h$ by continuity, so that $x_h$ is harmonic.

Suppose $x_H \geq x$, $Ax_H = 0$, then $x_H = (1+dA)^{-1}x_H \geq (I+dA)^{-1}x$, and taking
limits gives $x_{H} \geq x_{h}$.

REMARK. There exists $z$ with $(1+dA)^{-1}z : d > 0$ bounded if $A$ is linear ($z=0$) or $A^{-1}$ is locally bounded, by Theorem 3.2 of Calvert [3].

**Generalizations.**

Given $\varphi$ a proper convex function of $X$, we say $A$ is $d\varphi$ accretive if for $x, y$ in $D(A)$, there is $f$ in $d\varphi(x-y)$ with $\text{Re}(f, Ax-Ay) \geq 0$.

The basic Lemmas 1 and 2 hold in this context. If $d\varphi$ satisfies conditions of the type of Chapter 3 of Browder [1], then the basic existence theorem holds, in the following form for nonlinear operators.

**Theorem 15.** Suppose $X$ a reflexive Banach space, $N$ an open neighborhood of $0$, $T: N \rightarrow X^*$ uniformly continuous, and there exists $c, k > 0$ with $\text{Re}(Tx, x) \geq c\|x\|^2$ and $\|Tx\| \leq k\|x\|$ for $x$ in $N$. Suppose $T$ is cyclically monotone, i.e. (Rockafeller [13]) for any n-tuple $x_1 \cdots x_n$, $\text{Re}\left(\sum_{i=1}^{n} (Tx_i, x_i-x_{i+1})+(Tx_n, x_n-x_1)\right) \geq 0$. Suppose $B: D(B) \subset X \rightarrow X$ is $T$ accretive, i.e. $\text{Re}(T(x-y), Bx-By) \geq 0$ for $x, y$ in $D(B)$, $x-y$ in $N$. Suppose $R(I+B)=X$.

Then for $x_0$ in $D(B)$ there exists a unique strongly continuous weakly $C^1$ function $x: [0, \infty) \rightarrow X$ with $x(0) = x_0$, \[ \frac{dx}{dt}(t) = -Bx(t). \] The strong derivative exists almost everywhere and equals $-Bx(t)$.

**Proof.** By Rockafeller [13], we have a $C^1$ convex function $\varphi: N \rightarrow R$, with derivative $T$, there are constants $a, b > 0$ with

$$a\|x\|^2 \geq \varphi(x) \geq c\|x\|^2$$

for $x$ in $N$. Using $\varphi(x)$ instead of $\|x\|^2$ we may easily generalize Theorem 9.15 of Browder [1]. Q. E. D.

University of Colorado

**Bibliography**

Semigroups in Banach space


