Normal parts of certain operators

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1. Only bounded operators $T$ on a Hilbert space $H$ will be considered. A compact set $X$ of complex numbers containing $sp(T)$ is said to be a spectral set of $T$ (von Neumann [8]) if $\|f(T)\| \leq \sup_{z \in X} |f(z)|$, where $f(z)$ is a rational function having no poles on $X$; cf. Riesz and Sz.-Nagy [12], p. 435. For any compact set $X$ let $C(X)$ denote the space of continuous functions on $X$ and $R(X)$ the uniform closure of the set of rational functions with poles off $X$. It was shown by von Neumann that if $X$ is a spectral set of $T$ and if $C(X) = R(X)$ then $T$ must be normal; see also Lebow [6], p. 73. It may be noted that $C(X) = R(X)$ holds when $X$ has Lebesgue plane measure 0; this result is due to Hartogs and Rosenthal (cf. Gamelin [4], p. 47).

An operator $T$ is said to be hyponormal if

$$\tag{1.1} T^*T - TT^* \geq 0.$$ 

It is well-known that a subnormal operator, that is, an operator having a normal extension on a larger Hilbert space, is hyponormal, but that the converse need not hold. Further, if $T$ is subnormal then $sp(T)$ is a spectral set of $T$. On the other hand, if $T$ is only hyponormal, this need not be the case; see Clancey [1].

Let $T$ be hyponormal and let $D$ denote an open disk satisfying

$$\tag{1.2} sp(T) \cap D \neq \emptyset.$$

In case the set $sp(T) \cap D$ has planar measure zero then $T$ has a normal part, that is,

$$\tag{1.3} T = T_1 \oplus N, \quad N = \text{normal};$$

see Putnam [9]. Whether every compact set $X$ with the property that

$$\tag{1.4} X \cap D \neq \emptyset \Rightarrow \text{meas}_2(X \cap D) > 0 \quad (D = \text{open disk})$$

is the spectrum of a completely hyponormal operator (hyponormal and having no non-trivial reducing space on which it is normal) is not known. In this connection, see [3], [11]. As to subnormal operators, however, the authors

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have shown in [2] that a compact set $X$ is the spectrum of a completely subnormal operator (subnormal and completely hyponormal) if and only if

\[ X \cap D \neq \emptyset \Rightarrow R(X \cap \overline{D}) \neq C(X \cap \overline{D}), \]

where $D$ denotes an open disk. (The closure of a set $A$ is denoted by $\overline{A}$.)

In case $T$ is subnormal, then polynomials in $T$ and, in fact, rational functions of $T$ are also subnormal. On the other hand, if $T$ is assumed only to be polynomially hyponormal, so that all polynomials in $T$ are hyponormal, it seems to be unknown whether all rational functions of $T$ must also be hyponormal. Further, it is also apparently not known whether $T$ must be subnormal if all rational functions of $T$ are hyponormal.

It may be noted that if $T$ is hyponormal (and invertible) then so also is its inverse; Stampfli [13]. Also, there exist hyponormal operators $T$ which are not subnormal but are such that all powers $T^2, T^3, \ldots$ are subnormal; Stampfli [14]. In addition, for every positive integer $n$ there exists a hyponormal operator $T$ which is not subnormal and such that all polynomials in $T$ of degree not exceeding $n$ are hyponormal; Joshi [5].

If $T$ is hyponormal then $\|T\| = \sup \{|z| : z \in \text{sp}(T)\}$. It follows that if all rational functions of $T$ are hyponormal then $\text{sp}(T)$ is a spectral set of $T$. Further, if $T$ is hyponormal and if all polynomials in $T$ are hyponormal and if, in addition, $\text{sp}(T)$ does not separate the plane, then all rational functions of $T$ are also hyponormal. This is easily deduced from Mergelyan's theorem. (See Lebow [6], p. 66, where it is shown that if $X$ is a compact set which does not separate the plane and if for an operator $T$, $\|p(T)\| \leq \sup_{z \in X} |p(z)|$ holds for any polynomial $p(z)$, then $X$ is a spectral set of $T$.)

It will be shown in the present paper that certain results on subnormal operators obtained in [2] and [10] can be extended to operators $T$ for which $\text{sp}(T)$ is a spectral set or to operators $T$ which are polynomially hyponormal.

**Theorem 1.** Let $\text{sp}(T)$ be a spectral set of $T$. Suppose that $D$ is an open disk satisfying (1.2) and for which

\[ R(\text{sp}(T) \cap \overline{D}) = C(\text{sp}(T) \cap \overline{D}). \]

Then $T$ has a normal part, so that (1.3) holds.

In the special case in which $T$ is subnormal, the above result was proved in [2].

For any simple closed curve $C$, not necessarily having zero Lebesgue plane measure, denote its open interior by $\text{int}(C)$ and its open exterior by $\text{ext}(C)$. The following generalizes a result of [10].

**Theorem 2.** Let $T$ be polynomially hyponormal. Let $C$ be a simple closed curve such that
and suppose that
\[(1.8)\quad \{\text{sp}(T) \cap C\} - \{\text{sp}(T) \cap \text{int}(C)\}^- \neq \emptyset.\]

Then T has a normal part, so that (1.3) holds.

It may be noted that if T is supposed only to be hyponormal, rather than polynomially hyponormal, then T may be completely hyponormal even though its spectrum is a subset of a simple closed curve; see \([10]\). In fact, T can be chosen so that \(T^*T - TT^*\) has rank one and hence is even irreducible; cf. \([10], [11]\).

A dual of Theorem 2 is the following.

\textbf{Theorem 2'.} Let T be hyponormal and invertible and suppose that \(T^{-1}\) is polynomially hyponormal. Let C be a simple closed curve for which
\[(1.7)'\quad \text{sp}(T) \subseteq \{C \cup \text{ext}(C)\}\]
and
\[(1.8)'\quad \{\text{sp}(T) \cap C\} - \{\text{sp}(T) \cap \text{ext}(C)\}^- \neq \emptyset.\]

Then T has a normal part.

The above is of course a corollary of Theorem 2 by virtue of the mapping \(w = 1/z\).

\section{Proof of Theorem 1.}

In view of (1.2) it is clear that one can choose concentric open disks \(D_1 \subset D_2 \subset D\) centered at \(z_0\) with corresponding radii \(r_1 < r_2 < r\) and such that \(\text{sp}(T) \cap D_i \neq \emptyset\). Let \(A\) denote the closed annulus with hole \(D\) and outer radius so large that \(A\) contains that part of \(\text{sp}(T)\) lying outside \(D\). Then put \(Y = A \cup \{\text{sp}(T) \cap \bar{D}\}\). Let \(f(z)\) be defined by; \(f(z) = 1\) on \(\bar{D}\), \(f(z) = (R-r)/(r_1-r_2)\) if \(|z-z_0| = R\) and \(r_1 < R < r_2\), and \(f(z) = 0\) outside \(D_r\). Thus \(f\) is continuous in the plane and, in particular, \(f|_Y \in C(Y)\). Further, in view of (1.6), it is clear that \(f|_Y\) is locally in \(R(Y)\) so that, by Bishop's theorem (see Gamelin \([4],\) p. 51 or Zalcman \([15],\) p. 124), \(f|_Y \in R(Y)\). (Cf. the similar argument in \([2]\).)

Hence there exists a sequence \(\{r_n(z)\}, n = 1, 2, \ldots,\) of rational functions in \(R(Y)\) converging uniformly on \(Y\) to \(f(z)\). Since \(\text{sp}(T)\), hence also \(Y\), is a spectral set of \(T\), it follows that \(\{r_n(T)\}\) converges in the uniform topology to an operator \(f(T)\). If \(\mathcal{D}\) is defined by
\[(2.1)\quad \mathcal{D} = (f(T)\mathcal{D})^-,
\]
then clearly \(\mathcal{D}\) is invariant under \(T\). Let \(T_0 = T|_{\mathcal{D}}\).

Next, we show that \(\mathcal{D}\) reduces \(T\). By von Neumann \([8],\) p. 266, the image of \(\text{sp}(T)\) under \(f\) is a spectral set of \(f(T)\). But this set is real, so that by
von Neumann's theorem $f(T)$ is self-adjoint. Since $T$ commutes with $f(T)$, so also does $T^*$, and hence $T_0$ reduces $T$. Since $\|r_n(T)-f(T)\| \to 0$ and, by the spectral mapping theorem, $sp(r_n(T))=r_n(sp(T))$, it follows that $sp(f(T)) \supset f(sp(T)) \neq \{0\}$, so that, in particular, $\mathfrak{H}_0 \neq 0$-space. (Since $f(sp(T))$ is a spectral set of $f(T)$ then, in fact, $sp(f(T))=f(sp(T))$.) Thus,

\begin{equation}
T = T_1 \oplus T_0, \quad T_0 = T|_{\mathfrak{H}_0}.
\end{equation}

It will next be shown that $T_0$ is normal.

Since the spectrum of $T$ is a spectral set it follows that for every $x \neq 0$ in $\mathfrak{H}$ there is a positive measure $\mu[x, x]$ supported on $sp(T)$ such that

\begin{equation}
\langle g(T)x, x \rangle = \int_{sp(T)} g(t)d\mu[x, x]
\end{equation}

for every $g$ in $R(sp(T))$; see Lebow [6], pp. 70-71. Since $zf(z)$ is in $R(sp(T))$, just as $f(z)$, there exists a sequence $\{s_n(z)\}$ of functions in $R(sp(T))$ converging uniformly to $zf(z)$ and hence $\{s_n(T)\}$ converges uniformly to an operator $S$. By (2.3),

\begin{equation}
(Sx, x) = \int_{sp(T)} f(t)d\mu[x, x] = (\int_{sp(T)} f(t)d\mu[x, x])^* = \langle f(T)Tx, x \rangle
\end{equation}

(cf. Lebow [6], p. 73). Hence $S = T* f(T)$ and so $T* f(T)$ commutes with $T$. Since $f(T)$ also commutes with $T$, then $T* T f(T) = T* f(T) T = T T* f(T)$, so that $T_0$ is normal, and the proof of Theorem 1 is complete.

3. Lemma. Let $\{T_n\}$ be a sequence of hyponormal operators converging uniformly to the (hyponormal) operator $T$, so that

\begin{equation}
\|T_n - T\| \to 0 \quad \text{as} \quad n \to \infty.
\end{equation}

Then $z \in sp(T)$ if and only if there exists a sequence $\{z_n\}$, $z_n \in sp(T_n)$, such that $z_n \to z_0$.

Proof. The "if" part clearly holds for any bounded operators $T_n$, $T$ satisfying (3.1). In order to prove the "only if," let $z \in sp(T)$. If the assertion is false, then there exists a constant $\delta > 0$ and a sequence $\{n_k\}$ of positive integers satisfying $n_1 < n_2 < \cdots$ for which $sp(T_{n_k}) \cap \{z : |z - z_0| < \delta\} = \emptyset$. Since $T_{n_k}$ is hyponormal, then $\|(T_{n_k} - z_0)Ix\| \geq \|(T_{n_k} - z_0)I* x\| \geq \delta \|x\|$ for all $x$ in $\mathfrak{H}$. On letting $n_k \to \infty$, one obtains similar inequalities with $T_{n_k}$ replaced by $T$, so that $z \not\in sp(T)$, a contradiction.

4. Proof of Theorem 2. By the Riemann mapping theorem, the set $C \cup \text{int}(C)$ can be mapped homeomorphically onto $|w| \leq 1$ by $w = f(z)$, where $f(z)$ is analytic in $\text{int}(C)$. By Mergelyan's theorem ([7]) there exist polyno-
mials \( \{p_n(z)\} \), \( n = 1, 2, \ldots \), such that \( p_n(z) \to f(z) \) uniformly on \( C \cup \text{int}(C) \). Since the operators \( p_n(T) \) are hyponormal, then \( p_n(T) \) converges in the uniform topology to a hyponormal operator \( f(T) \). According to the spectral mapping theorem, \( sp(p_n(T)) = p_n(sp(T)) \) and it now follows from the Lemma that \( sp(f(T)) = f(sp(T)) \). Further, if \( z_1 \) is in the set of (1.8), then \( f(z_1) \) is in \( sp(f(T)) \cap C' \), where \( C' = \{ w : |w| = 1 \} \), and \( f(z_1) \) is not in the closure of \( sp(f(T)) \cap \text{int}(C') \). It follows from [9] that \( f(T) \) has a normal part \( M = f(T) \prod \Phi_0 \), \( \Phi_0 \neq 0 \), so that \( f(T) = S \oplus M \), where \( M \) is normal on \( \Phi_0 \neq 0 \). Since Mergelyan's theorem can be used again (cf. [10]) to recover \( T \) as \( T = g(f(T)) = g(S) \oplus g(M) \), where \( g \) is the inverse of \( f \), it follows that \( g(M) \) is also normal (on \( \Phi_0 \)) and the proof is complete.

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References
