A formula on some odd-dimensional Riemannian manifolds related to the Gauss-Bonnet formula

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§ 1. Introduction.

Let \((M^{2n-1}, g)\) be a compact orientable Riemannian manifold of odd dimension. The generalised Gauss-Bonnet formula is

\[
\frac{(-1)^n}{2^n \pi^n n!} \int_M \sum_{i=1}^{2n} \Omega_i^1 \wedge \cdots \wedge \Omega_{2n}^1 = \chi(N),
\]

where \(\Omega_i^1\) denote the curvature forms and \(\chi(N)\) is the Euler-Poincaré characteristic. The left hand side of (1.1) is a differential geometric or Riemannian geometric quantity and the right hand side is a topological quantity. In (1.1), even dimensionality is essential.

For a compact orientable Riemannian manifold \((M^{2n-1}, g)\) of odd dimension, we have \(\chi(M) = 0\). This shows that \(M = M^{2n+1}\) admits a vector field \(\xi\) with no singular points. If we try to find some formula on \((M^{2n+1}, g)\) analogous to (1.1), some restriction on this \(\xi\) may be necessary and it might be hoped that the right hand side is a linear combination of Betti numbers.

We assume that \(e = e_0\) is a unit vector field. Let \(w_0\) be the 1-form dual to \(e_0\) with respect to \(g\). Then we have local fields of orthonormal vectors \(e_0, e_1, \ldots, e_{2n}\) and the dual \(w_0, w_1, \ldots, w_{2n}\). We call this frame field a \(\xi\)-frame field. By \(\Omega_{AB}\) \((A, B = 0, 1, \ldots, 2n)\) we denote the curvature forms with respect to the above frame field. By \(\beta_r(M)\) we denote the \(r\)-th Betti number of \(M\).

In this paper we have

**THEOREM A.** Let \((M^{2n+1}, g)\) be a compact Riemannian manifold admitting a unit Killing vector \(\xi\) and let \((e_0, e_i)\) be a \(\xi\)-frame field. Assume that

\[
\Omega_{0i} = w_i \wedge w_0, \quad i = 1, \ldots, 2n,
\]

and that each trajectory of \(\xi\) is of constant length \(l(\xi)\). Then

\[
\frac{(-1)^n}{l(\xi)^n} \int_M F(\Omega_{0i}, w_0) = \sum_{r=0}^n (n+1-r)(-1)^r \beta_r(M),
\]

where, putting \(dw_0 = \sum \varphi_{AB} w_A \wedge w_B\),

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Some Gauss-Bonnet formula

\[ (1.4) \quad F(\xi, \omega) = \sum_{i_1 \cdots i_{2n}} (\xi_{i_1} - \omega_{i_1}) \cdots (\xi_{i_{2n}} - \omega_{i_{2n}}) \omega_{i_1} \cdots \omega_{i_{2n}} \wedge \omega_0. \]

The condition (1.2) is independent of the choice of \( \xi \)-frame fields. In fact, let \( R \) be the Riemannian curvature tensor of \( (M^{2n+1}, g) \). Then (1.2) is equivalent to

\[ (1.2)' \quad R(X, \xi)Y = g(X, Y)\xi - g(\xi, Y)X \]

for any vector fields \( X \) and \( Y \) on \( M \). A Riemannian manifold \( (M^{2n+1}, g) \) admitting a unit Killing vector \( \xi \) satisfying (1.2) or (1.2)' is called a Sasakian manifold (or a normal contact Riemannian manifold). In particular, Sasakian manifolds with constant \( l(\xi) \) are canonically related to Hodge manifolds (i.e., Kählerian manifolds whose fundamental 2-form defines an integral cocycle). Contact manifolds are orientable.

A special case is as follows:

**Theorem B.** Let \( (M^3, g) \) be a compact 3-dimensional Riemannian manifold admitting a unit Killing vector \( \xi \) such that

\[ (1.5) \quad R(X, \xi) = g(X, \xi)\xi - X. \]

If each trajectory of \( \xi \) is of constant length \( l(\xi) \), then

\[ (1.6) \quad \frac{1}{l(\xi)} \left[ \int_M K(\xi^+) dM + 3 \operatorname{Vol}(M) \right] = 2 - \beta_1(M), \]

where \( K(\xi^+) \) means sectional curvature of the 2-plane orthogonal to \( \xi \) and \( \operatorname{Vol}(M) \) denotes the total volume of \( (M^3, g) \).

Two typical examples, Sasakian manifolds and Riemannian product manifolds, show a clear difference between expressions of linear combinations of Betti numbers (see (7.1) and (8.2)).

§ 2. Preliminaries.

For local fields of orthonormal vectors \( (e_A, A = 0, 1, \ldots, 2n) \) and the dual 1-forms \( (\omega_A) \) on a Riemannian manifold \( (M^{2n+1}, g) \), the structure equations are

\[ (2.1) \quad dw_A = \sum w_B \wedge w_{BA}, \quad (A, B = 0, 1, \ldots, 2n), \]

\[ (2.2) \quad dw_{AB} = \sum w_{AC} \wedge w_{CB} + \Omega_{AB}, \]

where \( w_{AB} \) and \( \Omega_{AB} \) denote the connection forms and curvature forms, respectively; \( w_{AB} + w_{BA} = 0 \) and \( \Omega_{AB} + \Omega_{BA} = 0 \).

Let \( (*e_A) \) be another frame field such that

\[ (2.3) \quad *e_A = \sum a_{AB} e_B, \quad a_{AB}(x) \in O(2n+1). \]

Then the curvature forms \( *\Omega_{AB} \) with respect to \( (*e_A) \) satisfy
\[ *Q_{AB} = \sum a_{AC}a_{BD}Q_{CD}. \]

§ 3. Sasakian structures.

Let \((M^{2n+1}, g)\) be a Riemannian manifold admitting a unit Killing vector \(\xi\) satisfying
\[ R(X, \xi)Y = g(X, Y)\xi - g(Y, \xi)X, \]
where \(R(X, Y)Z = \nabla_X Y \cdot Z - \nabla_Y X \cdot Z\) and \(\nabla\) denotes the Riemannian connection. Since \(\xi\) is a Killing vector, it satisfies \(\nabla_X (\nabla \xi)Y + R(X, \xi)Y = 0\) (this relation is equivalent to the fact that \(\xi\) is an infinitesimal affine transformation). Hence, the left hand side of (3.1) may be replaced by \(-\nabla_X (\nabla \xi)Y\).

Such a Riemannian manifold is called a Sasakian manifold or normal contact Riemannian manifold (cf. Sasaki-Hatakeyama [8], Hatakeyama-Ogawa-Tanno [6], etc.) and it is denoted by \((M^{2n+1}, \xi, g)\). For completeness we give a brief summary of relations of structure tensors (see [6], up to constant factors).

We define a \((1,1)\)-tensor field \(\varphi\) by \(\varphi = -\nabla \xi\), i.e., \(X \cdot \varphi = -\nabla_X \xi\). By \(\nabla_X (\varphi \xi) = 0\), we have \(\varphi \xi = -\varphi \xi = 0\). Next, by \(\nabla_X (\varphi \xi) = 0\), we have \((\nabla_X \varphi)\xi + \varphi \nabla_X \xi = \nabla_X (-\varphi \xi)\xi - \varphi \varphi X = 0\). The last equation and (3.1) give
\[ \varphi \varphi X = -X + g(\xi, X)\xi. \]

Considering the inner product of the both sides of (3.2) and \(Y\), and noticing that \(\varphi = -\nabla \xi\) is skew-symmetric with respect to \(g\), we have
\[ g(\varphi X, \varphi Y) + g(\xi, X)g(\xi, Y) = g(X, Y). \]

If \(\omega_0\) is the 1-form dual to \(\xi\) with respect to \(g\), i.e., \(\omega_0(X) = g(\xi, X)\), by \(\varphi = -\nabla \xi\) we have
\[ d\omega_0(X, Y) = 2g(X, \varphi Y). \]

\(\omega_0\) satisfies \(\omega_0 \wedge (d\omega_0)^n \neq 0\) and is called a contact form. With respect to local coordinates \((x^a)\), we have
\[ d\omega_0 = \sum \varphi_{AB} dx^A \wedge dx^B. \]

Sasakian manifolds (more generally contact manifolds) are orientable. Let \((\xi = e_0, e_1, \ldots, e_n; \omega_0, \omega_1, \ldots, \omega_n)\) be a \(\xi\)-frame field. Then it is not difficult to see that (3.1) is equivalent to \(\Omega_{\xi} = \omega_0 \wedge \omega_i (i = 1, \ldots, 2n)\), since
\[ \Omega_{AB} = (1/2) \sum R_{ABCD}w_C \wedge w_D, \]
where we have put \(R(e_C, e_D)e_B = \sum R_{ABCD}e_A\).

Let $(M^{2n+1}, \xi, g)$ be a Sasakian manifold and assume that $\xi$ is regular (cf. Boothby-Wang [1], etc.). Then we have the fibering

$$\pi: M^{2n+1} \longrightarrow M^{2n+1}/\xi = B^{2n},$$

where $(B^{2n}, J, G)$ is a Kählerian manifold (more precisely, Hodge manifold) with (almost) complex structure tensor $J$ and the Kähler metric tensor $G$ (see Hatakeyama [5], p. 181, etc.). $w_0$ is an infinitesimal connection form on this principal bundle. $J$ and $G$ satisfy

$$g(X, Y) = G(\pi X, \pi Y) \cdot \pi + w_0(X)w_0(Y),$$

where $u^*$ denotes the horizontal lift of a vector field $u$ on $B^{2n}$ with respect to $w_0$. Conversely, every Hodge manifold $(B^{2n}, J, G)$ gives a Sasakian manifold $(M^{2n+1}, \xi, g)$ with regular $\xi$. Furthermore, we have

$$dw_0(X, Y) = 2G(\pi X, J\pi Y) \cdot \pi = 2g(X, \varphi Y).$$

Let $(f_i, i=1, \ldots, 2n)$ be local fields of orthonormal vectors in $B^{2n}$. Then $(\xi = e_0, e_i = f_i)$ is a $\xi$-frame field and the Riemannian connection forms $w_{AB}$ with respect to $(e_A)$ are given by

$$w_{00} = 0,$$

$$w_{0i} = -w_{i0} = -\sum \varphi_{ij}w_j,$$

$$w_{ij} = \pi^*(w_{ij}) - \varphi_{ij}w_0,$$

where $\varphi_{ij} = g(e_i, \varphi e_j)$, and $w_{ij}$ are the connection forms on $(B^{2n}, G)$ with respect to $(f_i)$ (cf. Kobayashi [7], Proposition 2). The curvature forms $\Omega_{AB}$ are given by (cf. [7], Proposition 3)

$$\Omega_{00} = 0,$$

$$\Omega_{0i} = -\Omega_{i0} = -\sum \varphi_{ik}\varphi_{kl}w_k \wedge w_l,$$

$$\Omega_{ij} = \pi^*(\Omega'_{ij}) - \sum (\varphi_{ij}\varphi_{kl} + \varphi_{ik}\varphi_{jl})w_k \wedge w_l,$$

where $\Omega'_{ij}$ are the curvature forms on $(B^{2n}, G)$. Hence, we have

$$\pi^*(\Omega'_{ij}) = \Omega_{ij} - \varphi_{ij}dw_0 - \sum \varphi_{ik}w_k \wedge \varphi_{jl}w_l.$$

§ 5. The Theorem A.

Let $(M^{2n+1}, \xi, g)$ be a compact Sasakian manifold with regular $\xi$. In this case regularity is equivalent to the fact that all trajectories of $\xi$ have the common length $l(\xi)$. The Gauss-Bonnet formula (for example, see Chern [2, 3])
for a compact orientable Riemannian manifold \((B^{2n}, G)\) is
\[
\frac{(-1)^n}{2^{2n}n!} \int_B \sum_{i_1, \ldots, i_{2n}} \varepsilon_{i_1, \ldots, i_{2n}} \Omega_{i_1} \wedge \cdots \wedge \Omega_{i_{2n-1}} \wedge \varepsilon_{i_{2n-1}, i_{2n}} = \chi(B),
\]
where \(\varepsilon_{i_1, \ldots, i_{2n}}\) is a symbol which is 1 or \(-1\) according as \((i_1, \ldots, i_{2n})\) is an even or odd permutation of \((1, \ldots, 2n)\), and is zero otherwise. It is not difficult to see that
\[
\int_M \pi^*(\Theta) \wedge \omega_0 = l(\xi) \int_B \Theta
\]
for any \(2n\)-form \(\Theta\) on \(B^{2n}\). Therefore, we get
\[
\frac{(-1)^n}{l(\xi)2^{2n}n!} \int_M \sum_{i_1, \ldots, i_{2n}} \varepsilon_{i_1, \ldots, i_{2n}} \pi^*(\Omega_{i_1}) \wedge \cdots \wedge \pi^*(\Omega_{i_{2n-1}}) \wedge \Omega_{i_{2n-1}} \wedge \omega_0 = \chi(B).
\]

Theorem A. Let \((M^{2n+1}, \xi, g)\) be a compact Sasakian manifold with regular \(\xi\). Then
\[
\frac{(-1)^n}{l(\xi)2^{2n}n!} \int_M \sum_{i_1, \ldots, i_{2n}} \varepsilon_{i_1, \ldots, i_{2n}} \pi^*(\Omega_{i_1}) \wedge \cdots \wedge \pi^*(\Omega_{i_{2n-1}}) \wedge \Omega_{i_{2n-1}} \wedge \omega_0 = \chi(B),
\]
where \(\beta_r(M)\) denotes the \(r\)-th Betti number of \(M^{2n+1}\).

Proof. First we notice that the integrand is independent of the choice of \(\xi\)-frame fields. By (4.9) and (5.3), it suffices to show
\[
\chi(B) = \sum_{r=0}^n (n+1-r)(-1)^r \beta_r(M),
\]
The exact sequence of Gysin for \(\pi: M^{2n+1} \rightarrow B^{2n}\) is
\[
0 \rightarrow H^1(B; R) \xrightarrow{\pi^*} H^1(M; R) \rightarrow H^0(B; R)
\]
\[
L_0 \rightarrow H^1(M; R) \xrightarrow{\pi^*} H^1(B; R) \rightarrow H^0(B; R) \rightarrow \cdots
\]
\[
L_{p-2} \rightarrow H^p(M; R) \xrightarrow{\pi^*} H^p(B; R) \rightarrow H^{p-1}(B; R) \rightarrow \cdots,
\]
where \(H^p(M; R)\) (or \(H^p(B; R)\)) is the \(p\)-th cohomology group of \(M^{2n+1}\) (or \(B^{2n}\)) with real coefficient \(R\), and \(L_p\) sends \(\lambda \in H^p(B; R)\) to \(W \wedge \lambda \in H^{p+2}(B; R)\), \(W\) being the fundamental 2-form of the Kählerian manifold \((B^{2n}, J, G)\) (cf. Chern-Spanier [4], Serre [9]). Since \((B^{2n}, J, G)\) is Kählerian, \(L_p\) is an into isomorphism for \(p \leq (2n-2)/2\). Therefore \(\beta_p(M) = \beta_p(B)\) and
\[
\beta_p(M) = \beta_{p-2}(B), \quad 2 \leq p \leq n,
\]
\[
\beta_p(M) = \beta_{p-2}(B) - \beta_p(B), \quad n+1 \leq p \leq 2n.
\]
Some Gauss-Bonnet formula (see also Tanno [11]). Then we get

\[ \chi(B) = \sum_{i=0}^{n-1} (-1)^i \beta_i(B) \]

\[ = \sum_{p=0}^{n-1} 2(-1)^p \beta_p(B) + (-1)^n \beta_n(B) \]

\[ = \sum_{p=0}^{n-3} 2(-1)^p \beta_p(B) + 3(-1)^{n-2} \beta_{n-2}(B) \]

\[ + 2(-1)^{n-1} \beta_{n-1}(B) - (-1)^n \beta_n(B) \]

\[ = \sum_{p=0}^{n-3} 2(-1)^p \beta_p(B) + 3(-1)^{n-2} \beta_{n-2}(B) \]

\[ + 2(-1)^{n-1} \beta_{n-1}(B) + (-1)^n \beta_n(M) . \]

Continuing this step we have (5.5). q.e.d.

An orthonormal frame \( (e_0, e_1, \varphi e_1 = e_{n+1}, \ldots, e_n, \varphi e_n = e_{2n}) \) is called a \( \varphi \)-frame. With respect to a \( \varphi \)-frame, we have

\[ (\varphi_{AB}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -E \\ 0 & E & 0 \end{pmatrix} , \]

where \( E \) denotes an \( n \times n \) unit matrix.

§ 6. The special case where \( \dim M = 3 \).

(4.9) with \( i = 1 \) and \( j = 2 \) is

\[ \pi^*(Q_{ij}) = Q_{ij} - \varphi_{iA} d\varphi_{Aj} - \sum \varphi_{ik} w_k \wedge \varphi_{i1} w_1 . \]

With respect to a \( \varphi \)-frame field, we have

\[ (6.1)' \]

\[ \pi^*(Q_{ij}) = Q_{ij} - 2w_1 \wedge w_2 - w_1 \wedge w_3 . \]

Therefore

\[ (6.2) \]

\[ l(\xi) \int_M Q_{ij} = \int_M Q_{ij} \wedge w_0 - 3 \int_M w_1 \wedge w_2 \wedge w_3 . \]

That is,

\[ (6.3) \]

\[ l(\xi) \chi(B) = (-1/2\pi) \left[ \int_M Q_{ij} \wedge w_0 - 3 \text{Vol}(M) \right], \]

where \( \text{Vol}(M) \) denotes the total volume of \( (M^{2n+1}, g) \). By (1.2), we have \( R_{A012} = 0 \), etc., and hence we have

\[ (6.4) \]

\[ Q_{ij} = (1/2) \sum R_{ijkA} w_A \wedge w_B = R_{ijkA} w_1 \wedge w_2 \]

where \( -R_{ijk} = K(e_i, e_j) \) is the sectional curvature for the \( (e_i, e_j) \)-plane. Consequently, we get
THEOREM B. Let \((M^3, \xi, g)\) be a 3-dimensional compact Sasakian manifold with regular \(\xi\). Then

\[
\frac{1}{l(\xi)2\pi} \left[ \int_M K(e_i, \varphi e_i) dM + 3 \text{Vol} (M) \right] = 2 \beta_3(M) - \beta_1(M).
\]

A Riemannian manifold \((M^{2n+1}, g)\) admitting a unit Killing vector \(\xi\) satisfying (1.5) is called a K-contact Riemannian manifold. Every K-contact Riemannian manifold of 3-dimension is Sasakian (Tanno [10]), and so Theorem B in the introduction is equivalent to the above one.

EXAMPLE. A unit sphere \(S^{2n+1}\) admits the standard Sasakian structure \(\xi\) (Sasaki-Hatakeyama [8]). For \(S^3\), we have \(l(\xi) = 2\pi\), \(K(e_i, \varphi e_i) = 1\) and \(\text{Vol} (S^3) = 2\pi^2\). On the other hand, \(\beta_1(S^3) = \beta_2(S^3) = 0\) and \(\beta_3(S^3) = \beta_4(S^3) = 1\).

§ 7. Special case where \(\dim M = 5\).

If \(\dim M = 5\), we have

\[
\frac{1}{l(\xi)2\pi} \int_M 8[\pi^*(\Omega_{13}) \wedge \pi^*(\Omega_{14}) + \pi^*(\Omega_{12}) \wedge \pi^*(\Omega_{14}) + \pi^*(\Omega_{12}) \wedge \pi^*(\Omega_{13})] \wedge w_0
\]

\[
= 3\beta_5(M) - 2\beta_4(M) + \beta_3(M).
\]

If we take a \(\varphi\)-frame field, we have \(\pi^*(\Omega_{13}) = \Omega_{13} - w_3 \wedge w_4\), \(\pi^*(\Omega_{14}) = \Omega_{14} - w_2 \wedge w_3\), \(\pi^*(\Omega_{12}) = \Omega_{12} - w_1 \wedge w_4\), \(\pi^*(\Omega_{13}) = \Omega_{13} - 2w_1 \wedge w_4 - 3w_2 \wedge w_3\), \(\pi^*(\Omega_{14}) = \Omega_{14} - 2w_1 \wedge w_4 - 3w_2 \wedge w_3\), and \(\pi^*(\Omega_{13}) = \Omega_{13} - 2w_1 \wedge w_4 - 3w_2 \wedge w_3\). Hence, we have

\[
\frac{1}{4\pi^5 l(\xi)} \int_M [\Omega_{13} \wedge \Omega_{24} + \Omega_{13} \wedge \Omega_{45} + \Omega_{14} \wedge \Omega_{23} + 3w_1 \wedge w_3 \wedge w_4 \wedge \Omega_{24}
\]

\[
+ 3w_2 \wedge w_4 \wedge \Omega_{13} + 15w_1 \wedge w_2 \wedge w_4 \wedge w_3 \wedge w_1 \wedge \Omega_{12}
\]

\[
+ 2w_1 \wedge w_3 \wedge \Omega_{13} - w_1 \wedge w_4 \wedge \Omega_{14} - w_2 \wedge w_4 \wedge \Omega_{23}
\]

\[
+ 2w_2 \wedge w_4 \wedge \Omega_{14} - w_3 \wedge w_4 \wedge \Omega_{23}],
\]

\[
= 3 - 2\beta_5(M) + \beta_3(M).
\]

EXAMPLE. For \(S^5\), we have \(l(\xi) = 2\pi\), \(\Omega_{ij} = -w_i \wedge w_j (i, j = 1, \ldots, 4)\) and \(\text{Vol} (S^5) = 2\pi^2\). On the other hand, \(\beta_i(M) = 0 (i = 1, \ldots, 4)\) and \(\beta_5(M) = \beta_6(M) = 1\).


(i) If \((M, g)\) is of constant curvature \(k\), we have

\[
R(X, Y)Z = k[g(X, Z)Y - g(Y, Z)X].
\]

If a Killing vector \(\xi\) of non-zero constant length satisfies
Some Gauss-Bonnet formula

\[ R(X, \xi)Z = k [ g(X, Z) \xi - g(\xi, Z)X ], \quad k > 0, \]

then we can assume the length of \( \xi \) is 1 and we can change the Riemannian metric \( g \) by \( g^* = (1/k)g \) and \( \xi \) by \( \xi^* = \sqrt{k} \xi \), so that \((M, \xi, g^*)\) is a Sasakian manifold.

Every complete Riemannian manifold of constant curvature 1 and odd dimension admits a Sasakian structure (Wolf [13], Tanno [12]).

(ii) Let \( N \) be a 4-dimensional compact orientable Riemannian manifold with Betti numbers \( \beta_p(N) \). Let \( S \) be a circle of length \( l \) and let \( N \times S \) be the Riemannian product of \( N \) and \( S \). A unit tangent vector field on \( S \) defines a unit Killing vector \( \xi \) on \( M^5 = N \times S \) in the natural way. Its dual 1-form \( \omega_0 \) is parallel. Then

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\[ = \chi(N) = 5 \beta_0(M) - 3 \beta_2(M) + \beta_4(M), \]

where \( \beta_r(M) \) denotes the \( r \)-th Betti number of \( M = N \times S \) and we have used \( \beta_r(M) = \sum_{p+q=r} \beta_p(N) \beta_q(S) \). One sees the difference between the right hand sides of (7.1) and (8.2).

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References

