Point derivations on commutative Banach algebras
and estimates of the $A(X)$-metric norm

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§ 1. Introduction.

Let $A$ be a commutative Banach algebra with unit 1. We denote the set of all homomorphisms of $A$ onto $C$ by $\mathcal{M}(A)$, which is called the maximal ideal space of $A$. For $\phi \in \mathcal{M}(A)$, a point derivation on $A$ at $\phi$ is an (algebraic) linear functional $D$ on $A$ with the property that $D(fg)=\phi(f)D(g)+\phi(g)D(f)$ for all $f, g \in A$. In this paper we consider the point derivations which are defined as follows. Let $\hat{f}(\phi)=\phi(f)$ be the Gelfand transform and let $\{\phi_r, t_r\}$ be a pair of nets in $\mathcal{M}(A) \times C \setminus \{0\}$ with the following properties:

1. $\phi_r$ converges to $\phi$ in $\mathcal{M}(A)$ with the weak*-topology,
2. $t_r$ converges to 0 in $C$,
3. \[ \frac{\hat{f}(\phi_r)-\hat{f}(\phi)}{t_r} \] converges for any $f \in A$.

Then the limit $D(f)=\lim_r \frac{\hat{f}(\phi_r)-\hat{f}(\phi)}{t_r}$ defines a point derivation at $\phi$.

In section 2 considering this kind of point derivation we shall give an another proof of Browder's theorem; there exists a nonzero point derivation at $\phi$ if $\phi$ is not isolated in $\mathcal{M}(A)$ with the metric topology. Also we shall prove that there exists a nonzero continuous point derivation at $\phi$ if $\phi$ is not isolated in $\mathcal{M}(A)$ with the metric topology and the norm $\|\phi-\phi\|$ of the metric topology is equivalent to a semi-metric $|\phi(w_1)-\phi(w_1)| + \cdots + |\phi(w_n)-\phi(w_n)|$ of the weak* topology in some metric neighborhood of $\phi$ in $\mathcal{M}(A)$, where $w_1, \ldots, w_n \in A$.

In the remaining sections we shall consider the function algebra $A(X)$ on a compact plane set $X$. In this case we obtain more exact results. As is well known, these results are translated for the case $R(X)$ and the proofs for the case $R(X)$ are performed similarly. We state here the corresponding results for $R(X)$. Let $R_0(X)$ be the set of all rational functions with poles off $X$. $R(X)$ is the uniform closure of $R_0(X)$ on $X$. The maximal ideal space of $R(X)$ is identified with $X$. It is known that each of the following condi-
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There exists a constant \( k \) such that \( |f'(x)| \leq k \| f \| \) for all \( f \in R_0(X) \).

(2) (Wilken [5]): There exists a complex representing measure \( \mu \) for \( x \) such that
\[
\int \frac{d|\mu|(z)}{|z-x|} < \infty .
\]

(3) (Hallstrom [4]): \[
\sum_{n=0}^{\infty} \frac{4^n}{r^{2n}} \gamma(E_n(x; r)\backslash X) < \infty ;
\]
where \( E_n(x; r) = \{ z : \frac{r}{2^{n+1}} < |z-x| < \frac{r}{2^n} \} \), and \( \gamma \) is the analytic capacity. For a plane set \( D \), the analytic capacity of \( D \), \( \gamma(D) \), is defined by
\[
\gamma(D) = \sup \{ |f'(\infty)| : f \in \mathcal{A}(D) \},
\]
where \( \mathcal{A}(D) \) is the set of all functions on the Riemann sphere \( S^2 \) such that \( f \) is analytic off a compact subset of \( D \), \( \| f \|_{S^2} \leq 1 \) and \( f(\infty) = 0 \). In section 3 we shall give two other equivalent conditions:

(4) There exists a sequence \( x_n \in X \) which converges to \( x \) and has the property that \( \frac{f(x_n) - f(x)}{x_n - x} \) converges for any \( f \in R(X) \).

(5) \[
\lim_{z \to x} \frac{\|z-x\|_R}{|z-x|} < \infty ;
\]
where \( \| \cdot \|_R \) denotes the \( R(X) \)-metric norm, which is defined by \( \| x-y \|_R = \sup \{ |f(x) - f(y)| : f \in R(X), \| f \| \leq 1 \} \).

In section 4 we aim to estimates the \( A(X) \)-metric norm by the continuous analytic capacity. Although the estimates for \( A(X) \) are given in section 4 precisely, we write here the main corresponding three estimates for \( R(X) \).

Let \( x, y \in X \) and let \( d(x; r) = \{ z : |z-x| < r \} \). We denote the distance between a point \( z \) and a set \( T \) by \( d(z, T) \).

\[
\text{(R-2)} \quad \| x-y \|_R \geq \frac{1}{5} \left[ \frac{4 \sum_{n=0}^{\infty} \frac{2^n}{r^{2n}} \gamma(E_n(x; r)\backslash X)}{3 + 4 \sum_{n=0}^{\infty} \frac{2^n}{r^{2n}} \gamma(E_n(x; r)\backslash X)} \right] - \frac{\gamma(A(x; r)\backslash X)}{d(y, A(x; r)\backslash X)} .
\]

Let \( 0 < \sigma < 1 \) and \( C \) be a universal constant. If \( |x-y| < \frac{r}{4} \), then

\[
\text{(R-4)} \quad \| x-y \|_R \leq C \left[ \frac{4 \sqrt{|x-y|}}{r-2|x-y|} + \frac{4 \sqrt{|x-y|}}{r-2\sqrt{|x-y|}} \right] + \frac{4}{\sigma} \sum_{n=0}^{\infty} \frac{2^n}{r^{2n}} \gamma(E_n(x; r)\backslash X) + \frac{4}{1-\sigma} \sum_{n=0}^{\infty} \frac{2^n}{\sigma|x-y|} \gamma(E_n(y; \sigma|x-y|)\backslash X) ,
\]
where \( \left\lfloor \frac{1}{2} \log_2 \frac{r}{|x-y|} \right\rfloor \) denotes the maximum integer which does not exceed
\[ \frac{1}{2} \log_2 \frac{r}{|x-y|}. \] If \(|x-y| < \frac{r}{8}\), then

\[ \|x-y\|_R \leq C |x-y| \left[ \frac{1}{r-2} \frac{1}{|x-y|} + \frac{48}{\sigma} \sum_{n=0}^\infty \frac{4^n}{r^\frac{r}{2}} \gamma(E_n(x; x) \setminus X) \right. \]

\[ \left. + \frac{1}{1-\sigma} \frac{4}{|x-y|} \sum_{n=0}^\infty \frac{2^n}{\sigma |x-y|} \gamma(E_n(y; x) \setminus X) \right]. \]

In section 5 we shall prove the following results by the application of the above estimates. Let \(x_n \in X\) be a sequence which converges to \(x\). Then \(x_n\) converges to \(x\) in the \(R(X)\)-metric topology if and only if

\[ \lim_{n \to \infty} \frac{1}{2} \sum_{k=0}^\infty \frac{2^k}{\sigma |x-x_n|} \gamma(E_k(x_n; x-x_n) \setminus X) = 0 \]

for any fixed \(0 < \sigma < 1\). And \(x_n\) has the property (r.4) if and only if

\[ \lim_{n \to \infty} \frac{1}{2} \sum_{k=0}^\infty \frac{2^k}{\sigma |x-x_n|} \gamma(E_k(x_n; x-x_n) \setminus X) < \infty \]

for any fixed \(0 < \sigma < 1\).

The notations \(\mathcal{A}(x; r)\), \(E_n(x; r)\), and \(d(z, T)\) remain valid throughout the paper.

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The argument in section 3 is an analogue of Browder [1] and the estimates (A-1) and (A-2) in section 4 are essentially due to Curtis [2].

§ 2. Sequential derivations.

Let \(A\) be a commutative Banach algebra and \(\mathcal{M}(A)\) the maximal ideal space. For \(\phi \in \mathcal{M}(A)\), \(D_\phi(A)(T_\phi(A))\) denotes the set of all (continuous) point derivations on \(A\) at \(\phi\), and \(A_\phi\) denotes the kernel of \(\phi\). It is easy to see that a linear functional \(D\) on \(A\) is a point derivation at \(\phi\) if and only if \(D(fg) = 0\) for all \(f, g \in A_\phi\) and \(D(1) = 0\). Thus \(D_\phi(A)\) is identified with the algebraic dual space of \(A_\phi/A_\phi^0\), where \(A_\phi^0 = \{ f_1 g_1 + \cdots + f_k g_k : f_i, g_i \in A_\phi \}\).

2.1. DEFINITION. Let \(\phi \in \mathcal{M}(A)\) and let \(\{\phi_r, t_r\}\) be a pair of nets in \(\mathcal{M}(A) \times C \setminus \{0\}\) with the properties (1.1), (1.2) and (1.3). Then we say that \(\{\phi_r, t_r\}\) is a sequential derivation at \(\phi\) for \(A\) and that \(D(f) = \lim_{r \to \infty} \frac{f(\phi_r) - f(\phi)}{t_r}\)

is the point derivation defined by \(\{\phi_r, t_r\}\).

2.2. THEOREM. If \(\{\phi_n, t_n\}_{n=1}^\infty\) is a sequential derivation at \(\phi\), then the point derivation \(D\) defined by \(\{\phi_n, t_n\}\) is continuous; more precisely, it follows that
In particular, \( \phi_n \) must converge to \( \phi \) in the metric topology.

**Proof.** Regarding \( \phi_n \) and \( \phi \) as bounded linear functionals on the Banach space \( A \), we have \( \left( \frac{\phi_n - \phi}{t_n} \right)(f) = \frac{\hat{f}(\phi_n) - \hat{f}(\phi)}{t_n} \) for \( f \in A \). Hence, the theorem follows from the uniform boundedness theorem.

2.3. **Corollary (Browder [1]).** If \( \phi \) is not isolated in \( \mathcal{M}(A) \) with the metric topology, then there exists a nonzero point derivation at \( \phi \).

**Proof.** By the hypothesis we can take a sequence \( \phi_n \) in \( \mathcal{M}(A) \) such that \( \phi_n \to 0 \) and \( 0n \) converges to \( \phi \) in the metric topology. If \( D_\phi(A) = \{0\} \), then \( A_\phi = A_{\phi}^* \). Therefore any element \( f \) of \( A \) can be represented in the form \( f - \hat{f}(\phi) = g_1h_1 + \cdots + g_kh_k \) for some \( g_i, h_i \in A_\phi \). Hence, we have

\[
\lim_{n \to \infty} \left| \frac{\hat{f}(\phi_n) - \hat{f}(\phi)}{\|\phi_n - \phi\|} \right| \leq \lim_{n \to \infty} \frac{\sum_{i=1}^{k} |\phi_n(g_i) - \phi(g_i)||h_i|}{\|\phi(h_i)| = 0.}
\]

Since this holds for all \( f \in A \), we have

\[
\lim_{n \to \infty} \left| \frac{\hat{f}(\phi_n) - \hat{f}(\phi)}{\|\phi_n - \phi\|} \right| \leq \lim_{n \to \infty} \frac{\sum_{i=1}^{k} |\phi_n(g_i) - \phi(g_i)||h_i|}{\|\phi(h_i)| = 0.
\]

Therefore \( \{\phi_n, \|\phi_n - \phi\|^2\} \) must be a sequential derivation. Since \( \lim_{n \to \infty} \frac{\|\phi_n - \phi\|}{\|\phi_n - \phi\|^2} = \infty \), we have a contradiction.

Now we shall consider a pair \( \{\phi_r, t_r\} \) of nets under a slight weak condition.

2.4. **Lemma.** Let \( \{\phi_r, t_r\} \) be a pair of nets in \( \mathcal{M}(A) \times C \setminus \{0\} \) with the properties (1.1), (1.2) and

\[
(2.1) \quad p(f) = \lim_{r} \left| \frac{\hat{f}(\phi_r) - \hat{f}(\phi)}{t_r} \right| < \infty \quad \text{for all } f \in A.
\]

Then \( p \) is a semi-norm on \( A \). If a linear functional \( D \) on \( A \) satisfies

\[
(2.2) \quad |D(f)| \leq p(f) \quad \text{for all } f \in A,
\]

then \( D \) is a point derivation at \( \phi \). And if \( \lim_{r} \frac{\|\phi_r - \phi\|}{t_r} < \infty \), then the point derivation \( D \) is continuous.

**Proof.** It is clear that \( p \) is a semi-norm. If \( D \) is a linear functional on \( A \) with the property (2.2), then for any \( f, g \in A_\phi \) it holds that

\[
|D(fg)| \leq p(fg) = \lim_{r} \left| \frac{\hat{f}(\phi_r) - \hat{f}(\phi)}{t_r} \right| g(\phi_r) + \frac{\hat{g}(\phi_r) - \hat{g}(\phi)}{t_r} \hat{f}(\phi) \]

\[
\leq p(f)|\phi(g)| + p(g)|\phi(f)| = 0,
\]
and \(|D(1)| \leq p(1) = 0\). Hence \(D\) is a point derivation at \(\phi\). Since \(p(f) \leq \lim \frac{\|\phi_r - \phi\|}{|r|} \|f\|\), the last statement holds.

As an application of the Hahn-Banach extension theorem, we have the following corollary.

2.5. COROLLARY. If there is an element \(w\) of \(A\) such that \(p(w) \neq 0\), then there exists a nonzero point derivation \(D\) at \(\phi\) such that \(D(w) = p(w)\) and \(|D(f)| \leq p(f)\) for all \(f \in A\).

2.6. THEOREM. Let \(w_1, \ldots, w_n \in A\). Let \(\{\phi_r\}\) be a net in \(\mathcal{M}(A)\) which converges to \(\phi\) and has the property

\[
(2.3) \lim_{r \to 0} \frac{\|\phi_r - \phi\|}{|\phi_r(w_1)| + \cdots + |\phi_r(w_n)|} < \infty.
\]

Then there exists a nonzero continuous point derivation at \(\phi\). Moreover, if (2.3) diverges for any lack of elements \(w_1, \ldots, w_n\), then \(\text{dim} \mathcal{T}_0(A) \geq n\).

PROOF. Removing elements out of \(w_1, \ldots, w_n\) within the property (2.3) as we can, it suffices to verify the last statement. Hence we assume (2.3) and that for each \(k = 1, \ldots, n\),

\[
\lim_{r \to 0} \frac{\|\phi_r - \phi\|}{|\phi_r(w_1)| + \cdots + |\phi_r(w_n)|} = \infty.
\]

Then there exists a subnet \(\phi_{r_k}\) for each \(k\) such that

\[
\lim_{r_k} \frac{\|\phi_{r_k} - \phi\|}{\sum_{i \neq k} |\phi_{r_k}(w_i)|} = \infty,
\]

and this yields

\[
\lim_{r_k} \frac{\|\phi_{r_k} - \phi\|}{|\phi_{r_k}(w_k)|} = \lim_{r_k} \frac{1}{\sum_{i \neq k} |\phi_{r_k}(w_i)|} \frac{\sum_{i \neq k} |\phi_{r_k}(w_i)|}{\sum_{i \neq k} |\phi_{r_k}(w_i)|} \frac{1}{\|\phi_{r_k} - \phi\|} \frac{\sum_{i=1}^n \|\phi_{r_k}(w_i)\|}{\sum_{i=1}^n \|\phi_{r_k}(w_i)\|} < \infty.
\]

Therefore we can define continuous semi-norms \(p_k\) on \(A\) by

\[
p_k(f) = \lim_{r_k} \frac{|\phi_{r_k}(f) - \phi_f(\phi)|}{|\phi_{r_k}(w_k)|} \quad \text{for} \quad f \in A.
\]

Since \(p_k(w_k) = 1\), there exist continuous point derivations \(D_k\) at \(\phi\) by Corollary 2.5 such that

\[
D_k(w_k) = 1 \quad \text{and} \quad |D_k(f)| \leq p_k(f) \quad \text{for} \quad f \in A.
\]

Finally, \(D_1, \ldots, D_n\) are linearly independent, for
This completes the proof.

2.7. COROLLARY. Let \( \phi \) be a non-isolated point of \( \mathcal{M}(A) \) in the metric topology. If there exist \( w_1, \ldots, w_n \in \mathcal{A} \) with the property:

\[
\lim_{\phi \to \phi} \frac{\|\phi - \psi\|}{\|\psi(w_1) + \cdots + \psi(w_n)\|} < \infty,
\]

then there exists a nonzero continuous point derivation at \( \phi \).

For \( w_1, \ldots, w_n \in A \), \( (w_1, \ldots, w_n)(\phi, \psi) = \sum_{i=1}^{n} |\phi(w_i) - \phi(w_i)| \) is a semi-metric on \( \mathcal{M}(A) \) for the weak* topology. The semi-metric \( (w_1, \ldots, w_n)(\phi, \psi) \) and the metric of the norm \( \|\phi - \psi\| \) are said to be equivalent on a subset \( M \) of \( \mathcal{M}(A) \) if and only if for \( \phi, \psi \in M \)

\[
K\|\phi - \psi\| \leq (w_1, \ldots, w_n)(\phi, \psi) \leq \max_{1 \leq i \leq n} \|w_i\| \cdot \|\phi - \psi\|,
\]

where \( K \) is some constant and the last inequality holds always.

2.8. COROLLARY. Let \( w_1, \ldots, w_n \in A \). If the semi-metric \( (w_1, \ldots, w_n)(\phi, \psi) \) and the metric of the norm \( \|\phi - \psi\| \) are equivalent on a metric open set \( U \) of \( \mathcal{M}(A) \), then there exists a nonzero continuous point derivation at any non-isolated point \( \psi \) of \( U \).

The following lemma is for the next section.

2.9. LEMMA. Let \( \psi \in \mathcal{M}(A) \). Let \( \{\psi_r, t_r\} \) be a pair of nets in \( \mathcal{M}(A) \times \mathcal{C} \setminus \{0\} \) with the properties (1.1), (1.2) and \( \lim_r \frac{\|\psi_{t_r} - \psi\|}{|t_r|} < \infty \). Suppose there exists a dense subset \( A_0 \) of \( A \) such that

\[
\lim_r \frac{f(\psi_{t_r}) - f(\psi)}{|t_r|} \text{ exist for all } f \in A_0.
\]

Then \( \{\psi_r, t_r\} \) is a sequential derivation at \( \psi \) for \( A \).

The proof is formal and will be omitted.

§ 3. Point derivations for \( A(X) \).

From now on \( X \) denotes a compact subset of the complex plane \( \mathcal{C} \), and \( A(X) \) denotes the uniform closed algebra of all continuous functions on \( X \) which is analytic in the interior of \( X \). The interior of \( X \) will be denoted by \( X^\circ \). Let \( x \in X \). Let \( A(X; x) \) be the set of all functions of \( A(X) \) which admit analytic continuation to some neighborhood of \( x \). Then \( A(X; x) \) is a uniformly dense subalgebra of \( A(X) \), and this implies that the maximal ideal space of \( A(X) \) is \( X \) ([3], Chap. II, Th. 1.8 and Cor. 1.10). When the functional \( f \mapsto f'(x) \) is continuous on \( A(X; x) \), the unique continuous extension on \( A(X) \) of this
functional is a continuous point derivation on $A(X)$ at $x$, and so we may use the notation $f'(x)$ also for all $f \in A(X)$. We can easily verify that any continuous point derivation on $A(X)$ at $x$ is a constant multiple of $f \mapsto f'(x)$ if there exists a nonzero continuous point derivation on $A(X)$ at $x$.

Now we prove the equivalence of the following conditions which were stated for $R(X)$ in section 1:

(a.0) There exists a nonzero continuous point derivation on $A(X)$ at $x$.

(a.1) There exists a constant $k$ such that $|f'(x)| \leq k\|f\|$ for all $f \in A(X) \setminus \{x\}$.

(a.2) There exists a complex representing measure $\mu$ such that

$$\int \frac{d|\mu|}{|z-x|} \leq \infty.$$ 

(a.3) $\sum_{n=0}^{\infty} \frac{4^n}{r^n} \alpha(E_n(x); r, X^n) < \infty$; where $\alpha$ denotes the continuous analytic capacity (see section 4).

(a.4) There exists a sequential derivation of the form $\{x_n, x_n-x\}$ at $x$ for $A(X)$.

(a.5) $\lim_{z \to x} \frac{\|z-x\|^4}{|z-x|} < \infty$;

where $\|z-x\|^4$ denotes the metric norm for $A(X)$, i.e.,

$$\|z-x\|^4 = \sup \{|f(z) - f(x)| : f \in A(X), \|f\| \leq 1\}.$$

The equivalence of (a.0) and (a.1) follows from the comments at the beginning of this section. Hence the equivalence of the conditions (a.0)~(a.3) is a formal modification of Wilken’s [5] and Hallstrom’s [3]. By Theorem 2.2, (a.4) implies (a.1). Furthermore, Corollary 2.7 and Lemma 2.9 imply:

3.1. Theorem. $\{x_n, x_n-x\}$ is a sequential derivation at $x$ if and only if

$$\lim_{n \to \infty} \frac{\|x_n-x\|^4}{|x_n-x|} < \infty.$$

This shows the equivalence of (a.4) and (a.5). Hence, to complete the equivalence, it suffices to show that (a.2) implies (a.4).

Let $m$ be the two-dimensional Lebesgue measure on the complex plane $\mathbb{C}$. Let $\mu$ be a (regular Borel) measure on $X$. For each $x \in \mathbb{C}$, we put

$$\tilde{\mu}(z) = \int \frac{d|\mu|}{|w-z|}$$

$$\tilde{\beta}(z) = \int \frac{d\mu}{w-z} \quad \text{when} \quad \tilde{\mu}(z) < \infty.$$

An application of Fubini’s theorem shows that $\tilde{\mu}$ is locally integrable with
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respect to \( m \), in particular, \( \hat{\mu}(z) < \infty \) a.e. \((m)\). If \( \hat{\mu}(x) < \infty \), then \( \frac{\mu(w)}{w-x} \) is a measure on \( X \). Hence, for each \( z \in C \), we also put

\[
(3.1) \quad \hat{\mu}(z, x) = \left( \frac{\mu(w)}{w-x} \right)^{(z)} = \int \frac{d\mu(w)}{|w-z||w-x|},
\]

\[
(3.2) \quad \mu(z, x) = \left( \frac{\mu(w)}{w-x} \right)^{(z)} = \int \frac{d\mu(w)}{(w-z)(w-x)} \quad \text{when } \hat{\mu}(z, x) < \infty.
\]

Let \( \mu \) be a complex representing measure for \( x \), i.e., \( \int f d\mu = f(x) \) for all \( f \in A(X) \). When \( \hat{\mu}(z) < \infty \), we put

\[
(3.3) \quad c = \int \frac{w-x}{w-z} d\mu(w) = 1 + (z-x)\hat{\mu}(z).
\]

If \( c \neq 0 \), then we can easily see that the point \( z \) belongs to \( X \) and

\[
(3.4) \quad d\nu(w) = \frac{w-x}{c(w-z)} d\mu(w)
\]
is a complex representing measure for \( z \).

3.2. LEMMA (Browder [1]). Let \( \mu \) be a measure on \( X \), and let \( x \in C \). For each positive integer \( n \), let \( A_n = \{ z : |z-x| \leq \frac{1}{n} \} \). Then

\[
\frac{1}{m(A_n)} \int_{A_n} |z-x| \hat{\mu}(z) \, dm(z) \to |\mu|(|x|) \quad \text{as } n \to \infty.
\]

3.3. LEMMA. Let \( x \in X \), let \( \mu \) be a complex representing measure with the property (a.2). Let \( \varepsilon > 0 \) and \( \delta = \varepsilon/(\varepsilon+2\|\mu\|+2) \). For each \( z \in X \),

\[
|z-x| \hat{\mu}(z, x) \leq \delta \quad \text{and} \quad |z-x| \hat{\mu}(z) \leq \delta \min(1, 1/\hat{\mu}(x))
\]

imply

\[
\left| \frac{f(z)-f(x)}{z-x} - f'(x) \right| \leq \varepsilon \|f\| \quad \text{for all } f \in A(X).
\]

PROOF. Our assumption is \( \hat{\mu}(z) < \infty \). Thus we can define a measure \( \mu' \) by

\[
d\mu'(w) = \left( \frac{1}{w-x} - \hat{\mu}(x) \right) d\mu(w).
\]

Then we have \( \int f d\mu'(w) = f'(x) \) for all \( f \in A(X) \); indeed, it suffices to show this only for \( f \in A(X; x) \), but this will follow easily. Take \( z \in X \) in our assumption. Since \( \delta < 1 \), \( c = 1 + (z-x)\hat{\mu}(z) \neq 0 \). Thus we can define a complex representing measure \( \nu \) for \( z \) by (3.4). Now

\[
\frac{d\nu(w)-d\mu(w)}{z-x} = \frac{1}{z-x} \left( \frac{w-x}{c(w-z)} - 1 \right) d\mu(w)
\]

\[
= \frac{1}{z-x} \frac{w-x-(1+(z-x)\hat{\mu}(z))(w-z)}{c(w-z)} d\mu(w)
\]
Moreover,
\[
\frac{dv(w) - d\mu(w)}{z-x} = -d\mu'(w)
\]
\[
= \left[ \frac{1}{c(w-z)} - \frac{1}{w-x} - \left( \frac{\hat{\mu}(z)}{c} - \hat{\mu}(x) \right) \right] d\mu(w)
\]
\[
= \left[ \frac{w-x-(1+(z-x)\hat{\mu}(z))(w-z)}{c(w-z)(w-x)} - \left( \frac{1}{c(z-x)} - \frac{1}{z-x} \right) d\mu(\zeta) \right] d\mu(w)
\]
\[
= \frac{z-x}{c} \left[ \frac{1}{(w-z)(w-x)} - \frac{\hat{\mu}(z)}{w-x} - \left( \frac{1}{(z-x)(z-x)} - \frac{\hat{\mu}(z)}{z-x} \right) d\mu(\zeta) \right] d\mu(w)
\]
\[
= \frac{z-x}{c} \left[ \frac{1}{(w-z)(w-x)} - \frac{\hat{\mu}(z)}{w-x} - (\hat{\mu}(z)-\hat{\mu}(z)\hat{\mu}(x)) \right] d\mu(w).
\]
Thus, for \( f \in A(X) \),
\[
\left| \frac{f(z) - f(x)}{z-x} - f'(x) \right| = \left| \int f(w) \left( \frac{dv(w) - d\mu(w)}{z-x} - d\mu'(w) \right) \right|
\]
\[
\leq \|f\| \frac{|z-x|}{|c|} \left[ \hat{\mu}(z, x) + |\hat{\mu}(z)| \hat{\mu}(x) + (|\hat{\mu}(z, x)| + |\hat{\mu}(z)\hat{\mu}(x)|) \|\mu\| \right]
\]
\[
\leq \|f\| \frac{|z-x|}{1-|\hat{\mu}(z)|} (\hat{\mu}(z, x) + \hat{\mu}(z)\hat{\mu}(x))(1+\|\mu\|)
\]
\[
\leq \|f\| \frac{2\delta}{1-\delta} (1+\|\mu\|) = \varepsilon \|f\|.
\]
This proves the lemma.

3.4. **Theorem.** Suppose there exists a nonzero continuous point derivation on \( A(X) \) at \( x \). For any \( \varepsilon > 0 \), we put
\[
D_\varepsilon(x) = \left\{ z \in X : \left| \frac{f(z) - f(x)}{z-x} - f'(x) \right| \leq \varepsilon \|f\| \quad \text{for all } f \in A(X) \right\},
\]
and
\[
\Delta_n = \left\{ z \in C : |z-x| \leq \frac{1}{n} \right\}.
\]
Then
\[
\lim_{n \to \infty} \frac{m(D_\varepsilon(x) \cap \Delta_n)}{m(\Delta_n)} = 1.
\]

**Proof.** Let \( \mu \) be a complex representing measure for \( x \) with the property (a.2), i.e., \( \hat{\mu}(x) < \infty \). Clearly, \( |\mu|(|\{x\}|) = 0 \), and \( \frac{\mu(w)}{w-x}(|\{x\}|) = 0 \). Let \( b = \min(1, 1/\hat{\mu}(x)) \). We put
\[
K_n = \left\{ z \in \Delta_n : |z-x| \hat{\mu}(z, x) \leq \delta \right\}
\]
Point derivations on commutative Banach algebras

\[ L_n = \{ z \in \mathcal{A}_n : |z-x| \bar{\rho}(z) \leq b\theta \} . \]

Then, by Lemma 3.3,

\[ m(D(x) \cap \mathcal{A}_n) \geq m(L_n \cap \mathcal{K}_n) \]

\[ \geq m(\mathcal{A}_n) \left( - \frac{1}{\delta} \int_{\mathcal{A}_n} |z-x| \bar{\rho}(z, x) \, dm(z) \right) \]

\[ + \frac{1}{b\theta} \int_{\mathcal{A}_n} |z-x| \bar{\rho}(z) \, dm(z) \) .

Hence we obtain

\[ 1 \geq \frac{m(D(x) \cap \mathcal{A}_n)}{m(\mathcal{A}_n)} \geq 1 - \frac{1}{\delta} \frac{1}{m(\mathcal{A}_n)} \int_{\mathcal{A}_n} |z-x| \left( \mu(w) \frac{w}{w-x} \right) \bar{\rho}(z) \, dm(z) \]

\[ - \frac{1}{b\theta} \frac{1}{m(\mathcal{A}_n)} \int_{\mathcal{A}_n} |z-x| \bar{\rho}(z) \, dm(z) . \]

Now the theorem follows from Lemma 3.2.

3.5. COROLLARY. If there exists a nonzero continuous point derivation at \( x \in X \), then there exists a sequence \( x_n \) in \( X \) such that \( x_n \) converges to \( x \) and, as linear functionals on \( A(X) \),

\[ \frac{f(x_n) - f(x)}{x_n - x} \to f'(x) \quad \text{(uniformly)} . \]

That completes the equivalence of the conditions (a.0)\( \sim \) (a.5).

3.6. COROLLARY. Let \( D : f \to f'(x) \) be the point derivation at \( x \). Then

\[ \| D_x \| = \lim_{z \to x} \frac{\| z-x \|^A}{|z-x|} . \]

PROOF. Let \( M = \lim_{z \to x} \frac{\| z-x \|^A}{|z-x|} \). There is a sequence \( x_n \) in \( X \) with

\[ \lim_{n} \frac{\| x_n - x \|^A}{|x_n - x|} = M . \]

Since the linear functional \( f \to \frac{f(x_n) - f(x)}{x_n - x} \) strongly converges to \( D_x \) by Lemma 2.9, \( \| D_x \| \leq M \). The reverse inequality follows from the above Corollary.

\[ \S \ 4. \ \text{Estimates of the } A(X)-\text{metric norm.} \]

Let \( D \) be a plane set. The continuous analytic capacity of \( D \), \( \alpha(D) \), is defined by

\[ \alpha(D) = \sup \{ |f'(\infty)| : f \in \mathcal{AC}(D) \} . \]

where \( \mathcal{AC}(D) \) is the set of all continuous functions \( f \) on the Riemann sphere \( S^2 \) such that \( f \) are analytic off a compact subset of \( D \), \( \| f \|_{S^2} \leq 1 \) and \( f(\infty) = 0 \).

For integers \( t, N, M(N < M) \), and it may be \( M = \infty \) and a positive number \( r \), we employ the notation \( \alpha_t^M(X; r) \) which is defined by
as it were, this means continuous analytic capacity at \( x \) of degree \( t \) with radius \( r \) and ratio \( 1/2 \) from \( N \) to \( M \). We shall use the notation \( \alpha_i(x; r) \) instead of \( \alpha^{\text{an}}_i(x; r) \). Let us attend to the following facts:

1. \( x \in X \) is a peak point for \( A(X) \) if and only if \( \alpha_1(x; r) = \infty \) (Melnikov),
2. there exists a nonzero continuous point derivation on \( A(X) \) at \( x \) if and only if \( \alpha_0(x; r) < \infty \) (Hallstrom, cf. (a.3)).

To obtain one side estimates, we repeat the argument in Curtis [2] which is based only on the above definition of \( \alpha \) and the following lemma.

4.1. **Lemma** (see [2], [3]). Let \( K \) be a compact plane set and \( f \) a continuous function on \( S^2 \) which is analytic off \( K \) and vanishes at \( \infty \). Then

\[
|f(z)| \leq \frac{\alpha(K)}{d(z, K)} \|f\|_{\infty}
\]

where we admit the right hand to attain the value \( \infty \) when \( z \in K \).

4.2. **Estimate** (essentially due to Curtis [2]). For \( x, y \in X \), it holds

\[
(A-1) \quad \|x-y\| \leq \frac{\alpha(\mathcal{D}(x; r) \setminus X^0)}{r+\alpha(\mathcal{D}(x; r) \setminus X^0)} - \frac{\alpha(\mathcal{D}(x; r) \setminus X^0)}{d(y, \mathcal{D}(x; r) \setminus X^0)}
\]

\[
(A-2) \quad \|x-y\| \leq \frac{1}{5} \frac{\alpha_0(x; r)}{3+\alpha_0(x; r)} - \frac{\alpha(\mathcal{D}(x; r) \setminus X^0)}{d(y, \mathcal{D}(x; r) \setminus X^0)}
\]

**Proof.** We shall only prove (A-2). Let \( \varepsilon > 0 \) and let \( M \) be a positive integer, and we set \( E_n = E_n(x; r) \). Then there exist compact sets \( K_n \subset E_n \setminus X^0 \) and functions \( f_n \in \mathcal{AC}(K_n) \) for \( 0 \leq n \leq M \) such that

\[
\alpha(E_n \setminus X^0) - \frac{\varepsilon}{M+1} - \frac{r}{2^n} \leq f_n'(\infty) \leq \alpha(E_n \setminus X^0).
\]

We define a function \( g \) on \( S^2 \) by

\[
g(z) = \sum_{n=0}^{M} \frac{2^n}{r} f_n'(\infty) - (z-x)f_n(z)
\]

\[
= \sum_{n=0}^{M} \frac{2^n}{r} f_n'(\infty) - \sum_{n=0}^{M} \frac{2^n}{r} (z-x)f_n(z).
\]

Since \( g(\infty) = 0 \), \( g \) is continuous on \( S^2 \) and analytic off \( K = \bigcup_{n=0}^{M} K_n \). Hence \( g \in A(X) \), and \( g(x) = \sum_{n=0}^{M} \frac{2^n}{r} f_n'(\infty) \). Since \((z-x)f_n(z)\) is analytic off \( K_n \), by the maximum modulus principle,

\[
\left\| \frac{2^n}{r} (z-x)f_n(z) \right\|_{K_n} \leq 1.
\]
On the other hand, applying Lemma 4.1 to $f_n$,

$$|f_n(z)| \leq \frac{\alpha(K_n)}{d(z, K_n)} \leq \frac{\alpha(E_n \setminus X^o)}{d(z, E_n)}.$$ 

Hence, if $\frac{r}{2^{j+1}} < |z - x| < \frac{r}{2^j}$, then the distance from $z$ to $E_n$ is at least $\frac{r}{2^{j+1}} - \frac{r}{2^{j+2}}$ for $n \neq j-1, j, j+1$, and hence for such $n$

$$\left|2^n \frac{r}{r} (z-x)f_n(z)\right| \leq 2^n \frac{r}{2^j} \frac{\alpha(E_n \setminus X^o)}{r} \leq 2^{n+2} \frac{r}{r} \frac{\alpha(E_n \setminus X^o)}{2^{j+1} - 2^{j+2}}.$$

Now, for $z \in \bigcup_{n=0}^M E_n$, we have

$$|g(z)| \leq \sum_{n=0}^M \frac{2^n}{r} f_n(\infty) + \sum_{n=0}^M \left|2^n \frac{r}{r} (z-x)f_n(z)\right|$$

$$\leq \sum_{n=0}^M \frac{2^n}{r} \alpha(E_n \setminus X^o) + 4 \sum_{n=0}^M \frac{2^n}{r} \alpha(E_n \setminus X^o) + 3$$

$$= \frac{5}{4} \alpha_1(M(x; r) + 3).$$

Again, by the maximum modulus principle, this estimate holds for all $z \in S^2$, and applying Lemma 4.1 to $g$, we obtain

$$|g(z)| < \frac{\alpha(K)}{d(z, K)} \left(\frac{5}{4} \alpha_1(M(x; r) + 3).\right)$$

Since $\|g\|_{\infty} \leq \frac{5}{4} \alpha_1(M(x; r) + 3)$, we have for $y \in X$

$$\|x-y\|_A \leq \frac{\sum_{n=0}^M \frac{2^n}{r} f_n'(\infty)}{\frac{5}{4} \alpha_1(M(x; r) + 3)} - \frac{\alpha(K)}{d(y, K)}$$

$$\leq \frac{1}{5} \frac{\alpha_1(M(x; r) - 4\varepsilon)}{\alpha_1(M(x; r) + 3)} - \frac{\alpha(J(x; r) \setminus X^o)}{d(y, J(x; r) \setminus X^o)}.$$

Now let $\varepsilon \to 0$ and $M \to \infty$, we have the estimate (A-2). One can also prove the estimate (A-1) by the similar modification of [2], Theorem 3.2.

To obtain the opposite estimate, we need the following theorem ([3], Chap. VIII, Th. 12.6).

(Melnikov’s Estimate) Let $J$ be an open annulus of conformal radius $r$, and let $K$ be a compact subset of $J$ and $f$ a continuous function on $J$ which is analytic in $J \setminus K$. Then

$$\left|\int_{J} f(z)dz\right| \leq \frac{c}{1-r} \|f\|_J \alpha(K \cap J),$$
where $c$ is a universal constant and $bJ$ denotes the boundary of $J$.

The estimate is not so simple, that we first make some calculations. Let $x, y$ be distinct points in $X$. We denote by $A(X; x, y)$ the set of all functions of $A(X)$ which admit analytic continuation to some neighborhood of $x$ and $y$. Let $f \in A(X; x, y)$. We extend $f$ to a continuous function on $S^2$ such that $f$ is analytic in some neighborhood of $x$ and $y$, and the norm $\|f\|_{S^2}$ is sufficiently near to $\|f\|_X$. Now let $r_1$ be a positive number such that $f$ is analytic in $\mathcal{A}(x; r_1)$. Our aim is to estimate $|f(x) - f(y)|$. Let $I = \{z : |z - x| = r_1\}$ and

$$g(z) = \frac{f(z) - f(y)}{z - y}.$$ 

Then $g$ is continuous on $S^2$ and analytic wherever $f$ is. Thus it follows from Cauchy's integral formula

$$f(z) - f(y) = \frac{x - y}{2\pi i} \int_{I} \frac{g(\zeta)}{\zeta - x} d\zeta.$$ 

Now we fix a number $\delta$ such that $0 < \delta < |x - y|$, and take a continuously differentiable function $h$ on $S^2$ such that $h$ is supported on $\mathcal{A}(y; \delta)$, $\left| \frac{\partial h}{\partial \zeta} \right| \leq \frac{4}{\delta}$, $0 \leq h(z) \leq 1$ on $S^2$ and $h(z) = 1$ when $z \in \mathcal{A}(y; \frac{\delta}{2})$. In order to estimate (4.3) we consider the integral of the form

$$G(w) = \frac{1}{\pi} \int_{S^2} \frac{g(z) - g(w)}{z - w} \frac{\partial h}{\partial \zeta}(z) d\zeta d\eta,$$

where $z = \xi + i\eta$. From [3], Chap. II, Lemma 1.7, $G$ is analytic wherever $g$ is and analytic in $S^2 \setminus \mathcal{A}(y; \delta)$, and $G - g$ is analytic wherever $g$ is and analytic in $\mathcal{A}(y; \frac{\delta}{2})$. Moreover, a crude estimate yields

$$|G(w)| \leq |g(w)| + \frac{1}{\pi} \left\| g \right\|_{A(v_1; \frac{\delta}{2})} \left| \frac{\partial h}{\partial \zeta} \right| \int_{S^2} \frac{d\zeta d\eta}{|z - w|}.$$ 

Since the last integral attains the maximum $2\pi \delta$ when $w = y$, the definition of $g$ yields

$$|G(w)| \leq |g(w)| + \frac{1}{\pi} \left\| g \right\|_{A(v_1; \frac{\delta}{2})} \frac{4}{\delta} \cdot 2\pi \delta = |g(w)| + \frac{32}{\delta^2} \left\| f \right\|.$$

And, by the maximum modulus principle,

$$\|G - g\|_{S^2} = \left\| G - g \right\|_{A(v_1; \frac{\delta}{2})} \leq \left\| g \right\|_{A(v_1; \frac{\delta}{2})} + \frac{32}{\delta^2} \left\| f \right\|.$$ 

$$\|G - g\|_{S^2} = \left\| G - g \right\|_{A(v_1; \frac{\delta}{2})} \leq 2\left\| g \right\|_{A(v_1; \frac{\delta}{2})} + \frac{32}{\delta^2} \left\| f \right\|.$$
If \( w \in \mathcal{J}(y; \delta) \), attending to that \( h(w) = 0 \) in (4.4), we have

\[
|G(w) - g(w)| \leq |g(w)| + \frac{1}{\pi} \|g\| \left\| a(y; \frac{4}{\delta} r) \right\| \frac{\partial h}{\partial \overline{z}} \left| \frac{\delta^2}{|w-y|-\delta} \right|
\]

\[
\leq \frac{2\|f\|}{|w-y|-\delta} + \frac{2\|f\|}{\delta^2} \frac{4}{\delta} \frac{\delta^2}{|w-y|-\delta}.
\]

Therefore,

(4.7) \[
|G(w) - g(w)| \leq \frac{18\|f\|}{|w-y|-\delta} \quad \text{for} \quad |w-y| > \delta.
\]

Now return to (4.3),

\[
\int_{r} \frac{g(\zeta)}{\zeta - x} \, d\zeta = \int_{r} \frac{(g - G)(\zeta)}{\zeta - x} \, d\zeta + \int_{r} \frac{G(\zeta)}{\zeta - x} \, d\zeta.
\]

If necessary, we shrink \( r \) sufficiently so that \( \mathcal{J}(x; r) \cap \mathcal{J}(y; \delta) = \emptyset \). Since \( G(z) \) is analytic in \( S \setminus \mathcal{J}(y; \delta) \), an application of Cauchy’s integral formula yields

\[
\int_{r} \frac{g(\zeta)}{\zeta - x} \, d\zeta = \int_{r} \frac{(g - G)(\zeta)}{\zeta - x} \, d\zeta + \int_{|\zeta-y|=\delta} \frac{G(\zeta)}{\zeta - x} \, d\zeta.
\]

We compute the integrals separately.

(The first integral): Let \( r > |x-y| + \delta \), and we fix a integer \( k > 0 \) such that \( \frac{r}{2^{k-1}} > |x-y| + \delta \). For sufficiently large integer \( M \geq k \), we may assume

\[
I = \left\{ z ; |z-x| = \frac{r}{2^{m+1}} \right\}.
\]

Then

\[
\int_{r} \frac{(g - G)(\zeta)}{\zeta - x} \, d\zeta = \int_{|\zeta-x|=r} \frac{(g - G)(\zeta)}{\zeta - x} \, d\zeta + \sum_{n=0}^{M} \int_{E_n(x; r)} \frac{(g - G)(\zeta)}{\zeta - x} \, d\zeta.
\]

Divide the second term into two parts at \( k \), apply (4.6) for \( i < k \) and the first term, and apply (4.7) for \( i \geq k \). Then Melnikov’s estimate yields

\[
\left| \int_{r} \frac{(g - G)(\zeta)}{\zeta - x} \, d\zeta \right| \leq \frac{2\pi r \cdot 18\|f\|}{r(r-|x-y|-\delta)} + 2\epsilon \sum_{n=0}^{k-1} \frac{18\|f\|}{r^{2^{n+1}}-|x-y|-\delta} \frac{\alpha(E_n(x; r) \setminus X^\delta)}{r^{2^{n+1}}}
\]

\[
+ 2\epsilon \sum_{n=k}^{M} \frac{40}{\delta} \|f\| \frac{\alpha(E_n(x; r) \setminus X^\delta)}{r^{2^{n+1}}}
\]

(The second integral): This time we can write as follows;
\[
\int_{|z-y|=\delta} \frac{G(\zeta)}{\zeta-x} \, d\zeta = \sum_{n=0}^{\infty} \int_{E_n(y;\delta)} \frac{G(\zeta)}{\zeta-x} \, d\zeta,
\]
since the terms of the right hand are vanishing for sufficiently large \(n\); because \(\frac{G(\zeta)}{\zeta-x}\) is analytic in some neighborhood of \(y\). Thus, by (4.5) and the definition of \(g\), Melnikov's estimate yields
\[
\left| \int_{|z-y|=\delta} \frac{G(\zeta)}{\zeta-x} \, d\zeta \right| \leq \frac{1}{|x-y| - \delta} \cdot 2c \sum_{n=0}^{\infty} \left( \frac{2 \|f\|}{\delta} + \frac{32 \|f\|}{\delta} \right) \alpha(E_n(y;\delta)\setminus X^u)
\]
\[
\leq \frac{2c}{|x-y| - \delta} \sum_{n=0}^{\infty} 18 \|f\| \frac{2^{n+1}}{\delta} \alpha(E_n(y;\delta)\setminus X^u).
\]
We put these together, and let \(M \to \infty\). Since \(A(X; x, y)\) is uniformly dense in \(A(X)\), we have the following estimate.

4.3. ESTIMATE. Let \(x, y \in X\). Let \(r > 0\), \(\delta > 0\) such that \(r > |x-y| + \delta\), and \(0 < \delta < |x-y|\). Let \(k\) be a positive integer such that \(\frac{r}{2^{k+1}} > |x-y| + \delta\), then
\[
(A-3) \quad \|x-y\|^4 \leq C|x-y| \left[ \frac{1}{r - |x-y| - \delta} \right.
\]
\[
+ 4 \sum_{n=0}^{k+1} \frac{1}{2^{n+1}} \|E_n(x; r)\| \alpha(E_n(x; r)\setminus X^u)
\]
\[
+ \frac{1}{\delta} \alpha^h(x; r) + \frac{1}{|x-y| - \delta} \alpha_i(y; \delta) \bigg] ,
\]
where \(C\) is a universal constant, for instance, we take \(C = 36\pi c\).

Now we shall derive two versions of \((A-3)\) which have meaning in the cases (4.1), (4.2) respectively.

4.4. ESTIMATE. Let \(x, y \in X\), \(0 < \sigma < 1\) and \(r > 0\). If \(|x-y| < \frac{r}{4}\), then
\[
(A-4) \quad \|x-y\|^4 \leq C \left[ \frac{|x-y|}{r - 2|x-y|} + \frac{\sqrt{|x-y|}}{\sqrt{r - 2 \sqrt{|x-y|}}} \alpha_i(x; r) + \frac{1}{\sigma} \alpha^h(x; r)
\]
\[
+ \frac{1}{1-\sigma} \alpha_i(y; \sigma|x-y|) \bigg] ,
\]
where \(k = \left[ \frac{1}{2} \log_2 \frac{r}{|x-y|} \right] ;\) that is, \(k\) is the maximum integer with \(\frac{1}{2^k} \geq \sqrt{\frac{|x-y|}{r}}\).

PROOF. Let \(\delta = \sigma|x-y|\) in \((A-3)\). Then we have only to consider the second term in \((A-3)\). This is converted as follows;
\[
|x-y|^4 \cdot 4 \sum_{n=0}^{k-1} \frac{1}{2^{n+1}} \frac{2^n}{r} \alpha^h(x; r)\setminus X^u
\]
4.5. Estimate. Let \( x, y \in X, 0 < \sigma < 1 \) and \( r > 0 \). If \( |x-y| < \frac{r}{8} \), then

\[
\|x-y\|^4 \leq C \frac{1}{r-2} \left[ \frac{1}{|x-y|} + \frac{6}{\sigma} \alpha_s(x; r) + \frac{1}{1-\sigma} \alpha_1(y; \sigma|x-y|) \right].
\]

**Proof.** Let \( k \) be the maximum integer such that \( k > \frac{4}{2n+1} |x-y| \). We must be concerned with the second and the third terms in (A-3). Since

\[
2n+1 - |x-y| - \delta > 2n+1 - 2 \cdot \frac{r}{2}\cdot 4 \cdot 2^k \geq \frac{1}{2} \cdot \frac{r}{2n+1},
\]

for \( n < k \), the second term is converted as follows;

\[
4 \sum_{n=0}^{k-1} \frac{r}{4n+1} \left[ \frac{1}{|x-y|} - \delta \right] \frac{2^n}{r} \alpha(E_n(x; r) \setminus X^o) \leq 4 \sum_{n=0}^{k-1} \frac{2^{n+2}}{r} \alpha(E_n(x; r) \setminus X^o) \leq \frac{2}{\sigma} \alpha_s(x; r).
\]

Also, the third term is converted as follows; let \( \delta = \sigma |x-y| \),

\[
\frac{1}{\delta} \cdot 4 \sum_{n=k}^{\infty} \frac{2^n}{r} \alpha(E_n(x; r) \setminus X^o) = \frac{4}{\delta} \sum_{n=k}^{\infty} \left( \frac{2^n}{r} \right) \alpha(E_n(x; r) \setminus X^o) \cdot \frac{r}{2n+1} \leq \frac{4}{\delta} \frac{|x-y|}{8} \sum_{n=k}^{\infty} \frac{2^{n+2}}{r} \alpha(E_n(x; r) \setminus X^o) \leq \frac{4}{\sigma} \alpha_1(x; r).
\]

**Comment.** Our estimates are unintelligible. However, if we consider when \( x \) is fixed and \( \sigma \) is a constant, then the first three terms in (A-4) and the first two terms in (A-5) are determined by the usual metric \( |x-y| \), and hence we have only to worry about the last term containing \( \alpha_1(y; \sigma|x-y|) \) for (A-4) and (A-5).

§ 5. Applications of the estimates.

We use the following simple facts:

\[
(5.1) \quad \alpha(\mathcal{A}(x; r)) = r.
\]

\[
(5.2) \quad \text{If } D_1 \subset D_2, \text{ then } \alpha(D_1) \leq \alpha(D_2).
\]

The first application is to show the following well known theorem (see [2], [3]).
5.1. **Theorem.** Let \( x \in X \).

(a) If \( \lim_{r \to 0} \frac{\alpha(D(x; r) \setminus X^o)}{r} > 0 \), then \( x \) is a peak point for \( A(X) \).

(b) If \( \alpha(x; r) = \infty \) for some \( r > 0 \), then \( x \) is a peak point for \( A(X) \).

**Proof.** (a) By the assumption, there exist \( r_n \downarrow 0 \) such that

\[
\lim_{n \to \infty} \frac{\alpha(D(x; r_n) \setminus X^o)}{r_n} > 0.
\]

Let \( y \in X \) be a distinct point from \( x \), and set \( C_n = \alpha(D(x; r_n) \setminus X^o) \). Then, by \( \langle A-1 \rangle \),

\[
\lim_{n \to \infty} \frac{C_n}{r_n} = \lim_{n \to \infty} \frac{\alpha(D(x; r_n) \setminus X^o)}{r_n} = \lim_{n \to \infty} \frac{\alpha(D(x; r_n) \setminus X^o)}{r_n} > 0,
\]

since \( d(y, D(x; r_n) \setminus X^o) \to |x-y| \) and \( C_n = \alpha(D(x; r_n) \setminus X^o) \leq r_n \to 0 \). This shows that \( x \) is isolated in \( X \) with the \( A(X) \)-metric topology. Therefore \( x \) must be a peak point for \( A(X) \) by a Corollary of Theorem 2 in Browder’s [1]. (b) will follow from the similar argument with the use of \( \langle A-2 \rangle \) instead of \( \langle A-1 \rangle \).

5.2. **Theorem.** Let \( x \in X \) and let \( x_n (\neq x) \) be a sequence in \( X \).

(a) If \( x_n \) converges to \( x \) in the \( A(X) \)-metric topology, then for any \( \sigma > 0 \)

\[
\lim_{n \to \infty} \alpha(D(x_n; \sigma|x_n-x|) \setminus X^o) = 0.
\]

(b) If \( \{x_n, x_n-x\} \) is a sequential derivation for \( A(X) \) at \( x \), then for any \( \sigma > 0 \)

\[
\lim_{n \to \infty} \alpha(D(x_n; \sigma|x_n-x|) \setminus X^o) = 0.
\]

**Proof.** (a) By the assumption \( x \) is not a peak point for \( A(X) \). Hence \( \lim_{r \to 0} \frac{\alpha(D(x; r) \setminus X^o)}{r} = 0 \) by Theorem 5.1 (a), so the conclusion yields to the inequality

\[
\frac{\alpha(D(x_n; \sigma|x_n-x|) \setminus X^o)}{|x_n-x|} \leq (1+\sigma) \frac{\alpha(D(x; (1+\sigma) |x_n-x|) \setminus X^o)}{(1+\sigma)|x_n-x|}.
\]

(b) In this case there is a nonzero bounded point derivation on \( A(X) \) at \( x \). Hence \( \lim_{r \to 0} \frac{\alpha(D(x; r) \setminus X^o)}{r} = 0 \) ([4], Th. 2). The remains are similar.

5.3. **Theorem.** Let \( x \in X \). Let \( x_n \) be a sequence in \( X \) which converges to \( x \) in the natural topology of \( C \).

(a) Suppose \( x \) is not a peak point for \( A(X) \). \( x_n \) converges to \( x \) in the \( A(X) \)-metric topology if and only if

\[
\lim_{n \to \infty} \alpha(x_n; \sigma|x_n-x|) = 0;
\]
where $\sigma$ is any fixed number $0 < \sigma < 1$.

(b) Suppose there exists a nonzero continuous point derivation on $A(X)$ at $x$. Then $\{x_n, x_n-x\}$ is a sequential derivation for $A(X)$ at $x$ if and only if

$$\lim_{n \to \infty} \frac{\alpha_1(x_n; |x_n-x|)}{|x_n-x|} < \infty;$$

where $\sigma$ is any fixed number $0 < \sigma < 1$.

PROOF. (a) We shall prove “if only” part. From (A-2),

$$\|x_n-x\|^4 \geq \frac{1}{5} \frac{\alpha_1(x_n; |x_n-x|)}{3+\alpha_1(x_n; |x_n-x|)} - \frac{\alpha(D(x_n; \sigma |x_n-x|) \backslash X^0)}{|x_n-x| - \sigma |x_n-x|}.$$

Since we have seen $\lim_{n \to \infty} \frac{\alpha(D(x_n; \sigma |x_n-x|) \backslash X^0)}{|x_n-x|} = 0$ in Theorem 5.2 (a), the second term in the right hand tends to 0 as $n \to \infty$, and the assumption is $\|x_n-x\|^4 \to 0$, thus $\alpha_1(x_n; |x_n-x|)$ must converge to 0. “if” part is an easy consequence of the estimate (A-4) and the last comment in the previous section.

(b) We shall prove “if only” part. From (A-2),

$$\|x_n-x\|^4 \geq \frac{1}{5} \frac{\alpha_1(x_n; |x_n-x|)}{3+\alpha_1(x_n; |x_n-x|)} - \frac{\alpha(D(x_n; \sigma |x_n-x|) \backslash X^0)}{|x_n-x| - \sigma |x_n-x|}.$$

We have seen in Theorem 5.2 (b) that the second term in the right hand tends to zero. And since $x_n$ converges to $x$ in the metric topology, $\alpha_1(x_n; |x_n-x|)$ converges to 0 by (a). Thus Theorem 3.1 implies the first half. The latter half follows from (A-5) and the last comment in the previous section.

Addendum. After this paper was submitted for publication, James Liming Wang sent to the author his paper “An approximate Taylor’s theorem for $R(X)$” (Aarhus Univ. Preprint Series, 1972/73, No. 59). With another remarkable facts he showed independently in it that the arguments in section 3 are valid for $t$-th order point derivation.

Suggested by his paper, the author obtained the estimates in the case of $t$-th order point derivation. In particular, it was proved that the linear functional $\frac{f(x_n) - f(x)}{x_n-x}$ on $A(X)$ converges uniformly to $f'(x)$ if and only if

$$\lim_{n} \frac{\alpha_1(x_n; |x_n-x|)}{|x_n-x|} = 0,$$

for any fixed $0 < \sigma < 1$. 
References


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