An odd characterization of some simple groups

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§ 1. Introduction.

In his lecture note [10], G. Higman proposed characterizations of finite simple groups in terms of properties concerning with odd primes, which are called odd characterizations. He discussed the recent results on odd characterizations and also illustrated some techniques which are used in such characterizations. For instance, the classification problem of $\mathfrak{CO}_n$-groups is the most famous example (Fletcher [3], [4]). However, as a more typical situation for such problems, we consider the structure of the centralizers of elements of order 3. Actually, we are interested in such centralizers in known sporadic simple groups which are given in Conway [2] and Tits [12] and we notice that those have some particular properties.

The purpose of this paper is to prove a result in this direction:

**Theorem.** Let $G$ be a finite simple group and assume that for any element $s$ of $G$ of order 3, $C_G(s) \cong Z_3 \times Z_3$ or $Z_3 \times A_4$. Then $G \cong A_6$, $A_7$, $L_3(4)$, $L_3(7)$, $U_3(5)$, or $M_{22}$.

All groups considered in this paper are finite. Most of our notation is standard and taken from Gorenstein [6]. For each $p$-subgroup $P$ of a group $G$, $H(P)$ denotes the set of $p'$-subgroups of $G$ normalized by $P$, and $H^*(P)$ denotes the set of the maximal elements in $H(P)$. For any 2-group $T$, $m(T)$ is the maximum rank of an abelian subgroup of $T$.

§ 2. Preliminary lemmas.

**Lemma 2.1.** The following hold:

1. $L_3(4) \cong GL(8, 2)$;
2. $M_{10}$, $PGL(2, 9) \cong GL(7, 2)$;
3. Let $X$ be an elementary abelian group of order $2^8$ and $G$ be a subgroup of Aut $(X)$ isomorphic to $A_6$, $M_{10}$ or $PGL(2, 9)$. Assume that $|C_X(s)| \leq 4$ for each element $s$ of order 3. Then $|X| = |C_X(t)|^2$ for any involution $t$ of $G$.

**Proof.** (1) Suppose $L = L_3(4)$ acts faithfully on an elementary abelian group $V$ of order $2^8$. Since $L$ has an elementary abelian Sylow 3-subgroup of
order 9 and all elements of $L$ of order 3 are conjugate, we have that $|C_X(s)| = 4$ for any element $s$ of $L$ of order 3. Clearly, a Sylow 7-subgroup of $L$ centralizes an element $v \in V^*$. Since $|L:C_L(v)| = |v^L| < |V| = 2^8$, we have $|C_L(v)| > 70$. Thus $C_L(v)$ has no normal 7-complement, as otherwise $|O_{C_L(v)}(V)| \leq 8$. Thus $C_L(v)$ contains a Frobenius group of order 21. But then for some element $s$ of $L$ of order 3, $|C_L(v)| = 16$, a contradiction.

(2) This follows from the fact that all elements of order 3 are conjugate in either group.

(3) It follows easily that a Sylow 3-subgroup of $G$ acts fixed-point-free on $X$. Assume first that the involution $t$ normalizes a Sylow 3-subgroup $S$ of $G$. Since $X = \langle C_X(s) | s \in S^* \rangle$ and $|C_X(s)| = |C_X(s) \cap C_X(t)|^2$ for each $s \in S^*$, we see that $|X| = |C_X(t)|^2$. Thus we may assume that $t$ normalizes no Sylow 3-subgroup of $G$. Then we have that $G$ is isomorphic to $PGL(2, 9)$ and $X$ is of order 28. Note that $t$ inverts a Sylow 5-subgroup $P$ of $G$ and $N_G(P)$ is dihedral of order 20. It will suffice to show that $P$ acts fixed-point-free on $X$. Suppose false. Then $|C_P(t)| = 16$. Since $G$ has 36 Sylow 5-subgroups, we have $|C_P(t)^*| \cdot |G:N_G(P)| > |V|^*$. This means that there is $v \in V^*$ which is centralized by two distinct Sylow 5-subgroups, and so $C_P(v)$ is not 5-closed. Thus $C_P(v)$ contains a subgroup $A$ isomorphic to $A_5$. We may assume that $A$ contains $P$. As $N_G(P) \leq A$, there is an involution $u$ in $N_G(P) \cap C(t)$. We have $|C_P(t) \cap C(u)| = 4$. Since $ut$ and $t$ are conjugate and $[V, P] \leq C_P(t)u$, we have $|C_P(t)| = |C_P(tu)| = 2^8$, and so $C_P(t) \leq C_P(t)$. Thus $t \in C_P(v)$. But since $N_G(A) \leq G^c$, we have that $G = \langle A, t \rangle \leq C_P(v)$, contrary to (2).

The following lemma and the proof are by Goldschmidt [5], Corollary 4.

**Lemma 2.2.** Let $X$ be an elementary abelian 2-subgroup of a finite group $G$ and $T$ a Sylow 2-subgroup of $N_G(A)$. Assume that for each element $t$ of $T - X$, $m(X) > m(T/X) + m(C_X(t))$. Then $T$ is a Sylow 2-subgroup of $G$ and $A$ is strongly closed in $T$.

**Proof.** Let $Y$ be a subgroup of $T$ conjugate to $X$ in $G$. If $X \neq Y$, then for $y \in Y - X$, $C_X(y) \supseteq X \cap Y$. Thus $m(T/X) + m(C_X(y)) \geq m(X/Y/X) + m(X \cap Y)$. Since $XY/X \cong Y/X \cap Y$, we have that $m(T/X) + m(C_X(y)) \geq m(Y) = m(X)$, a contradiction. Hence $X = Y$, and so $X$ is weakly closed in $T$. In particular, $T$ is a Sylow 2-subgroup of $G$. Next, suppose $X$ is not strongly closed in $T$. Let $t$ be an involution of $T - A$ conjugate to an element of $X$. Among all $g \in G$ such that $t^g \in X$, choose $g$ in such a way that $|X \cap X^{t^g-1}|$ is maximal. Set $Y_0 = X \cap X^{t^g-1}$, $Y_1 = \langle Y_0, t \rangle$ and $Y_2 = C_X(t \mod Y_0)$. Clearly, $Y_0 < Y_2 \leq N_G(Y_1)$. Since $\langle X^{t^g-1}, Y_2 \rangle \leq N_G(Y_1)$, the weak closure of $X$ implies that $Y_2 \leq N_G(X^{t^g-1})$ for some $a \in N_G(Y_1)$. Thus $Y_2^{ag} \leq T$ for some $n \in N_G(X)$. We have that $\langle Y_0^{ag}, t^{ag} \rangle = Y_0^{ag} = Y_0^{ag} \leq X$, and so $Y_0^{ag} \leq X \cap X^{ag}$. Thus the maximality of $|Y_0|$ implies that $Y_0^{ag} = X \cap X^{ag}$. By a change of notation, we may assume
Odd characterization

that \( Y^s \leq T \). We have \( Y^s = X \cap X^s \leq X \cap Y^s \leq Y^s \), and so \( Y^s = Y^s \cap X \). Thus \( Y^s / Y^s \cap Y^s = Y^s / Y^s \), \( X / X^s \leq T / X \), and so \( m(Y^s / Y^s) \leq m(T / X) \). Set \( Y = C_X(Y) \).

Since \( X \) is elementary, \( \left[ X, Y \right] \leq Y \). Furthermore, \( X / Y^s \cong [X / Y^s, Y] \cong [X, Y] Y^s / Y^s \leq Y / Y^s \), and so \( m(X / Y^s) \leq m(Y / Y^s) \).

Hence

\[
m(X) = m(Y) + m(Y^s / Y) + m(X / Y^s)
\]

\[
\leq m(Y) + m(Y^s / Y) + m(Y / Y^s)
\]

\[
= m(Y) + m(Y^s / Y^s)
\]

\[
\leq m(Y) + m(T / X),
\]

contrary to the assumption. The lemma is proved.

**Lemma 2.3** (Fletcher [3]). *\( \pi \theta \)-groups with elementary abelian Sylow 3-subgroup of order 9 are \( 3 \)-closed or isomorphic to one of \( A_4, M_{19}, PGL(2, 9), L_3(4) \), where \( M_{19} \) is a subgroup of the Mathieu group \( M_{11} \) of index 11.

**Lemma 2.4** (Harada [9]). Let \( G \) be a simple group which contains an elementary abelian subgroup of order 16 such that \( A \) is a Sylow 2-subgroup of \( C_3(A) \) and \( N_3(A) / C_3(A) \) is isomorphic to \( A_6 \) or \( A_4 \). Then \( G \) is of sectional 2-rank 4. In particular, \( G \) is isomorphic to \( M_{22}, M_{23}, MC_3, L_4(q), q \equiv 5 \pmod{8} \), or \( U_4(q), q \equiv 3 \pmod{8} \).

**Lemma 2.5** (Smith-Taylor [11]). Let \( G \) be a finite group with noncyclic abelian Sylow \( p \)-subgroup \( P \). Assume that \( |N_3(P) : PC_3(P)| = 2 \). Then \( O^p(G) < G \) or \( G \) is \( p \)-solvable.

### § 3. The proof of the theorem.

Throughout the remainder of this paper, \( G \) denotes a simple group satisfying the assumption of our theorem, that is, \( C_3(s) \) is isomorphic to \( Z^3 \times A_4 \) for each element \( s \) of \( G \) of order 3. If \( G \) is a \( \pi \theta \)-group, then it follows from Fletcher theorem that \( G \) is isomorphic to \( A_6 \) or \( L_3(4) \). Thus we may assume that \( C_3(s) \cong Z_3 \times A_4 \) for an element \( s \) of \( G \) of order 3. Let \( S \) be a Sylow 3-subgroup of \( G \). Then clearly \( S \cong Z^3 \times A_4 \) and \( C_3(S) = S \). We argue by induction on \( |G| \).

**Lemma 3.1.** The following hold:

1. \( N_3(S) \) is a Frobenius group such that \( |N_3(S) : S| = 4 \) or 8.
2. For any element \( s \) of order 3, \( C_3(s) \cong S_5 \) or \( S_9 \).

**Proof.** Since \( G \) is simple and \( S \) is abelian, it follows from Burnside's transfer theorem and Smith-Taylor's theorem that \( |N_3(S) : S| \geq 4 \). Since \( C_3(s) \cap N_3(S) = S \) for any \( s \in S^\ast \), \( N_3(S) \) is a Frobenius group, proving (1). Since an involution of \( N_3(S) \) inverts \( S \), (2) follows easily.

**Lemma 3.2.** The following hold:
Let $s \in S^*$, $V = O_2(C_0(s))\neq 1$. Then $C_0(V) = O_2(C_0(V))\langle s \rangle$.

Let $X \in H^*(S)$. Then $X$ is a 2-group and $N_0(X)/X$ is a C00-group. Furthermore, $|X| = 4$ or $C_0(X) \leq X$.

Let $X \in H(S)$ and $\langle s_1 \rangle, 1 \leq i \leq 4$, be the four subgroups of $S$ of order 3. Then

$$X = C_X(s_1)C_X(s_2)C_X(s_3)C_X(s_4),$$

and

$$|X| = \prod_i |C_X(s_i)|.$$

Let $X \in H(S)$ and let $t$ be an involution of $N_0(S)$. Then $t$ normalizes $X$ and $|X| = |C_X(t)|^2$.

**Proof.** (1) By Lemma 3.1, we have that $\langle s \rangle \in SyZ(C_0(V))$ and $C_0(V) \cap C_0(V) = \langle s \rangle \times V$. Thus $C_0(V)$ has a normal 3-complement by Burnside's transfer theorem. Since $O_3(C_0(V))$ is normalized by $S$, (1) holds.

(2) Set $N = N_0(X)$ and $N = N/X$. $C_N(s) = \bar{C}_N(s)$ for any $s \in S^*$. Since $C_N(s)$ has a normal 3-complement, if follows from the maximality of $X$ that $\bar{C}_N(s)$ is a 3-group, proving (2).

(3) This is well known (See [6], Theorem 5.3.16).

(4) Since $t$ normalizes $O_2(C_0(s))$ for any $s \in S^*$, we have that $t \in N_0(X)$. Since $t$ inverts $S$, it follows from (3) that $|X| = |C_X(t)|^2$. The lemma is proved.

**Lemma 3.3.** Let $X \in H^*(S)$. Assume that $X$ is abelian. Then one of the following holds:

(1) $X$ is a four-group;
(2) $X$ is strongly closed in a Sylow 2-subgroup of $G$;
(3) $X \cong Z_2^4$ and $N_0(X)/X \cong A_4$.

**Proof.** Set $N = N_0(X)$ and $\bar{N} = N/X$. Let $T$ be a Sylow 2-subgroup of $N$. We may assume that $|X| > 4$. By Lemma 2.1(1), Lemma 3.2(2) and Fletcher's theorem, we have that $\bar{N}$ is 3-closed or isomorphic to $A_4$, $PGL(2, 9)$ or $M_{10}$. By Lemma 2.2(3), for any involution $t$ of $T - X$, $m(X) = 2m(C_X(t)) \geq 4$. Thus if $m(X) \geq 6$ or $m(T/X) = 1$, then $m(X) > m(T/X) + m(C_X(t))$. By Lemma 2.2, (2) holds. Assume $|X| = 16$. Then $\bar{N} \cong A_4$ by Lemma 2.1(2), and so (3) holds. The lemma is proved.

**Lemma 3.4.** Let $X \in H^*(S)$. Assume that $X$ is not abelian. Then there is a subgroup $Y$ of $X$ of order 16 such that $C_0(Y) = Y$ and $N_0(Y)/Y \cong A_4$.

**Proof.** Let $u$ be an element of $N_0(S)$ of order 4, $t = u^2$, and $\langle s_i \rangle$, $1 \leq i \leq 4$, be the four subgroups of $S$ of order 3. Set $V_i = O_2(C_0(s_i))$ for each $i$. Then $t \in N_0(V_i) - C_0(V_i)$ for each $i$. We may assume that $V_i = V_2$ and $V_i'' = V_4$. Since $X' \neq 1$, $|X| = 2^4$ or $2^8$. We may assume that $V_2 \leq X' \setminus Z(X)$.

We shall first assume that $X$ is of order $2^4$. Then $X' = V_2$. By Lemma 3.2, $N = N(V_2) = XS\langle t \rangle$. Since $V_2'' = V_4 \neq V_2$ and $u$ normalizes $[V_i, V_j]$, we see
Odd characterization

that \([V_1, V_2]\neq V_3\), and so \(X\neq V_1 V_2 V_3\). Thus \(V_4 \leq X\). Set \(Y=V_3 V_4\), \(L=N_G(Y)\) and \(\bar{L}=L/Y\). Clearly \(\langle X, S, u \rangle \leq L\) and \(N_G(S)=\bar{S}\langle u \rangle\). Since \(u\) does not normalize \(X\), we have that \(O_3(\bar{L})=O_3(L)=1\). Let \(\bar{L}_0\) be a minimal normal subgroup of \(\bar{L}\). Then \(\bar{L}_0\) is simple and satisfies the assumption of our theorem. Since \(C_G(Y)=Y\), we have that \(\bar{L} = \bar{L}_0 \cong A_7\), as required.

Next assume that \(X\) is of order \(2^6\). Since \(X=\langle O_3(C_G(s)) \mid s \in S^*\rangle\), we have that \(N_G(S) \leq N_G(X)\), and so \(X' \cap Z(X) = V_3 V_4\). Since \(X\) is not abelian, we have that \(X' = Z(X) = \Phi(X) = V_1 V_4\) and \(N_G(S) = \langle S \rangle\). There are \(v_1 \in V_1\) and \(v_2 \in V_2\) such that \([v_1, v_2] = 1\). Acting \(S\) on the relation, we see that any element of \(V_1 \cup V_2\) commute with no element of \(V_3 \cup V_4\). Thus involutions of \(X\) are contained in \(X' V_1 \cup X' V_2\). Since \(|C_{X/X}(t)| = |C_X(t)| = 4\), any involution of \(T = T-X\) is conjugate to \(t\) in \(T\). Since \(|C_X(t)| = 16\), elementary abelian subgroup of \(T\) of order \(2^6\) are only \(V_1 X'\) and \(V_2 X'\). In particular, \(X\) is characteristic in \(T\), and so \(T\) is a Sylow 2-subgroup of \(G\). We shall show that if two elements of \(X\) are conjugate in \(G\), then they are conjugate in \(N_G(X)\). Assume \(a, b \in X\), \(b = a^t\), \(g \in G\). We will show that \(a \sim b\) in \(N(X)\). We may assume that \(a, b \in V_1 X'\) and \(C_T(a) = \Phi(C_T(b))\). Since \((V_1 X')^g = V_1 X'\) or \(V_2 X'\), we have that \(g \in N_G(V_1 X')\). Set \(L = N_G(V_1 X')\). Then \(N_L(S) = \langle S \rangle\). By Smith-Taylor’s theorem, \(L\) is 3-solvable. Thus \(L \leq N_G(X)\), and so \(g \in N(X)\). We proved that if \(a \sim b\) in \(G\) for involutions \(a, b \in X\), then \(a \sim b\) in \(N_G(X)\). In particular, if \(x\) is an involution of \(X - X'\), then \(|C_G(x)| = 2^6\) and \(m(C_T(x)) = 6\). By Harada’s transfer theorem ([8], Lemma 16), \(t\) is conjugate to an element of \(X\). Take an element \(g\) of \(G\) such that \(t^g = x \in X\) and \(C_T(t) = T\). Let \(v_i\), \(1 \leq i \leq 4\), be involutions of \(V_i \cap C_G(t)\). Then \(C = C_T(t) = \langle u, v_1, v_2, v_3 \rangle\). Since \(v_i u = v_i\), we have that \([v_1, v_2] = v_3 v_4\). Thus \(|C| = 2^6\) and \(m(C) = 4\). Hence \(x \in X'\). In particular, \(t \neq v_i\). Since \(v_i \in X - X'\) and \(t \sim v_i\), \(t \sim (tv_i)^g = xv_i x^g \in X - X'\). This is a contradiction. The lemma is proved.

We can now establish our theorem. Let \(X \in H^*_s(S)\). If \(X\) is of order 4, then since \(X = \text{Syl}_2(C_G(X))\) by Lemma 3.2(1), we have that a Sylow 2-subgroup of \(G\) is dihedral or semi-dihedral. Such simple groups are known by [1] and [7]. From this, we can easily show that \(G \cong A_5\), \(L_3(7)\) or \(U_4(5)\). By Goldschmidt theorem [5], there is no strongly closed abelian subgroup in a Sylow 2-subgroup of \(G\). Thus if \(|X| > 4\), then there is an elementary abelian subgroup \(Y\) such that \(A \in \text{Syl}_2(C_G(A))\) and \(N_G(A) / A \cong A_4\) or \(A_7\) by Lemma 3.3 and 3.4 and so Harada’s theorem [9] implies that \(G \cong M_{22}\). Adding the \(\mathcal{CO}_{\theta}\) -cases, \(G\) is isomorphic to one of the groups in the conclusion of our theorem.

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References


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