Homomorphism of the Lie algebras of vector fields

By Kazuo MASUDA

(Received June 14, 1975)
(Revised Oct. 29, 1975)

Introduction.

The Lie algebra Λ(M) formed by vector fields on a smooth manifold M gives an important example of infinite dimensional Lie algebra and has a geometric significance for the manifold theory. A basic theorem states that the Lie algebra structure of Λ(M) completely determines the underlying smooth structure of M. Namely, for two smooth manifolds M and N and for any Lie algebra isomorphism φ of Λ(M) onto Λ(N), we can find a diffeomorphism Φ of M onto N satisfying \( \phi = \Phi^* \) ([2], [5]). Our investigation starts from the observation of this theorem. In this paper we shall consider not only isomorphisms but also homomorphisms of Λ(M) into Λ(N) and study the relation between M and N.

There is a non-trivial homomorphism of Λ(M) into Λ(N) when N has some bundle structure over a product manifold of copies of M. We shall prove that if N is compact and there is a non-trivial homomorphism, then N is necessarily related to M in such a manner; hence, in particular, we have \( \dim M \leq \dim N \), and M is also compact. We deduce these results from the fact that any homomorphism is, in a sense, of local character and can be expressed by the use of the jets of vector fields.

We shall describe an outline of this paper. In §1 we shall determine the local form of a homomorphism. Let φ be a homomorphism of Λ(M) into Λ(N). For a generic point \( q \) of N we can find a finite number of points \( p_1, \ldots, p_t \) of M and charts \( \{ U_v; (x_v) = (x_{v,1}, \ldots, x_{v,n}) \} \) near \( p_v \) and \( \{ U; (x, y) \} \) near \( q \) with

\[
(x_{v,1}, \ldots, x_{v,t}, y) = (x_1, \ldots, x_{v,1}, y_1, \ldots, y_{d-n}),
\]

which satisfy the following property:

For any \( X \in \Lambda(M) \) with \( X = \sum f_i(x_v) \partial_{x_v} \) on each \( U_v \) we have

\[
\phi(X)(x, y) = \sum_{v=1}^t \sum_{i=1}^n \left( f_i(x_v) \partial_{x_v} + \sum_{\alpha < \beta} \frac{D^n}{\alpha!} f_i(x_v) Y_{\beta}^\alpha(y) \right)
\]

on \( U \) for some integer \( h \) and vector fields \( Y_{\beta}^\alpha(y) = \sum_{j} Y_{\beta j}^\alpha(y) \partial_{x_j} \) where \( \partial_{x_j} \) de-
notes the vector field $\frac{\partial}{\partial x^i}$.

If $N$ is compact and $\varphi$ is non-trivial, then $N$ has a rather restricted structure subject to $M$. The relation between $M$ and $N$ will be clarified in § 2 as follows: For any positive integer $l$, let $M_l$ be a smooth manifold formed by all the sets of distinct $l$ points of $M$ and put $N_0 = \{ q \in N \mid \varphi(X) \text{ vanishes at } q \}$ for any $X \in \mathcal{A}(M)$. Then $N$ is a finite disjoint union of $N_0$ and some topological fibre bundles $N_l$ over $M_l$. The bundle $N_l$ is closely related to the jet bundle of the tangent bundle of $M^l = M \times \cdots \times M$. Actually, we can construct many examples of homomorphisms which yield such situations. However we have no example such that $N_0 \neq \emptyset$ or $N_l$ is not a smooth bundle. Since the behaviour of $\varphi(X)$ near $q$ depends only on the behaviour of $X$ near $p_1, \cdots, p_l$, we can consider the germ of $\varphi$ at $(q; p_1, \cdots, p_l)$. We say $\varphi$ is transitive at $q$ if the image $\varphi(\mathcal{A}(M))$ is transitive at $q$. In § 3 we shall show that the classification of the transitive germs can be reduced to that of certain subalgebras of $\bigoplus g(n, h)$ where $g(n, h)$ is the finite dimensional Lie algebra formed by the $h$-jets of vector fields on $R^n$ vanishing at $0$. In § 4 we shall prove that any homomorphism of $\mathcal{A}(M)$ into $\mathcal{A}(N)$ is necessarily continuous in the $C^\infty$-topology. As a consequence, when $N$ is compact, it follows from [4; Theorem 1.3.2] that $\varphi$ induces a local homomorphism between the diffeomorphism groups of $M$ and $N$. This establishes an analogy to the corresponding theorem known for finite dimensional Lie algebras and Lie groups.

Some of our results were announced in [3].

§ 1. Local normal form of a homomorphism.

For any smooth manifold $M$, we denote by $\mathcal{A}(M)$ the Lie algebra formed by all the smooth vector fields on $M$ under the usual bracket operation. Let $\varphi: \mathcal{A}(M) \to \mathcal{A}(N)$ be a Lie algebra homomorphism. In this section we shall give an explicit expression of $\varphi$ in terms of local coordinate systems on $M$ and $N$. For this purpose, we first establish the following theorem concerning a characterization of the subalgebra of $\mathcal{A}(M)$ with finite codimension, essentially due to I. Amemiya [1]. We consider $\mathcal{A}(M)$ as a $C^\infty(M)$-module under the usual multiplication. For any point $p$ of $M$, we put $\mathcal{A}_p = \{ f \in C^\infty(M) \mid f(p) = 0 \}$.

**Theorem.** Let $\mathcal{B}$ be a proper subalgebra of $\mathcal{A}(M)$ with codim $\mathcal{B} = d < \infty$. Then we can find a finite number of points $p_1, \cdots, p_l$ of $M$ such that the relation

$$\bigoplus_{p=1}^l \mathcal{A}_{p_i}(M) \supset \mathcal{B} \supset \bigoplus_{p=1}^{l+1} \mathcal{A}_{p_i+1}(M)$$

holds for $h = 2((d-nl)^2 + d-nl) + 1$ where $n = \dim M$. Moreover we have $l \leq d/n$.

In order to prove this theorem, we need two lemmas. For any open set
$U$ of $M$, we put $\mathcal{A}_U = \{X \in \mathcal{A}(M) \mid \text{supp } X \subseteq U\}$.

**Lemma 1.** Let $\mathcal{B}$ be as in Theorem 1. Suppose that there are $Z \in \mathcal{A} = \mathcal{A}(M)$ and $g \in C^\infty(M)$ such that $Z(g) \equiv 1$ on $U$. Then we can find a non-trivial polynomial $P$ with $\deg P \leq 2(d^2 + d)$ such that $P(g) \mathcal{A}_U \subseteq \mathcal{B}$.

**Proof.** Put $\mathcal{B}' = \{X \in \mathcal{B} \mid [X, Y] \subseteq \mathcal{B} \text{ for every } Y \in \mathcal{A}\}$. For any $X \in \mathcal{B}$, $\text{ad } X : Y \mapsto [X, Y]$ induces a linear transformation $T_X : \mathcal{A} / \mathcal{B} \to \mathcal{A} / \mathcal{B}$. Since $\mathcal{B}'$ is the kernel of the map $X \mapsto T_X$ of $\mathcal{B}$ into the space of endomorphisms of $\mathcal{A} / \mathcal{B}$, we have $\text{codim } \mathcal{B}' \leq d^2 + d$. Let $\mathcal{P}$ be the space of all polynomials and put $\mathcal{P}' = \{P \in \mathcal{P} \mid gP(g)Z, P(g)Z \subseteq \mathcal{B}'\}$. Then we have $\text{codim } \mathcal{P}' \leq 2(d^2 + d)$ since $\mathcal{P}'$ is the kernel of the map $\mathcal{P} \to \mathcal{A} / \mathcal{B} \oplus \mathcal{B}'$ induced by the map $P \mapsto gP(g)Z \oplus P(g)Z$. Hence we can find a non-trivial polynomial $P \in \mathcal{P}'$ with $\deg P \leq 2(d^2 + d)$. For any $X \in \mathcal{A}_U$ we have

$$\mathcal{B} \ni [P(g)Z, gX] = g[P(g)Z, X] + P(g)Z(g)X,$$

and hence

$$\mathcal{B} \ni [P(g)Z, gX] = g[P(g)Z, X] - X(g)P(g)Z.$$

Substituting $X(g)Z \in \mathcal{A}_U$ for $X$ in (*), we obtain $\mathcal{B} \ni 2X(g)P(g)Z$, which combined with (*), gives $\mathcal{B} \ni P(g)X$. This completes the proof.

**Lemma 2.** Let $U_\nu \ (\nu = 1, 2, \cdots)$ be open sets of $M$ such that $\bigcup U_\nu$'s are disjoint and locally finite and let $(x_\nu) = (x_\nu^1, \cdots, x_\nu^l)$ be a coordinate system on $U_\nu$. Then there are a finite number of integers $\nu_1, \cdots, \nu_l \ (l \leq 2(d^2 + d))$ such that $\mathcal{B} \ni \mathcal{A}_{U_\nu}$ for $U' = \bigcup_{\nu = 1} U_\nu - \bigcup_{i=1} U_{\nu_i}$.

**Proof.** Hereafter we shall denote by $\partial_\nu$ the vector field $\partial / \partial x_\nu^i$. Choose $Z \in \mathcal{A}(M)$ and $g \in C^\infty(M)$ such that $Z = \partial_\nu$ and $g = x_\nu^i + \text{constant}$ on every $U_\nu$. We may assume that $\nu < g < \nu + 1$ on $U_\nu$. Since $Z(g) = 1$ on $U = \bigcup U_\nu$, by Lemma 1 we have $P(g) \mathcal{A}_U \subseteq \mathcal{B}$ for some polynomial $P$ with $\deg P \leq 2(d^2 + d)$. We can take integers $\nu_1, \cdots, \nu_l$ for which we have $P(g) \not\equiv 0$ on $U' = U - \bigcup_{i=1} U_{\nu_i}$. Then for any $Y \in \mathcal{A}_U$, there is $X \in \mathcal{A}_U \subset \mathcal{A}_U$ such that $Y = P(g)X$ and hence $Y \in \mathcal{B}$, which completes the proof.

**Proof of Theorem 1.** We say a point $p$ of $M$ is singular if for any neighborhood $U$ of $p$ we have $\mathcal{B} \ni \mathcal{A}_U$. Then by Lemma 2 the number of singular points is at most $2(d^2 + d)$. Let $\{p_1, \cdots, p_i\}$ be the set of singular points. We show that $\mathcal{B} \ni \bigcup_{i=1} \mathcal{A}_{p_i}^m \mathcal{A}_U$ where $m = 2(d^2 + d)$ and $U$ is some neighborhood of the set $\{p_1, \cdots, p_i\}$. For each $\nu$ let $(x_\nu)$ be a coordinate system on some neighborhood $U_\nu$ of $p_\nu$ with $p_\nu = (0)$. Choose $Z \in \mathcal{A}(M)$ and $g \in C^\infty(M)$ such that $Z = \partial_\nu$ and $g = x_\nu^i$ on each $U_\nu$. Then by Lemma 1 there is a polynomial $P(t) = t^p(1 + at + \cdots) \ (p \leq m)$ for which we have $P(g) \mathcal{A}_U \subseteq \mathcal{B}$ for $U' = \bigcup U_\nu$. 


Lie algebras of vector fields

Since $1 + a_i + \cdots \neq 0$ on some neighborhood $U'$ of $\{p_1, \cdots\}$, we have $g^p A_{U'} \subset B$ and hence $g^m A_{U'} \subset B$. For any $g \in C^\infty(M)$ which is a homogeneous polynomial of $(x_i^2)$ of degree 1 on each $U_i$, we have the same relation $g^m A_{U'} \subset B$ for some $V$. Since any homogeneous polynomial of degree $m$ is a linear combination of some $m'$th powers of homogeneous polynomials of degree 1, we have $g^{m'} A_{U'} \subset B$ for some $U$ as desired. Next, we prove that $B \supset \cap M_{p \in V} A(M)$. For any $p \in M - \{p_1, \cdots\}$, by definition, we have $B \supset A_{U_p}$ for some neighborhood $U_p$ of $p$. Then $(U) \cup (U_p)_p$ covers $M$. According to the dimension theory, $M$ admits a finite open covering $\{U, U_1, \cdots, U_{n+1}\}$ such that for each $i \leq n+1$, $U_i = \bigcup U_{ij}$ where $U_{ij}, U_{i2}, \cdots$ satisfy the conditions in Lemma 2 and each $U_{ij}$ is contained in some $U_p$. By Lemma 2 there are a finite number of integers $j_1, j_2, \cdots$ such that $B \supset A_{U_{ij}}$ for $U_i = U_i - \bigcup U_{ij}$. Since $B \supset A_{U_{ij}}$, using the partition of unity subordinate to the finite covering $\{U, U_1, U_{11}, \cdots, U_{ij}, \cdots\}$ of $M$, we have $B \supset A(M)$ as desired. Next, we show that $\cap M_{p \in V} A(M) \supset B$. Assume the contrary. Then there is $Z \in B$ which does not vanish at some $p$. We can take a coordinate system $(x^1, \cdots, x^n)$ on some neighborhood $U$ of $p$ such that $Z = x^1$ on $U$. Choose $g \in C^\infty(M)$ satisfying $g = x^1$ on $U$. Then by Lemma 1 we have $B \supset P(g) A_U$ for some polynomial $P$. For any $Y \in A_U$, we have $B \supset P(g) Y Z = P'(g) Z(g) Y + P(g) [Y, Z]$ and hence $B \supset -P'(g) Z(g) Y = -P'(g) Y$, which implies $B \supset A_U$. Applying the same argument successively, we have $B \supset A_U$, which contradicts the fact that $p$ is singular. Therefore we have $\cap M_{p \in V} A(M) \supset B$. Since codim $\cap M_{p \in V} A(M)$ is $n$, we have $l \leq d/n$. We must show $B \supset A_{U_p}$, which completes the proof of Lemma 1. Then the argument similar to the proof of Lemma 1 shows that there is a polynomial $P$ with $P(0) = 0$ and deg $P \leq 2(e^2 + e) + 1 = h$ where $e = d - n \cdot l = \text{codim } B$ in $\cap M_{p \in V} A(M)$ such that we have $P(g) \cap M_{p \in V} A_U \subset B$ for $U = \bigcup U_p$. Then the same argument as above shows that $B \supset \cap M_{p \in V} A(M)$, which completes the proof of Theorem 1.

Now, let $\varphi: A(M) \to A(N)$ be a non-trivial Lie algebra homomorphism. Throughout this paper we assume that $M$ and $N$ are connected and have no boundary and dim $M = n$ and dim $N = d$ are positive. Put $A_q(N) = M_q A(N)$ and $N^+ = \{q \in N \mid \varphi^{-1} A_q(N) \neq A(M)\}$. Then $q \in N$ belongs to $N^+$ if and only if there is $X \in A(M)$ such that $\varphi(X)(q)$, the value of $\varphi(X)$ at $q$, $\neq 0$. Hence $N^+$ is non-empty open subset of $N$. For any $q \in N^+$ we have codim $\varphi^{-1} A_q(N) \leq \text{codim } A_q(N) = d < \infty$, hence by Theorem 1, there are points $p_1, \cdots, p_l$ of $M$ such that

$$\bigcap_{v=1}^l M_{p_v} A(M) \supset \varphi^{-1} A_q(N) \supset \bigcap_{v=1}^l M_{p_v}^{h+1} A(M)$$

holds for $h = 2((d - n l)^2 + d - n l) + 1$. 

Lemma 3. For any $X \in \mathcal{A}(M)$, $\varphi(X)(q)$ is determined by the $h$-jets of $X$ at $p_1, \ldots, p_l$.

Proof. For each $v$ let $(x_v) = (x_{v1}, \ldots, x_{vn})$ be a coordinate system on some neighborhood $U_v$ of $p_v$ and $(a_v)$ the coordinates of $p_v$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, choose $Y_v \in \mathcal{A}(M)$ such that $Y_v^a = x_v^a \partial x_v^a$ on some neighborhood of $p_v$ and $\text{supp } Y_v^a \subset U_v$. We assume that $U_v$'s are disjoint. If $X = \sum f(x) \partial x_v^a$ on each $U_v$, we have

$$X = \sum_{v, \alpha} \frac{D^\alpha}{\alpha!} f(x) \sum_{\beta \in \mathbb{N}^n} \binom{\alpha}{\beta} (-a_v)^{\alpha - \beta} Y_v^\beta \in \cap_{v \neq v'} \mathcal{A}(M) \subset \varphi^{-1} \mathcal{A}(N).$$

Here we denote by $D^\alpha$ the differential operator $\partial^{\alpha_1}(\partial x^n)^{\alpha_n}$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Therefore we have

$$\varphi(X)(q) = \sum_{v, \alpha} \frac{D^\alpha}{\alpha!} f(x) \sum_{\beta \in \mathbb{N}^n} \binom{\alpha}{\beta} (-a_v)^{\alpha - \beta} \varphi(Y_v^\beta)(q),$$

which completes the proof.

Note that the set $\{p_1, \ldots, p_l\}$ is uniquely determined by (1). The set $\{p_1, \ldots, p_l\}$ is denoted by $\varphi(q)$ for $q \in N^*$. Let $k \leq d/n$ be the maximal number of $p_v$'s when $q$ ranges over $N^*$, and for each $l < k$, let $N_l$ be the set of points $q'$ of $N^*$ such that the number of the corresponding points is $l$.

Example 1. Let $\varphi : \mathcal{A}(R^n) \rightarrow \mathcal{A}(R^n)$ be a homomorphism given by

$$\varphi(f(x)\partial x) = f(x)\partial x + f(y)\partial y + (f'(x) + f'(y))\alpha(z)\partial z,$$

where $\alpha(z)$ is a smooth function. For a point $q = (a, b, c) \in R^3$ we have

$$\varphi^{-1} \mathcal{A}_{a,b,c}(R^3) = \{f(x)\partial x \mid f(a) = f(b) = (f'(a) + f'(b))\alpha(c) = 0\}$$

and hence

$$\mathcal{M}_a \cap \mathcal{M}_b \mathcal{A}(R^3) \supset \varphi^{-1} \mathcal{A}_{a,b,c}(R^3) \supset \mathcal{M}_a \cap \mathcal{M}_b \mathcal{A}(R^1).$$

Therefore we obtain

$$\varphi(a, b, c) = \{a, b\} \text{ when } a \neq b, = \{a\} \text{ when } a = b,$$

$$N_1 = \{(x, x, z) \in R^3\} \text{ and } N_2 = \{(x, y, z) \in R^3 \mid x \neq y\}.$$

Now we shall study the set $\varphi(q)$.

Lemma 4. For each $l \leq k$, $N_k \cup N_{k-1} \cup \cdots \cup N_1$ is an open subset of $N^*$. Let $q$ be a point of $N_l$ with $\varphi(q) = \{p_l, \ldots, p_l\}$ and $U_v$ a neighborhood of $p_v$ for each $v$ such that $U_v \cap U_{v'} = \emptyset$ for $v \neq v'$. Then there are a neighborhood $U$ of $q$ and a continuous map $\tilde{\varphi} : U \cap N_l \rightarrow \times U_1 \times \cdots \times U_l \subset M^1 \times \cdots \times M$ such that for any $q' \in U \cap N_l$, $\tilde{\varphi}(q') = (p_l, \ldots, p_l)$ implies $\varphi(q') = \{p_l, \ldots, p_l\}$.

Proof. Fix $l \leq k$ and $q \in N_l$. It suffices to show that there is a neighbor-
hood \( U \) of \( q \) such that for any \( q' \in U \) there are points \( p'_\nu \) of \( U'_\nu \) for all \( \nu \) satisfying \( \phi(q') \supset \{ p'_0, \ldots, p'_\nu \} \). Assume the contrary. Then there is a sequence of points \( \{ q_i \} \) converging to \( q \) such that for some fixed \( \nu' \), \( \phi(q_i) \cap U_{\nu'} = \{ p_i, \ldots, p_{i_m} \} \cap U_{\nu'} = 0 \) for all \( i \) \((0 \leq m \leq k) \). Choose \( X \in \mathcal{A}(M) \) satisfying \( \text{supp} \ X \subset U_{\nu'} \) and \( X(p_{\nu'}) = 0 \). Then by the definition of \( \phi \) we have

\[
X \in \bigcap_{\nu = 1}^m \mathcal{A}_{p_{\nu'}} \mathcal{A}(M) \subset \varphi^{-1} \mathcal{A}_{q_i}(N) \quad \text{and} \quad X \in \bigcap_{\nu = 1}^l \mathcal{A}_{p_{\nu}} \mathcal{A}(M) \supset \varphi^{-1} \mathcal{A}_{q}(N) .
\]

Therefore we obtain \( \phi(X)(q_i) = 0 \) and \( \phi(X)(q) \neq 0 \), which is a contradiction. This completes the proof.

Now let \( q \) be a point of \( \text{Int} \ N_i \), the topological interior of \( N_i \) in \( N \). Then Lemma 3 implies the next

**Lemma 5.** Let \( U \) and \( U_{\nu} \) be the neighborhoods given in Lemma 4. Then for any \( X \in \mathcal{A}(M) \), \( \phi(X) \upharpoonright U \), the restriction of \( \phi(X) \) to \( U \), depends only on \( X \mid U_{\nu} \).

By this lemma we may restrict our consideration to \( U \) and \( \cup U_{\nu} \). In the following arguments we replace these neighborhoods by smaller ones if necessary. Choose a coordinate system \((x_1) = (x_1', \ldots, x_1') \) on \( U_{\nu} \) for each \( \nu \). Since \( \phi(\partial_{x_1}) \)'s are linearly independent and \( [\phi(\partial_{x_1}), \phi(\partial_{x_2})] = 0 \), we can choose a coordinate system \((x_1, y) = (x_1', \ldots, x_1', y_1, \ldots, y_d) \) on \( U \) such that \( \phi(\partial_{x_1}) = \partial_{x_1} \) for all \( i \) and \( \nu \).

**Lemma 6.** Let \( \tilde{\phi}(x_\nu, y) \) be the \((i, \nu)\)-component of \( \tilde{\phi}(x_\nu, y) \in U_1 \times \cdots \times U_l \) with respect to the above coordinate systems. Then \( \tilde{\phi}(x_\nu, y) = x_i + c_i(y) \) for some continuous function \( c_i \). Moreover \( \tilde{\phi} \) is a smooth submersion on some open dense subset of \( U \).

**Proof.** For any point \((a_*, b) \in U \), we have

\[
X = (x_1' - \tilde{\phi}_i(a_*, b)) \partial_{x_1} \in \varphi^{-1} \mathcal{A}_{(a_*, b)}(N)
\]

and hence

\[
0 = \phi(X)(a_*, b) = \sum_{s = 1}^{h+1} \left( \frac{h+1}{s} \right) (-\tilde{\phi}_i(a_*, b))^s (x_1') \partial_{x_1}(a_*, b) .
\]

We put \((x_1, \ldots, x_d) = (x_\nu, y)\) and \( Y = \sum Y^s \partial_{x_\nu} \) for \( Y \in \mathcal{A}(U) \). Put \( \tilde{\phi}_i(x_\nu, y) = x_i + c_i(y) \). Then \( c_i(a_*, b) \) satisfies the equations

\[
F_i(c_i, a_*, b) = \sum \left( \frac{h+1}{s} \right) (-a_i - c_i)^{h+1-s} \phi((x_1') \partial_{x_1})^s (a_*, b) = 0, \quad q = 1, \ldots, d .
\]

For any \( j \) and \( \mu \) we have

\[
\frac{\partial}{\partial a_i} \phi((x_1') \partial_{x_1})^s (a_*, b) = [\phi(\partial_{x_1}), \phi((x_1') \partial_{x_1})]^s (a_*, b)
\]

\[
= \delta s \phi((x_1') \partial_{x_1})^s (a_*, b)
\]
where \( \delta = 1 \) when \( j = i \) and \( \mu = \nu \) and \( \delta = 0 \) otherwise. Using this equation we have easily \( \frac{\partial}{\partial a_\mu} F^a_k(c'_i, a_*, b) = 0 \), and hence \( F^a_k \) is independent of \( (a_*) \). Since by Lemma 4 \( c'_i(a_*, b) \) is continuous and \( F^a_k \) is a polynomial with respect to \( c'_i \), it follows that \( c'_i \) is independent of \( (a_*) \) as desired. Now we prove the second part of the lemma. Let \( U_0 \) be a non-empty open subset of \( U \). Since \( (x_i' - \bar{\phi}'_i)(a_*, b)^{k+l} = \varphi^{-1}(A, a_*, b)(N) \) and \( \delta_j^a = \varphi^{-1}(A, a_*, b)(N) \) for any point \( (a_*, b) \) of \( U_0 \), we can find an integer \( m < h \) and a point \( (d_*, e) \) of \( U_0 \) such that \( (x_i' - \bar{\phi}'_i)(a_*, b)^{m+1} \varphi^{-1}(A, a_*, b)(N) \) for any point \( (a_*, b) \) of \( U_0 \) and \( (x_i' - \bar{\phi}'_i)(d_*, e)^{m+1} \varphi^{-1}(A, (d_*, e))(N) \). Then \( c'_i \) satisfies the equations \( F^a_k(c'_i, a_*, b) = 0 \) and we have

\[
\frac{\partial F^a_k}{\partial c'_i}(c'_i(d_*, e), d_*, e) = -(m+1)\varphi((x_i' - \bar{\phi}'_i)(d_*, e)^m \varphi^{-1}(A, a_*, b)(N))
\]

for some \( q \). Therefore by the inverse function theorem, \( c'_i \) is smooth on some neighborhood of \( (d_*, e) \). The same arguments for other \( (i, \nu) \)'s complete the proof.

Now we can prove the main theorem of this section. For each \( 1 \leq k \), put \( N^*_k = \{ q \in \text{Int } N_1 \mid \bar{\phi} \text{ is smooth near } q \} \).

**Theorem 2.** \( \bigcup_{i=1}^k N^*_i \) is dense in \( N^+ \). Let \( q \) be a point of \( N^*_i \) with \( \varphi(q) = \{ p_1, \ldots, p_l \} \) and \( (x_\nu) = (x_i, \ldots, x_\nu) \) a coordinate system on some neighborhood \( U_\nu \) of \( p_\nu \) for each \( \nu \). Then there is a coordinate system \( (x_\nu, y) = (x_i, \ldots, x_\nu, y) = (x_i, \ldots, x_\nu, y^1, \ldots, y^{d-n-1}) \) on some neighborhood \( U \) of \( q \) satisfying the following properties:

i) \( \bar{\phi}(U) \subset U_1 \times \cdots \times U_\nu, \bar{\phi}(x_\nu, y) = (x_\nu) = (x_i, \ldots, x_\nu) \) and \( \varphi(\partial_{x_i}) = \partial_{x_i} \).

ii) For any \( X \in \mathcal{A}(M) \) with \( X U_\nu = \sum \partial f_i(x_\nu) \partial_{x_i} \) we have

\[
\varphi(X) U = \sum_{\nu=1}^l \sum_{i=1}^n f_i(x_\nu) \partial_{x_i} + \sum_{0 \leq |\alpha| < h} \sum_{\nu=1}^l \alpha^q f_i(x_\nu) Y_\nu(y).
\]

Here \( h = 2(d-nl)^2 + d - nl \) and \( Y_\nu \)'s are fixed vector fields such that

\[
[\varphi_\nu, \partial_{x_i}] = 0 \quad \text{for } \nu \neq \mu \quad \text{and} \quad [\varphi_\nu, \varphi_\mu] = \beta^i j Y_{\mu+1}^{\nu+1} - \alpha_y Y_{\mu+1}^{\nu+1}.
\]

In the right hand side of the second equation we put \( Y_{\nu} = 0 \) if \( |\gamma| > h \). We use \( i \) instead of the multi-index \( \alpha \) such that \( \alpha = \delta_y \) (Kronecker's \( \delta \)).

**Remark 1.** Note that the vector fields \( x_i \partial_{x_i} \)'s satisfy the same relation as (4). It is easy to show that for all \( Y_\nu \)'s satisfying the relation (4), the map \( \varphi : \mathcal{A}(U) \to \mathcal{A}(U) \) given by (3) is a homomorphism.

**Proof of Theorem 2.** It follows easily from Lemma 4 and Lemma 6 that
Let \( q \) be as in the theorem. Then we can choose a coordinate system \((x_*, y)\) satisfying i). For any point \((x_*, y)\) of \( U\), since \( \phi(x_*, y) = (x_*) \) we have, by (2),

\[
\phi(\sum_{i=1}^n f_i^*(x_*) \partial_{x_i}) = \sum_{i=1}^n \sum_{\nu=1}^d D^\nu f_i^*(x_*) Z^\nu_i(x_*, y),
\]

where \( Z \)'s are suitable smooth vector fields. Since \( \phi(\partial_{x_i}) = \partial_{x_i} \), we have \( Z^\nu_i = \partial_{x_i} \). We investigate \( Z \)'s for \( \alpha > 0 \). First, applying \( \phi \) to the equation

\[
[\partial_{x_i}, \sum_{i=1}^n f_i^*(x_*) \partial_{x_i}] = \sum_{i=1}^n D^i f^*(x_*) \partial_{x_i},
\]

we have \([\partial_{x_i}, Z^\nu_i] = 0\) so that \( Z^\nu_i(x_*, y) = Z^\nu_i(y) \). Putting \( Z^\nu_i(y) = \sum_{\mu=1}^d Z^\nu_i(y) \partial_{x_\mu} + \sum_p Z^\nu_p(y) \partial_{y_p} \), we show that \( Z^\nu_i \equiv 0 \). Assume the contrary. Then there is some \( Z^\nu_i \) such that \( Z^\nu_i(y) \neq 0 \) on some non-empty open set \( U' \subset U \). Applying \( \phi \) to the equation

\[
[\sum_{i=1}^n f_i^*(x_*) \partial_{x_i}, \sum_{i=1}^n g_i^*(x_*) \partial_{x_i}] = \sum_{i=1}^n D^i f^*(x_*) \partial_{x_i} - \sum_{i=1}^n \sum_{\nu=1}^d Z^\nu_i(x_*) D^i g^*(x_*) \partial_{x_i},
\]

we have

\[(*) \quad [\sum_{i=1}^n (f_i^* \partial_{x_i} + \sum_{0 < \alpha \leq \ell} \partial_{x_\alpha} D^\alpha f_i^* \sum_{\mu=1}^d Z^\mu_j \partial_{x_\mu} + \sum_p Z^p \partial_{y_p}), \sum_{i=1}^n (g_i^* \partial_{x_i} + \sum_{0 < \alpha \leq \ell} \partial_{x_\alpha} D^\alpha g_i^* Z^\nu_i)]
\]

\[
= \sum_{i=1}^n (f_i^* D^i g^* \partial_{x_i} + \sum_{0 < \alpha \leq \ell} \sum_{\tau = 0}^\alpha (\alpha) D^\tau f_i^* D^{\alpha-\tau} g_i^* Z^\nu_i) + \sum_{0 < \alpha \leq \ell} \sum_{\tau = 0}^\alpha (\alpha) D^\tau g_i^* D^{\alpha-\tau} f_i^* Z^\nu_i.
\]

We claim that \( Z^\nu_i \equiv 0 \) on \( U' \) for all \( \alpha \) and \( i \). Suppose it is not true. Then there is some \( Z^\nu_i \) such that \( Z^\nu_i \equiv 0 \) on \( U' \) if \( |\alpha| \geq |\tilde{\alpha}| \), or if \( |\alpha| = |\tilde{\alpha}| \) and \( \alpha_f > \tilde{\alpha_f} \). Comparing the coefficients of \( D^\alpha f_i^* D^\beta g_i^* \partial_{x_i} \) in both sides of (*) we have \( Z^\nu_i \equiv 0 \) on \( U' \), which is a contradiction. Therefore we obtain \( Z^\nu_i \equiv 0 \) on \( U' \) for all \( \alpha \) and \( i \). Next, comparing the coefficients of \( D^\alpha f_i^* D^\beta g_i^* \partial_{x_i} \) in both sides of (*) we have \( Z^\nu_i \equiv 0 \) on \( U' \), which is a contradiction. Thus we have proved that \( Z^\nu_i \equiv 0 \) and hence that \( Z^\nu_i(y) = \sum_p Z^p \partial_{y_p} \). Putting \( Z^\nu_i(y) = \frac{1}{\alpha!} Y_i^\nu(y) \), it is easy to show that \( \phi \) is a homomorphism if and only if (4) holds. This completes the proof of Theorem 2.

**Example 2.** Let \( G(n, h) \) be a Lie group consisting of all \( h \)-jets at 0 of diffeomorphisms of \( R^n \) fixing the origine 0 and \( g(n, h) \) its Lie algebra. We
assume that the diffeomorphisms act on $\mathbb{R}^n$ from the right, i.e., $(gh)(p) = h(g(p))$ for any diffeomorphisms $g, h$ and any point $p$ of $\mathbb{R}^n$. Then we can take \( \{x^i\partial_{x^i} | 0 < |\alpha| \leq h, i \leq n\} \) as a basis of $g(n, h)$ with the usual bracket operation and the exponential mapping $g(n, h) \rightarrow G(n, h)$ is given by $\exp tX = \text{the } h\text{-jet of } \exp tX \text{ at } 0$ where $\exp tX$ is the 1-parameter group of local transformations generated by $X = \sum X^i x^\alpha \partial_{x^i}$. Let $J^{h-1}TM^I$ be the $(h-1)$-jet bundle of the tangent bundle $TM^I$. It is a $G$-bundle where $G = \bigoplus G(n, h)$. Let $P$ be its associated principal $G$-bundle. For a right $G$-manifold $F$, put $N = F \times_G P$. Then $\text{Diff} (M)$, the group of all diffeomorphisms of $M$, acts on $N$ naturally, namely there is a homomorphism $\Phi : \text{Diff} (M) \rightarrow \text{Diff} (N)$, hence we get a homomorphism $\varphi = \Phi_\ast : \mathcal{A}(M) \rightarrow \mathcal{A}(N)$. $\varphi$ is given as follows. For $X \in \mathcal{A}(M)$ let $\exp tX$ be the 1-parameter group of transformations generated by $X$ (we assume that the manifolds are compact). Then $\varphi(X) = \Phi_\ast(X)$ is the infinitesimal transformation of $\Phi(\exp tX)$. Put $Y^\alpha = \rho_\ast(x^\alpha \partial_{x^i}) \in \mathcal{A}(F)$ where $\rho : G \rightarrow \text{Diff} (F)$ is a homomorphism induced by the action of $G$ on $F$ and $\rho_\ast : \bigoplus G(n, h) \rightarrow \mathcal{A}(F)$ is a homomorphism induced by $\rho$. Let $U_1 \times \cdots \times U_i \times F = U \subset N$ be a local trivial structure of $N$. Then it is easy to show that $\varphi$ is given by the formula (3) in Theorem 2. Hence the map $\tilde{\varphi}$ is the projection map $N \rightarrow M'$ in this case.

Remark 2. Let $V_1 \times \cdots \times V_i \times F = V \subset N$ be another local trivial structure of $N$ and let $g$ be a diffeomorphism of $W = (U_1 \cap V_1) \times \cdots \times (U_i \cap V_i) \times F$ induced by the transition function of $N$, and let $\varphi_U$ and $\varphi_V$ be homomorphisms of $\mathcal{A}(M)$ into $\mathcal{A}(U_1 \times \cdots \times U_i \times F)$ and $\mathcal{A}(V_1 \times \cdots \times V_i \times F)$ respectively given by the formula (3). Then we have $g_\ast(\varphi_U(W)) = \varphi_V(X) | W$ for all $X \in \mathcal{A}(M)$. We shall use this fact in the proof of Theorem 3 and Theorem 3'.

In Example 2, the map $\tilde{\varphi}$ can be defined globally $N \rightarrow M'$, but this is not true in general as shown in the next example.

Example 3. Let $\sigma$ be a free involution of a manifold $F$ and $\tau$ a free involution of $M \times M \times F$ given by $\tau(x, y, z) = (y, x, \sigma(z))$. Let $\varphi : \mathcal{A}(M) \rightarrow \mathcal{A}(M \times M \times F)$ be a homomorphism given by

$$\varphi(\Sigma f^i(x) \partial_{x^i})(x, y, z) = \Sigma f^i(x) \partial_{x^i} + \Sigma f^i(y) \partial_{y^i}.$$ 

Since $\tau \varphi(X) = \varphi(X)$, $\varphi$ induces a homomorphism $\mathcal{A}(M) \rightarrow \mathcal{A}(M \times M \times F/\tau)$. Clearly in this case the map $\tilde{\varphi} : M \times M \times F/\tau \rightarrow M \times M$ does not exist globally.

§ 2. Bundle structure of $N_1$.

In this section we shall show that $N_1$ is a (topological) fibre bundle with the projection map $\phi$ and study its bundle structure. It will be seen that $N_1$ is closely related to $N = F \times_G P$ in Example 2 (cf. Theorem 3). Now, put $M(l)$
Lie algebras of vector fields 515

Since the symmetric group \( S_l \) acts freely on \( M(l) \), we obtain a smooth manifold \( M_l = M(l) / S_l \) which consists of all the sets of distinct \( l \) points of \( M \). Put \( M\{k\} = \bigcup_{i=1}^{k} M_i \) and give it the quotient topology induced by the natural map \( M^k \rightarrow M\{k\} \).

**Proposition 1.**

i) Let \( X \) be a vector field on \( M \) with compact support and \( q \) a point of \( N \). Suppose that at \( q \) \( \text{Exp} \, t\varphi(X) \) (cf. Example 2) is defined for \( 0 \leq t \leq 1 \). Then for any \( Y \in \mathcal{A}(M) \) we have \( \varphi((\text{Exp} \, X)_*Y) = (\text{Exp} \, \varphi(X))_*\varphi(Y) \) at the point \( \text{Exp} \, \varphi(X)q \). Moreover if \( q \in N^+ \) we have \( \text{Exp} \, \varphi(X)q \in N^+ \) and \( \phi(\text{Exp} \, \varphi(X)q) = \text{Exp} \, X\phi(q) \).

ii) \( \phi \) is a continuous map of \( N_i \) into \( M_i \). If \( M \) is not compact then \( N = N \) and \( \phi \) is a continuous map of \( N \) into \( M\{k\} \).

**Remark 3.** When \( \dim M = \dim N \), the part ii) remains true even if \( M \) is compact. We do not know whether this fact holds in general.

**Proof of Proposition 1.** First we prove i) under the following additional assumption:

\( (*) \quad q \in N_k^+ \) and for each \( \nu \), \( X(p_\nu) \neq 0 \) or \( X \equiv 0 \) on some neighborhood of \( p_\nu \).

Here \( \phi(q) = \{ p_1, \ldots, p_k \} \).

Choose \( U_v, (x_v), U \) and \( (x*, y) \) as in Theorem 2. We may assume that \( X|U_v = \partial_{x_v} \) for \( \nu \leq s \) and \( X|U_v \equiv 0 \) for \( \nu > s \) for some \( s \) and hence that \( \varphi(X)|U = \partial_{x_1} + \cdots + \partial_{x_s} \). For brevity we put \( p_\nu = \text{Exp} \, tXp_\nu \) and \( q_\nu = \text{Exp} \, \varphi(X)q \). We can extend the coordinate system \( (x_*, y) \) to some open set \( U' \) containing all the points \( q_t (0 \leq t \leq 1) \) so that \( \text{Exp} \, \varphi(X)(x_*, y) = (x_1^t + t, x_1^t, \ldots, x_s^t + t, \ldots, x_k^t, y) \). Similarly we can extend \( (x_\nu) (1 \leq \nu \leq s) \) to some open set \( U'_\nu \). Then we have \( \varphi(X)|U' = \partial_{x_1} + \cdots + \partial_{x_s} \) and \( X|U'_\nu = \partial_{x_\nu} \). Note that \( U'_s \) are not necessarily disjoint. For each \( t (0 \leq t \leq 1) \), consider the following statement:

\[ C_t : \quad q_t \in N_k^+, \quad \varphi(q_t) = \text{Exp} \, tX\phi(q) = \{ p_{1t}, \ldots, p_{kt} \} \quad \text{and the coordinate systems} \quad (x_\nu) \quad \text{and} \quad (x_*, y) \quad \text{satisfy i) of Theorem 2 on some neighborhoods of} \quad p_{1t} \quad \text{and} \quad q_t \quad \text{respectively.} \]

Since the set \( \{ t \mid C_t \text{ is true} \} \) is open and contains a sufficiently small \( t \), to prove \( C_t \) it suffices to show that if \( C_t \) is true for \( t < s \) then \( C_s \) is true. Take \( Y_t \in \mathcal{A}(M) (1 \leq \nu \leq k) \) such that \( Y_t = \partial_{x_\nu} \) on some neighborhood of \( p_{1t} \) and \( \text{supp} \, Y_t \supseteq p_{1t} \) for \( \mu \neq \nu \). Then \( \varphi(Y_t) = \partial_{x_\nu} \) on some neighborhood \( V \) of the set \( \{ q_t \mid s - \epsilon < t < s \} \) for some \( \epsilon > 0 \) and hence \( \varphi(Y_t)(q_t) \neq 0 \), which implies that \( q_t \in N^+ \). Further we have \( \varphi(q_t) \supseteq p_{1t} \). Really if \( \varphi(q_t) \not\supseteq p_{1t} \) we can choose \( Y_t \) so that \( \text{supp} \, Y_t \cap \varphi(q_t) = \emptyset \) and hence that, in view of Lemma 3, \( \varphi(Y_t)(q_t) = 0 \), which is a contradiction. Since \( p_{1t} \)'s are distinct and \( k \) is, by definition, the maximal number of \( p_\nu \)'s, it follows that \( \varphi(q_t) = \{ p_{1t}, \ldots, p_{kt} \} \) and hence \( q_t \in N_k^+ \). By Lemma 5 we may restrict our consideration to some neighborhoods of \( p_{1t} \) and \( q_t \). Since \( L_{\varphi(q_t)(x_\nu)}(\partial_{x_\nu}) \)
\([\varphi(X), \varphi(\partial_x)] = \varphi(X, \partial_x) = 0\) (we denote by \(L_{\varphi(X)}\) the Lie derivative with respect to \(\varphi(X)\)) and \(\varphi(\partial_x) = \partial_x\) on \(V\), we obtain \(\varphi(\partial_x) = \partial_x\). By Lemma 6 we have \(\varphi(x, y) = x + c'(y)\) for some \(c'\), and since \(\varphi(x, y) = x + c'\) on \(V\), it follows that \(\varphi(x, y) = x + c'\). Therefore we have \(q_i \in N_k\) and the coordinate systems \((x_i)\) and \((x_i, y)\) satisfy i) of Theorem 2, which completes the proof of \(C_\varphi\). It remains to prove that \(\varphi((\exp X)_* Y) = (\exp \varphi(X))_* \varphi(Y)\) at \(q_i\), but this is clear since \(\exp X\) and \(\exp \varphi(X)\) are parallel translations and \(\varphi\) is given by (3) in Theorem 2.

Now we prove i) in general case. If \(q \in \tilde{N}_k\) we can, by Theorem 2, choose a sequence \(\{q_i\}\) in \(\tilde{N}_k\) converging to \(q\) such that each \(q_i\) satisfies the assumption in i) and the assumption (*). Then we have \(\varphi((\exp X)_* Y) = (\exp \varphi(X))_* \varphi(Y)\) at \(q_i\) and hence at \(q_i\). It follows that \(\varphi^{-1}(A_k(N)) = (\exp X)_*(\varphi^{-1}A_k(N))\). Therefore if \(q \in N^+\) then we have \(q_1 \in N^+\) and \(\varphi(q) = \exp X \varphi(q)\). Note that we have \(q_1 \in \tilde{N}_k\) for \(0 \leq t \leq 1\) since \(q_t \in N_k\) converges to \(q_1\) and hence that if \(q \in N^+\) then \(q_1 \in N^+\). Applying the above argument to the manifold \(N^+\), we can prove i) for \(q \in \tilde{N}_k\) and similarly for \(q \in \tilde{N}_k\) converging to some point \(q^*\) of \(N\), we can prove i) for \(q \in \tilde{N}_k\) and similarly for \(q \in \tilde{N}_k\) converging to a point \(q^*\) of \(N\). Therefore \(L(K)\) is relatively compact in \(M\). Assume the contrary. Then there is a sequence \(\{q_i\}\) in \(K \cap N^+\) converging to some point \(q^*\) of \(K\) such that the set \(\{p_i\}\) is discrete where \(\varphi(q_i) = \{p_i, p_{i+1}, \ldots\}\). We may assume that \(p_i \neq p_j\) for all \(i\) and \(j < i\). By Lemma 3 we can choose \(Y \in A(M)\) such that \(|\varphi(Y)(q)| \geq i\), which is a contradiction. Here \(|\cdot|\) denotes the norm of the vector with respect to some metric on \(M\). Therefore \(L(K)\) is relatively compact. Next, let \(\{q_i\} \subset N^+\) be a sequence converging to a point \(q\) of \(N\) and \(K\) a compact neighborhood of \(q\). We show \(q \in N^+\). Assume the contrary. Put \(\varphi(q_i) = \{p_{i1}, p_{i2}, \ldots\}\). Since \(L = L(K)\) is relatively compact, we may assume that the sequence \(\{p_{i1}\}\) converges to some point \(p_i\) of \(L\). Since \(M\) is not compact, we can choose \(Y \in A(M)\) such that \(\text{supp } Y\) is compact and \(\exp Y(U) \cap L = \emptyset\) for some neighborhood \(U\) of \(p_i\). Since by assumption \(q \in N^+\), we have \(\varphi(Y)(q) = 0\). So we may assume that at \(q_i\) \(\exp tY\varphi(Y)\) is defined for \(0 \leq t \leq 1\) and \(\exp Y\varphi(Y)q_i \in K\) for all \(i\). Then by i) we have \(\varphi(\exp \varphi(Y)q_i) = \exp Y\varphi(Y)q_i = \exp Yp_{i1} = \ldots\). By the definition of \(L\) we have \(\exp Yp_{i1} \in L\), which contradicts the fact that \(\exp Y(U) \cap L = \emptyset\). Thus we have \(q \in N^+\). This implies that \(N^+\) is closed. Since \(N^+\) is open and \(N\) is connected, we have \(N^+ = N\). Next, we show that \(\varphi(q_i) = \{p_{i1}, \ldots\} \rightarrow \varphi(q) = \{p_1, \ldots, p_m\} \subset M(k)\). Assume the contrary. Then we have two cases:

1) There is a subsequence \(\{q_i\}^*\) of \(\{q_i\}\) such that the sequence \(\{p_{i1}\}\) converges to a point \(p\) with \(p \neq p_\nu\) for \(\nu = 1, 2, \ldots, m\).
2) There is a neighborhood $U$ of $p_1$ such that $U \ni p_{i\nu}$ for all $i$ and $\nu$.

In case 1, we can choose $Y \in \mathcal{A}(M)$ such that $\text{supp } Y$ is compact and does not contain $p_\nu$ for $\nu = 1, \cdots, m$ and that $\text{Exp } Y(U) \cap L = \emptyset$ for some neighborhood $U$ of $p$. Then by Lemma 3 we have $\varphi(Y)(q) = 0$, which yields a contradiction by the same argument as above. In case 2, we can choose $Y \in \mathcal{A}(M)$ such that $\text{supp } Y \subset U$ and $Y(p_1) \neq 0$. Then we have $\varphi(Y)(q_i) = 0$ for all $i$ and $\varphi(Y)(q) \neq 0$, which is a contradiction. Therefore we have that $\psi(q_i) \to \psi(q)$ in $M\{k\}$, which completes the proof of ii).

**COROLLARY 1.** Suppose that $N$ is compact. Then $M$ is also compact and $\varphi$ is injective. Moreover each non-empty $N_1$ is a (topological) fibre bundle over $M_1$ with the projection map $\psi$.

**PROOF.** First we show that $M$ is compact. Assume the contrary. Then by Proposition 1 $\varphi$ is continuous. Let $q$ be a point of $N_1 (\neq \emptyset)$ with $\varphi(q) = \{p_1, \cdots, p_k\}$. For any point $\{p'_1, \cdots, p'_l\}$ of $M_1$, there are $X_i's \in \mathcal{A}(M)$ such that $\text{supp } X_i's$ are compact and $\text{Exp } X_i(p_i, \cdots, p_k) \to \{p'_1, \cdots, p'_l\}$ in $M(k)$ as $i \to \infty$ (recall that $M$ is connected). Since $\varphi(N)$ is compact and $\varphi(\text{Exp } \varphi(X_i)q) = \text{Exp } X_i\varphi(q) \supseteq \varphi(N)$, we have $\{p'_1, \cdots, p'_l\} \subset \varphi(N)$ and hence $\varphi(N) = M\{k\}$. Since $\varphi(N)$ is compact, so is $M$, which is a contradiction. Therefore $M$ is compact.

Next, let $q$ be a point of $N_1 (\neq \emptyset)$ with $\varphi(q) = \{p_1, \cdots, p_l\}$. For any point $\{p'_1, \cdots, p'_l\}$ of $M_1$, there is $X \in \mathcal{A}(M)$ such that $\varphi(\text{Exp } \varphi(X)q) = \text{Exp } \varphi(q) = \{p_1, \cdots, p_l\}$. Thus $\psi : N_1 \to M_1$ is surjective. In particular $\psi : N_1 (\neq \emptyset) \to M_1$ is surjective. The injectivity of $\varphi$ follows easily from this fact and the definition of $\psi$. It is clear that $\text{Exp } \varphi(X)$ gives a homeomorphism of $\psi^{-1}\{p_1, \cdots, p_l\}$ onto $\psi^{-1}\{p'_1, \cdots, p'_l\}$.

Now we give the local trivial structure of $N_1$. Let $U_\nu$ be a neighborhood of $p_\nu$ and $(x_\nu)$ a coordinate system on some neighborhood of $U_\nu (\nu \leq \ell)$. We assume that $U_\nu's$ are disjoint and diffeomorphic to the unit disk $\{x \in \mathbb{R}^n \mid |x|^2 \leq 1\}$ by these coordinate systems. Choose $X_\nu \in \mathcal{A}(M)$ with $X_\nu|U_\mu = \partial x_\mu$ and $X_\nu|U_\mu = 0$ for $\mu \neq \nu$. Let $(a_\nu)$ be the coordinates of $p_\nu$ and put $F_\nu = \psi^{-1}\{p_1, \cdots, p_l\}$. Then the local trivial structure of $N_1$ is given by the map $\chi_\nu : U_1 \times \cdots \times U_l \times F_\nu \to \psi^{-1}(U_1 \times \cdots \times U_l) \subset N_1$ defined by $\chi_\nu(x_1) \times \cdots \times (x_l) \times y = \text{Exp } \varphi(\sum (x_i - a_i)X_i)y$.

Here we consider $U = U_1 \times \cdots \times U_l$ as a subset of $M_l$. This completes the proof of Corollary 1.

To express the transition functions of the bundle $N_1$, we need some definitions. We assume that $M$ is oriented and $\text{dim } M = n \geq 3$. Let $\mathcal{U} = \{U\}$ be an open covering of $M$ such that each intersection of finite $U's$ is a disk and $(x_U) = (x_V)$ a coordinate system on $U$. Let $G^+(n, h)$ be the connected component of $G(n, h)$ (cf. Example 2) containing the identity element $1$ and $\widetilde{G}^+(n, h)$ its universal covering Lie group. Since $G^+(n, h)$ is homotopically equivalent to $\text{SO}(n)$, $\widetilde{G}^+(n, h)$ is a double covering of $G^+(n, h)$ and homotopically equivalent to $\text{Spin}(n)$. For any $U$ and $V$ with $U \cap V \neq \emptyset$, let $J_{UV} : U \cap V \to G^+(n, h)$ be a
map given by \( J_{UV}(x_U) = \text{the h-jet of the coordinate transformation } x_V(x_U) \) at \( (x_U) \) and let \( \tilde{J}_{UV} : U \cap V \to \tilde{G}^+(n, h) \) be one of its liftings. Note that \( J_{UV} \) is a transition function of \( J^{h-1}TM \). Since \( J_{UV}(x_U)J_{VW}(x_V(x_U)) = J_{UVW}(x_U) \) for any \( (x_U) \in U \cap V \cap W \), there is an element \( \varepsilon_{UVW} \in Z_2 \subseteq \tilde{G}^+(n, h) \) such that \( \tilde{J}_{UV}(x_U)\tilde{J}_{VW}(x_V(x_U)) = \varepsilon_{UVW}\tilde{J}_{UVW}(x_U) \) for any \( (x_U) \). Here \( Z_2 \) is the inverse image of 1 by the covering map \( \tilde{G}^+(n, h) \to G^+(n, h) \). Note that \( \{\varepsilon_{UVW}\} \) gives the second Whitney class of \( M \), \( w_2(M) \in H^2(M; Z) \), and hence that if \( M \) has a spin structure we can choose the liftings \( \tilde{J}_{UV} \) so that each \( \varepsilon_{UVW} = 1 \). Let \( U^i = \{U\} = \{U_1 \times \cdots \times U_n \mid U_i \in \mathcal{U}\} \) be an open covering of \( M \) and \( (x_U) = (x_{U_1}, \ldots, x_{U_m}) \) a coordinate system on \( U \). Finally let \( \tilde{J}_{UV} : U \cap V \to \tilde{G}^+(n, h) \) be a map given by \( \tilde{J}_{UV}(x_U) = \bigoplus \tilde{J}_{UVU}(x_{UV}) \) and put \( \varepsilon_{UVW} = \bigoplus \varepsilon_{UVUW} \in Z_2 \subseteq \tilde{G}^+(n, h) \). With these notations we have the following

**Theorem 3.** Assume that \( N \) is compact and \( M \) is oriented with \( \dim M = n \geq 3 \). i) Let \( \tilde{N}_i \) be the lifting of the bundle \( N_i \) to \( M(l) \) by the map \( M(l) \to M(l)/S_i = M_i \). Then there is a topological fibre bundle \( \tilde{N}_i \) over \( M^l \) with \( \tilde{N}_i | M(l) = \tilde{N}_i \).

ii) Put \( h = 2((d-nl)^2 + d - nl) + 1 \) where \( d = \dim N \). Then \( G = \tilde{G}^+(n, h) \) acts on the fibre \( F_i \) of the bundle \( \tilde{N}_i \) from the right and hence we have a homomorphism \( \rho : G \to \text{Homeo}(F_i) \). The transition functions of \( \tilde{N}_i \) are given by \( g_{UV}(x_U) = \rho(\tilde{J}_{UV}(x_U))h_{UV} \), where \( h_{UV} \)'s are elements of the centralizer of \( \rho(G) \) in \( \text{Homeo}(F_i) \) satisfying the relation \( h_{UV}h_{VW} = \rho(\varepsilon_{UVW})h_{VW} \).

**Remark 4.** It would seem that the fibre \( F_i \) is a smooth submanifold (with corner) of \( N_i \). If this is true, it is easily seen that \( N_i \) is a smooth fibre bundle and that \( \text{Homeo}(F_i) \) can be replaced by \( \text{Diff}(F_i) \). Further note that for any smooth right \( \tilde{G}^+(n, h) \)-manifold \( F_i \) and for all \( h_{UV} \)'s \( \in \text{Diff}(F_i) \) satisfying the conditions in Theorem 3 ii), \( \{\varepsilon_{UVW}\} \) gives a smooth fibre bundle \( \tilde{N}_i \) over \( M^l \).

We can construct a local homomorphism \( \Phi : \text{Diff}(M) \to \text{Diff}(\tilde{N}_i) \) by using the local trivial structure of \( \tilde{N}_i \) and hence get a homomorphism \( \varphi = \Phi_* : \mathcal{A}(M) \to \mathcal{A}(\tilde{N}_i) \). If \( \rho(\bigoplus Z_2) = \{1\} \subseteq \text{Diff}(F_i) \) (which means that \( \bigoplus G^+(n, h) \) acts on \( F_i \)), \( \Phi \) can be extended to a global homomorphism \( \text{Diff}(M) \to \text{Diff}(\tilde{N}_i) \). In this case, since \( \rho(\varepsilon_{UVW}) = 1 \), we may put each \( h_{UV} = 1 \). The homomorphism \( \varphi \) obtained in this way is exactly the same one given in Example 2.

Since \( \varphi(\mathcal{A}(M)) \) is a subalgebra of \( \mathcal{A}(N) \) and by Proposition 1

\[ \exp t \varphi(X) \varphi(\mathcal{A}(M)) \subseteq \varphi(\mathcal{A}(M)) \]

it follows that for any point \( q \) of \( N_i \) there is a
leaf \( L \subset N_i \) containing \( q \). Clearly the map \( X \to \varphi(X)|L \) gives a homomorphism \( \mathcal{A}(M) \to \mathcal{A}(L) \). For this homomorphism we have a more precise theorem, namely

**Theorem 3'.**

i) \( L \) is a smooth fibre bundle over \( M_i \). Let \( \tilde{L} \) be the lifting of \( L \) to \( M(l) \). Then there is a smooth fibre bundle \( \tilde{L} \) over \( M^l \) with \( \tilde{L}|M(l) = \tilde{L} \).

Moreover there are a covering space \( \hat{M}^l \) of \( M^l \) and a closed subgroup \( H \) of \( G = \bigoplus \tilde{G}^+(n, k) \) such that \( \tilde{L} \) is a fibre bundle over \( \hat{M}^l \) with connected fibre \( H \setminus G \) (homogeneous space).

ii) The transition functions of the bundle \( \tilde{L} \) over \( \hat{M}^l \) are given by \( g_{U V}(x_U) = R(\tilde{f}_{U V}(x_U))L(k_{U V}) \), where \( k_{U V}'s \) are elements of the group \( H \setminus N(H) \) (\( N(H) \) is the normalizer group of \( H \) in \( G \)) satisfying the relation \( k_{V W}k_{U V} = \tilde{e}_{U V}k_{W V} \) and \( R \) and \( L \) are actions on \( H \setminus G \) induced by the right and the left translations of \( G \) respectively. Here \( U, V \) and \( W \) are elements of the open covering of \( \hat{M}^l \) induced by \( U \).

**Proof of Theorem 3 and Theorem 3'.** We investigate the bundle \( \hat{N}_i \). Let \( (p_1, \ldots, p_l) \) be a point of \( U \cap M(l) \) where \( U = U_1 \times \cdots \times U_l \subset U' \), and let \( U'_i \)'s be disjoint neighborhoods of \( p_i \)'s respectively. Then the local trivial structure of \( \hat{N}_i \) \( U_1 \times \cdots \times U_l \) given in Corollary 1 gives a foliation of \( \dim M \) of \( \hat{N}_i \) \( U_1 \times \cdots \times U_l \) and this foliation depends only on the coordinate system \( (x_U) \). Therefore we have a foliation of \( \hat{N}_i \) \( U_1 \times \cdots \times U_l \cap M(l) \) and each leaf is a covering space of \( U_1 \times \cdots \times U_l \cap M(l) = U \cap M(l) \). Since \( U \cap M(l) \) is simply connected by the assumption that \( \dim M \geq 3 \), each leaf is homeomorphic to \( U \cap M(l) \) and hence we get a local trivial structure of \( \hat{N}_i \) \( U \cap M(l) \). We first prove Theorem 3'. Since the groups generated by \( \text{Exp } X \)'s and \( \text{Exp } \varphi(X) \)'s for \( X \in \mathcal{A}(M) \) act transitively on \( M_i \) and \( L \) respectively and \( \varphi(\text{Exp } X(q)) = \text{Exp } X \varphi(q) \), it follows that \( L \) is a bundle over \( M_i \) and that, in view of Lemma 6, \( \varphi \) is a smooth submersion of \( L \) onto \( M_i \). Therefore \( L = L_i \) and the expression (3) of \( \varphi \) in Theorem 2 holds good everywhere. Now we study the bundle \( \tilde{L} \). Let \( (p_1, \ldots, p_l) \) be a point of \( U \cap M(l) \), \((a_U)\) its coordinates and \( F_L \) the fibre over \( (p_1, \ldots, p_l) \). We give the local trivial structure of \( \tilde{L}|U \cap M(l) = (U \cap M(l)) \times F_L \) as above. Choose \( X_\mu^u \in \mathcal{A}(M) \) such that \( \text{supp } X_\mu^u \supset p_\mu \) for \( \mu \neq v \) and \( X_\mu^u = (x_\mu^u - a_\mu^u) \partial_{x_\mu^u} \) on some neighborhood of \( p_\mu \). Put \( Y_\alpha^v = \varphi(X_\mu^u)|F_L \). Then \( Y_\alpha^v \) is a vector field on \( F_L \) by ii) of Theorem 2. Moreover for \( \varphi \in \mathcal{A}(M) \) with \( X|U = \sum f_{V_\nu}(x_{V_\nu}) \partial_{x_{V_\nu}} \) we have

\[
(5) \quad \varphi(X) = \sum_{i=1}^l \sum_{l=1}^l \left( f_{V_i}(x_{V_i}) \partial_{x_{V_i}} + \sum_{a \in A} D_{a} f_{V_i}(x_{V_i}) Y_{\alpha}^v \right)
\]

on \( \tilde{L}|U \cap M(l) = (U \cap M(l)) \times F_L \). For another \( V \in U \) we get a similar expression of \( \varphi \) with \( Y_\alpha^v \) replaced by \( Y_\alpha^\nu \in \mathcal{A}(F_L) \) where \( F_L \) is a fibre over some point \( (p'_1, \ldots, p'_l) \) of \( V \cap M(l) \). Then we have

**Lemma 7.** There is a diffeomorphism \( g : F_L \to F_L^* \) such that \( g_\alpha Y_\alpha^v = Y_\alpha^\nu \) for all \( \alpha, \nu \) and \( i \).
PROOF. Let \((x_v)\) be a coordinate system on some simply connected neighborhood \(\tilde{U}_v\) of \(\{p_v, p'_v\}\) which is identical with \((x_{uv})\) and \((x_{vy})\) on some neighborhoods of \(p_v\) and \(p'_v\) respectively. Then we have a local trivial structure of \(L|U_1 \times \cdots \times U_l \times M(l)\) and the expression (5) of \(\varphi\). This trivial structure gives the desired diffeomorphism.

By this lemma for all \(V \subseteq U^l\) we have the local trivial structure of \(L|V \cap M(l) = (V \cap M(l)) \times F_L\) and the expression (5) with the same fibre \(F_L\) and the vector fields \(Y_v\)’s. Now we investigate the transition function \(g_{uv}(x_v)\) of \(L\). Since \(Y_v\)'s satisfy the relation (4) in Theorem 2, the map \(x_v^g \partial _{x_v} \rightarrow Y_v\) gives a homomorphism \(\hat{\oplus} G(n, h) \rightarrow \mathcal{A}(F_L)\). Since \(\text{Exp} \, t \varphi(X_v)\) are defined for all \(t \in \mathbb{R}\), it follows that \(\text{Exp} \, t Y_v\) are also defined for all \(t \in \mathbb{R}\) and hence that there is a homomorphism \(\rho : G = \hat{\oplus} G(n, h) \rightarrow \text{Diff} (F_L)\), namely, \(G\) acts on \(F_L\). Then \(\rho(J_{uv}(x_v))\) gives a diffeomorphism of \((U \cap V) \times F_L\). Let \(\varphi_v\) be a homomorphism \(\mathcal{A}(M) \rightarrow \mathcal{A}(U \times F_L)\) given by (5) and \(\varphi_v\) a similar one. Then we have \(\rho(J_{uv}(x_v)) \ast \varphi_v(X) = \varphi_v(X)\) on \((U \cap V) \times F_L\) by Remark 2. (Remark 2 remains valid with \(\hat{\oplus} G(n, h)\) replaced by \(\hat{\oplus} G^*(n, h)\).) Put \(h_{uv}(x_v) = \rho(J_{uv}(x_v))^{-1} g_{uv}(x_v)\). Then we have \(h_{uv}(x_v) \ast \varphi_v(X) = \varphi_v(X)\) on \((U \cap V) \times F_L\) for all \(X \in \mathcal{A}(M)\). It follows easily that \(h_{uv}(x_v)\) is independent of \(x_v\) and \((h_{uv}) \ast Y_v\) for all \(x_v \in U \cap V\). Hence \(\{g_{uv}\}\) gives a bundle \(\tilde{L}\) over \(M^l\) as desired. The last part of \(i)\) of Theorem 3 follows from the facts that \(\tilde{L}\) is connected and that the action of \(G\) is transitive. Note that for this bundle \(\tilde{L}\) over \(M^l\), the same fact as in Lemma 7 holds. The action of \(G\) on \(H \setminus G\) is induced by the right translation and the centralizer of \(\rho(G)\) in \(\text{Diff} (H \setminus G)\) is isomorphic to the group \(H \setminus \mathcal{N}(H)\) and its action on \(H \setminus G\) is induced by the left translation of \(G\). This completes the proof of Theorem 3'. Since the diffeomorphism \(\text{Exp} \, \varphi(X_v)\) of \(N\) gives a homeomorphism of the fibre \(F_i\) of the bundle \(N\), Theorem 3 follows from the above argument.

When \(\hat{\oplus} G^*(n, h)\) acts on \(F_i\) or \(M\) has a spin structure, the relation \(h_{uv} h_{vw} = \rho(\varepsilon_{uvw}) h_{uw}\) reduces to \(h_{uv} h_{vw} = h_{uw}\) and hence \(\{h_{uv}\}\) gives a locally constant bundle over \(M^l\). We give an example such that some \(\rho(\varepsilon_{uvw}) \neq 1\).

**Example 4.** Assume that there is an element \(v \in \text{Tor} H^k(M; \mathbb{Z})\) reduced to \(u_2(M) \neq 0 \in H^k(M; \mathbb{Z})\). For example, \((4k+1)\)-dim real projective space satisfies this condition. Let \(\rho_i : \hat{G}^*(n, 1) \rightarrow GL(N, C)\) be a complex representation such that \(\rho_i(-1) = -I_N = -\text{identity}\) and let \(\rho_2 : GL(N, C) \rightarrow \text{Diff} (S^{2N-1})\) be a homomorphism induced by the action of \(GL(N, C) \subset GL(2N, R)\) on the sphere \(S^{2N-1}\) considered as the real Stiefel manifold \(V_{2N, 1}\). Put \(\rho = \rho_2 \rho_1 : \hat{G}^*(n, 1) \rightarrow \text{Diff} (S^{2N-1})\). Then \(\rho(-1) \neq 1\). Since \(v \in \text{Tor} H^k(M; \mathbb{Z})\), there is a locally con-
constant complex line bundle whose first Chern class is v, where locally constant means that the transition functions \( k_{UV} \)'s are constant. The assumption assures that there are complex numbers \( h_{UV} \)'s such that \( h_{UV}^* = k_{UV} \) and \( h_{UV} h_{UV}^* = \varepsilon_{UV} h_{UV} \). Here we consider \( \varepsilon_{UV} \in \mathbb{Z}_2 = \{1, -1\} \) as a complex number. Put \( h_{UV} = \rho_3(h_{UV} I_n) \).

Then it commutes with all the elements of \( \rho(G^+(n, 1)) \) and we have \( h_{UV} h_{UV}^* = \rho(\varepsilon_{UV}) h_{UV} \) and some \( \rho(\varepsilon_{UV}) \neq 1 \) as desired.

In Examples 1~4 we have \( \tilde{N}_k = N_i \), but this is not true in general. The local trivial structure given in Corollary 1 gives a foliation of \( N_i \cap M(l)/S_1 \). In general the behaviour of each leaf near \( N_{i-1} \) is not simple. Really we have

**EXAMPLE 5.** Let \( \varphi : \tilde{A}(R^n) \to \tilde{A}(R^n \times R^n \times R^n) \) be a homomorphism given by

\[
\varphi(\sum f^{i}(x) \partial_{x^i})(x, y, z) = \sum f^{i}(x) \partial_{x^i} + \sum f^{i}(y) \partial_{y^i}.
\]

For this homomorphism, we have \( \varphi(x, y, z) = \{x, y\} \) and hence \( \tilde{N}_k = N_i \). The leaf of the foliation of \( N_i \) (given by the natural coordinate system \((x) \) of \( R^n = M \)) is given by \( z = \text{constant} \). We shall deform this homomorphism. First put \((X, Y, Z) = (x - y, y, z) \). Then we have \( N_i = \{(X, Y, Z) \in R^n \times R^n \times R^n | X \neq 0\} \) and

\[
\varphi(\sum f^{i}(x) \partial_{x^i})(X, Y, Z) = \sum (f^{i}(X + Y) - f^{i}(Y)) \partial_{x^i} + \sum f^{i}(Y) \partial_{y^i} = \sum \int_0^1 \partial_{x^i} f^{i}(tX + Y) dt X^i \partial_{x^i} + \sum f^{i}(Y) \partial_{y^i}.
\]

Let \((R, \theta) = (R, \theta^n, \ldots, \theta^n) \) be a polar coordinate system of \( R^n \) such that \( R^2 = |X|^2 \) and \( X^i = R S^i(\theta) \) for some \( S^i \). Then \( \partial_{x^i} = S^i(\theta) \partial_R + \sum A^m(\theta) \frac{1}{R} \partial_{y^m} \) for some \( A^m \). Next, let \((\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{(X, Y, Z)} = (x, y, z) \) be another coordinate system of \( N_i \). Then the leaf is given by \( \tilde{Z} = \alpha(\tilde{X}) \). Really we have \( \tilde{N}_i \) is diffeomorphic to \( \{(\tilde{X}, \tilde{Y}, \tilde{Z}) \in R^n \times R^n \times R^n | |\tilde{X}| = r > 2\} \) by the map \((R(r), \theta, \tilde{Y}, \tilde{Z}) \to (r, \theta, \tilde{Y}, \tilde{Z}) \). In this coordinate system \( (r, \theta, \tilde{Y}, \tilde{Z}) \) we have on \( N_i \)

\[
X^i \partial_{x^i} = R(r)/R'(r) S^i S^j \partial_{x^j} + \sum S^j A^m \partial_{y^m} + S^i S^j R(r) \sum \partial_{y^m} \alpha^k(R(r), Z(R(r), \tilde{Z})) \partial_{z^k},
\]

where \( Z(R, \tilde{Z}) \) denotes the inverse of \( \tilde{Z} = \alpha(R, Z) \). Now we assume that \( R(r)/R'(r) \) and \( R(r) \partial \alpha^k(R(r), Z(R(r), \tilde{Z})) \) can be extended to smooth functions \( g(r) \) and \( h^k(r, \tilde{Z}) \) respectively such that \( g(r) = r \) and \( h^k(r, \tilde{Z}) = 0 \) for \( r \leq 1 \). For example \( R(r) = \exp(-\exp(1/(r - 2))) \) and \( \alpha^k(R, Z) = \log R + Z^k \) satisfy these conditions. Put

\[
P^i = g(r) S^i S^j \partial_{x^j} + \sum S^j A^m \partial_{y^m} + S^i S^j \sum h^k(r, \tilde{Z}) \partial_{z^k}.
\]
Then it is a smooth vector field with respect to the coordinate systems \((\bar{X}, \bar{Z})\) and \((r, \theta, \bar{Z})\). Let \(\varphi_1: \mathcal{A}(\mathbb{R}^n) \to \mathcal{A}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m)\) be a map given by

\[
\varphi_1(\sum f^i(x)\partial_{ax_i})(\bar{X}, \bar{Y}, \bar{Z}) = \sum \int_0^1 \partial_j f^i(v)dt f^i + \sum f^i(\bar{Y})\partial_{aY_i}
\]

where \(v = (v^1, \ldots, v^n) = (tR(r)S(\theta) + \mathbf{p}, \ldots)\). Then \(\varphi_1|N_2\) is a homomorphism. If \(|X| = r \leq 2\), then \(R(r) = 0\) and hence we have

\[
\varphi_1(\sum f^i(x)\partial_{ax_i}) = \sum \partial_j f^i(\bar{Y}) P^i + \sum f^i(\bar{Y})\partial_{aY_i}.
\]

It is easy to show that \(P_i\)'s satisfy the relation (4) in Theorem 2. Therefore by Remark 1 \(\varphi_i\) is a homomorphism. For this homomorphism \(\varphi_1\), we have \(\varphi(\bar{X}, \bar{Y}, \bar{Z}) = (\bar{Y}, v)\) and hence \(N_2 = \{(\bar{X}, \bar{Y}, \bar{Z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \mid |\bar{X}| > 2\}, \ N_1 = \{(\bar{X}, \bar{Y}, \bar{Z}) \mid |\bar{X}| \leq 2\}, \ N_2 = \mathbb{R}^m \cup \mathbb{R}^m\) and \(F_1 = D^n \times \mathbb{R}^m\). If we take \(\alpha(R, Z) = \log R + Z^2\), then the leaf is given by \(\bar{Z}^2 = \alpha(R, Z_0) = \log R + Z_0^2\) for some constant vector \(Z_0\).

§ 3. Classification of transitive germs of homomorphisms.

In this section we shall consider the classification of germs of homomorphisms. Let \(\varphi: \mathcal{A}(M) \to \mathcal{A}(N)\) be a homomorphism and \(q\) a point of \(N\) with \(\varphi(q) = \{p_1, \ldots, p_l\}\). Then by (1) in § 1 we have

\[
\bigcap_{v=1}^l \mathcal{H}_v \mathcal{A}(M) / \bigcap_{v=1}^l \mathcal{H}_v^{p+1} \mathcal{A}(M) \supseteq \varphi^{-1} \mathcal{A}_q(N) / \bigcap_{v=1}^l \mathcal{H}_v^{p+1} \mathcal{A}(M).
\]

The left hand side of this formula is isomorphic to the algebra \(g = \bigoplus \mathcal{A}(n, h)\) and hence the right hand side, denoted by \(B_q\), is considered as a subalgebra of \(g\). However, since the above isomorphism depends on the coordinate systems, \(B_q\) is not well defined as a subalgebra of \(g\). We say subalgebras \(B\) and \(B'\) of \(g\) are equivalent if \(\operatorname{Ad}(g)B = B'\) for some \(g \in \bigoplus G(n, h)\) and denote by \(B(n, h, l)\) the set of the equivalence classes of subalgebras of \(g\). Then \(B_q\) gives an element of \(B(n, h, l)\), denoted by \(B_q\) also. Now we say \(\varphi\) is transitive at \(q\) if \(\{\varphi(X) \in T_q N \mid X \in \mathcal{A}(M)\} = T_q N\) where \(T_q N\) denotes the tangent space of \(N\) at \(q\). Then we have

**Lemma 8.** If \(\varphi\) is transitive at \(q\), then there is a neighborhood \(U\) of \(q\) such that \(B_q = B_{q'}\) for all \(q' \in U\).

**Proof.** By i) of Proposition 1 we have \(\varphi((\Exp X)_* Y) = (\Exp \varphi(X))_* \varphi(Y)\) at the point \(q = \Exp \varphi(X) q\) and hence \(\varphi^{-1} \mathcal{A}_q(N) = (\Exp X)_* \varphi^{-1} \mathcal{A}_q(N)\). Let \(g \in \bigoplus G(n, h)\) be the \(h\)-jet of \(\Exp X\) at \(\{p_1, \ldots, p_l\}\). Then we have \(\operatorname{Ad}(g) B_q = B_{q'}\). The assumption of the lemma implies that \(\{\Exp \varphi(X) q \mid X \in \mathcal{A}(M)\}\) covers some neighborhood of \(q\).
Let \( \varphi \) be transitive at \( q \). Then \( q \in \text{Int } N_i \) and hence by Lemma 5 there are neighborhoods \( U \) and \( U_v \) of \( q \) and \( p_v \) respectively such that \( \varphi(X) \mid U \) depends only on \( X \mid U_v \). Therefore we may consider the germ of \( \varphi \) at \( (q; p_1, \ldots, p_i) \). We say the germs \( \varphi \) and \( \varphi' \) at \( (q; p_1, \ldots, p_i) \) are equivalent if there are diffeomorphisms \( g: \cup V_v \rightarrow \cup V'_v \) and \( h: V \rightarrow V' \), where \( V_v, V'_v \), \( V \) and \( V' \) are some neighborhoods of \( p_v \) and \( q \) respectively, such that \( g(p_1, \ldots, p_i) = \{ p_1, \ldots, p_i \} \), \( h(q) = q \) and \( h_\ast(\varphi(X) \mid V) = \varphi' \circ (g_\ast(X \mid U_v)) \mid V' \) for any \( X \in \mathcal{G}(M) \). We do not require that \( \varphi \) and \( \varphi' \) are the restrictions of the global homomorphisms \( \mathcal{A}(M) \rightarrow \mathcal{A}(N) \). We denote by \( H_i(n, l, d) \) the set of equivalence classes of transitive germs at \( (q; p_1, \ldots, p_i) \) (recall that \( \dim M = n \) and \( \dim N = d \)) and by \( B(n, h, l, e) \) the set of equivalence classes of the subalgebras of \( \bigoplus a(n, h) \) of codim \( e \). Then we have

**Theorem 4.** The correspondence \( \varphi \rightarrow B_q \) gives a bijection \( H_i(n, l, d) \rightarrow B(n, h, l, e) \) where \( e = d - nl \) and \( h = 2(e^2 + e) + 1 \).

**Proof.** We first show that the map is injective. Since codim \( (\cap \mathcal{G} p_v \mathcal{A}(\cup U_v)) \) in \( \mathcal{A}(\cup U_v) \) is equal to \( nl \) and \( \varphi \) is transitive at \( q \), it follows that codim \( B_q = d - nl = e \). By Lemma 6 and i) of Proposition 1, \( \varphi \) is a smooth submersion on some neighborhood of \( q \) and hence by Theorem 2 we have the expression (3) of \( \varphi \). We use the same notations as in Theorem 2. Let \( (a_s, b) \) be the coordinates of \( q \) and put \( F = \{(x, y) \in U \mid (x^*) = (a^*) \} \). Since the correspondence \( x_s \partial_x Y_s \rightarrow Y_s \) gives a homomorphism \( f: g = \bigoplus a(n, h) \rightarrow \mathcal{A}(F), G = \bigoplus G(n, h) \) acts locally on \( F \) in the following sense. There are a neighborhood \( V \) of \( \{1\} \times F \) in \( G \times F \) and a map \( g: V \rightarrow F \) such that \( g(\exp X, q') = \exp f(X)q' \) for \( (\exp X, q') \in V \). Since \( \varphi \) is transitive, this action is transitive and hence \( F \) is locally diffeomorphic to the germ of the homogeneous space \( H \backslash G \), where \( H \) is a subgroup of \( G \) whose Lie algebra is \( \{ \sum a^i_x \partial_x Y_s \mid a^i_x Y_s(b) = 0 \} = B_q \). More precisely, there are an open set \( F' \) of \( F \) containing \( q \) and a neighborhood \( W \) of \( 1 \) in \( G \) such that \( F' \) is diffeomorphic to \( H \backslash W \), where \( H \) is a connected component of \( H \backslash W \) containing \( 1 \). The right translation of \( G \) induces a homomorphism \( \varphi_1: g \rightarrow \mathcal{A}(H \backslash W) \) and \( \varphi_1(x_s \partial_x Y_s) \) corresponds to \( Y_s \mid F' \) by the above diffeomorphism. Since \( \varphi \) is determined by \( Y_s \)'s, it is determined by \( H \) and hence by \( B_q \). Thus the map \( \varphi \rightarrow B_q \) is injective. On the other hand, for any \( B \in B(n, h, l, e) \) we can construct \( H_w \backslash W \) and get \( \varphi_1(x_s \partial_x Y_s) \in \mathcal{A}(H_w \backslash W) \) and hence a homomorphism \( \varphi: \mathcal{A}(\cup U_v) \rightarrow \mathcal{A}(\cup U_v \times (H_w \backslash W)) \) given by the formula (3). This completes the proof.

**Example 6.** For \( (n, l, d) = (1, 1, 2) \) we have \( H_1(1, 1, 2) = B(1, 5, 1, 1) = \{ B_1, B_2, B_3 \} \). The subalgebras \( B_i \subset a(1, 5) \) and the corresponding transitive homomorphisms \( \varphi: \mathcal{A}(R^1) \rightarrow \mathcal{A}(R^3) \) are given as follows.
$B_1 = \{ \sum_{j=1}^5 a_j x^j \partial_x | a_1 = 0 \}$, \quad $\varphi(f(x) \partial_x) = f(x) \partial_x + f'(x) \partial_y$,

$B_2 = \{ \sum_{j=1}^5 a_j x^j \partial_x | a_1 + a_5 = 0 \}$, \quad $\varphi(f(x) \partial_x) = f(x) \partial_x + \left( f'(x) + \frac{1}{2!} f''(x) e^y \right) \partial_y$,

$B_3 = \{ \sum_{j=1}^5 a_j x^j \partial_x | a_1 + a_3 = 0 \}$, \quad $\varphi(f(x) \partial_x) = f(x) \partial_x + \left( f'(x) + \frac{1}{3!} f'''(x) e^y \right) \partial_y$.

In general, the cardinality of $B(n, h, l, e)$ is not finite. For example, in case $n \geq 2$, $l=1$ and $d=2n$, for any $t \in \mathbb{R}$ put

$B_t = \{ \sum_{i=0}^n \sum_{\alpha \in \mathbb{A}} a_{\alpha} x^\alpha \partial_x^\alpha | \sum_i a_i + t \sum_{i \notin j} a_j = 0 \text{ for } k = 1, \ldots, n \} \subset \mathfrak{a}(n, h)$

where $h = 2(n^2 + n) + 1$. Then $B_t = B_s$ in $B(n, h, l, e)$ if and only if $t = s$. The corresponding transitive homomorphism $\varphi_t : \mathcal{A}(\mathbb{R}^n) \to \mathcal{A}(\mathbb{R}^{2n})$ is given by

$$\varphi_t(\sum_i f^i(x) \partial_x^i)(x, y) = \sum_i f^i(x) \partial_x^i + \sum_i D^i f^i(x) \partial_y^i + \sum_{i \notin j} D^i f^i(x) e^{y^j} \sum_{k} (t + j_k) \partial_y^k.$$ 


In [4] H. Omori proved that if $M$ and $N$ are compact and $\varphi : \mathcal{A}(M) \to \mathcal{A}(N)$ is a homomorphism which is continuous in the $C^\infty$-topology, then $\varphi$ induces a local homomorphism $\text{Diff}(M) \to \text{Diff}(N)$. We shall show that any homomorphism $\varphi$ is continuous without the assumption of compactness of $M$ and $N$. Lemma 3 implies the continuity of $\varphi$ in the weak topology. Theorem 2 does not imply the continuity of $\varphi$, because in general the local coordinate system $(x, y)$ on a neighborhood $U$ of $q$ does not fit with the given one $(u) = (u^p)$ on an open set $U_1$ of $N$, that is, $D^i_x u^p$, $D^i_y u^p$, $D^i_x y^j$ and $D^i_y y^j$ are not necessarily bounded when $q$ tends to a point of $(N - N_1) \cap U_1$. Here $D^i_x$ denotes the differential operator with respect to $x$. Recall that the $C^\infty$-topology of $\mathcal{A}(N)$ is given by the seminorms $| \cdot |_{u, p}$ defined as follows. Let $(u) = (u^p)$ be a coordinate system on a relatively compact open set $U$ of $N$ which can be extended to some neighborhood of $\overline{U}$. Then for $Y \in \mathcal{A}(N)$ with $Y = \sum g^p(u) \partial_u^p$ on $U$, we put

$$|Y|_{u, p} = \sup_{\alpha \geq \gamma, r, s \in U, p} |D^\alpha g^p(u)|.$$ 

First we assume that $M$ is compact. Then there is a finite open covering $\{ V_\mu \}$ of $M$ satisfying the following properties:

i) Each $V_\mu$ is diffeomorphic to the unit disk $\{ x \in \mathbb{R}^n | |x|^2 < 1 \}$ by the coordinate system $(x_\mu) = (x_\mu^1, \ldots, x_\mu^n)$ on some neighborhood of $\overline{V}_\mu$. 

K. MASUDA
ii) Any set \( \{p_1, \ldots, p_k\} \subset \mathbb{M} \) is contained in some \( V_\mu \), where \( k \) is the integer defined in § 1.

To prove the continuity of \( \varphi \) it suffices to show the next

**Lemma 9.** For any seminorm \( | \cdot |_{\nu, \rho} \), there is a constant \( C \) such that for any \( X \in \mathbb{A}(\mathbb{M}) \) we have

\[
|\varphi(X)|_{\nu, \rho} \leq C \sum_{\mu} |X|_{\nu, \rho} a + b,
\]

where \( a = \lfloor d/n \rfloor \) is the integer part of \( d/n \) and \( b = 2a((d-n)^2 + d-n+1) - 1 \).

**Proof.** Let \( \varphi(X)|U = \sum \varphi^p(X)(u)\partial_{u^p} \). Now we estimate \( D^\beta \varphi^p(X)(u) \) for \( |\beta| \leq r \). For any \( q \in U \cap N \), choose an open set \( V_\mu \) containing \( \varphi(q) = \{p_1, \ldots, p_l\} \). Applying Theorem 2 to \( U_\nu = V_\mu \) and \( (x_v) = (x_p) \) (\( v = 1, \ldots, l \)), we can get a coordinate system \( (x_\nu, y) \) on some neighborhood \( U_q \) of \( q \) such that \( \varphi(x_\nu, y) = (x_\nu) = (x_1, \ldots, x_l) \in V_\nu \times \cdots \times V_\nu = \mathbb{M}^l \) and that for any \( X \in \mathbb{A}(\mathbb{M}) \) with \( X = \sum f_i(x^i)\partial_{x_i}^p \) on \( V_\mu \) we have

\[
\varphi(X)(x_\nu, y) = \sum_{\nu} \left( f_i(x^i)\partial_x^\nu + \sum_{\theta \in \mathbb{A}(\mathbb{M})} \frac{D^\alpha}{\alpha!} f_i(x^i) Y_{\theta}(y) \right)
\]
on \( U_q \). It follows that

\[
D^\beta \varphi^p(X)(u) = \sum_{\nu} \sum_{|\beta| \leq r} D^\gamma f_i(x^i) Z_{\beta}(u)
\]
on \( U_q \), where \( Z \)'s are smooth functions on \( U_q \). To eliminate \( Z \)'s we need the following lemma which will be proved at the end of this section.

**Lemma 10.** Let \( \Phi : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)[Z^\nu] \) be a map given by

\[
\Phi(f(x)) = \sum_{\nu} \sum_{|h| \leq h} D^h f(x^i) Z_{\nu}(x^i).
\]

Then we have

\[
\Phi(f(x)) = f(x_i)\Phi(1) + \sum_{j=1}^{l(h+1)-1} \sum_{j_1 \cdots j_k = 1}^{m} \int_0^1 \cdots \int_0^1 \partial_{j_1} \cdots \partial_{j_k} f(x(k)) dt(k)
\]

\[
\sum_{m=0}^b (-1)^m \sum_{t \leq t_1 < \cdots < t_m \leq h} x_{t_1} f_{t_1} \cdots x_{t_m} f(x_{t_1} \cdots + x_{t_m}) f(x_{t_1} + \cdots + x_{t_m}) dt(k),
\]

where

\[
x(k) = (1-t_1)x_1 + (1-t_2)x_2 + \cdots + (1-t_k)x_k + (1-t_k)x_{k+1} + \cdots + (1-t_k)x_{k+1},
\]

\[
dt(k) = t_k^{l+1} \cdots t_k^{l+1} \cdots t_{k-1} dt_k \cdots dt_1.
\]

Put \( \Phi(f(x)) = \Phi(f(x))(x_p) = \sum_{\nu} \sum_{|\beta| \leq r} D^\nu f(x_p) Z_{\beta}(u) \). Then we have \( \Phi(f(x))(x_p)(u) = D^\nu \varphi^p(f(x_p)\partial_{x^p})(u) \). Here we consider \( f(x_p)\partial_{x^p} \) as a vector field on \( M \) by extending it suitably. For \( u \in U_q \), the right hand side of the above equation is independent of this extension. Applying Lemma 10 to \( \Phi \) and substituting...
We have
\[ D\dot{\varphi}^p(f(x_\mu)\partial_{x_\mu})(u) = f(x_1)D\dot{\varphi}^p(\partial_{x_1})(u) + \sum_{k=1}^{k(h+r+1)-1} \sum_{\mathcal{J}} \int_0^1 \cdots \int_0^1 \partial_{j_1} \cdots \partial_{j_k} f(x(k)) \, dt(k) \]
\[ + \sum_{m=0}^k (-1)^m \sum_{\mathcal{I}} x_{i_1} \cdots x_{i_m} D\dot{\varphi}^p(x_{j_{m+1}}^{+j_{m+1}} \cdots j_k \partial_{x_\mu})(u) \big|_{x_{\nu} = x_{\nu}}. \]

Note that \(|x_\nu| < 1\), that \(x(k)|_{x_{\nu} = x_{\nu}} \in V_\mu\) and that \(D\dot{\varphi}^p(x_{j_{m+1}}^{+j_{m+1}} \cdots j_k \partial_{x_\mu})(u)\) is smooth on \(\bar{U}\) and hence bounded on \(\bar{U}\). If we fix the extension of \(x_{j_{m+1}}^{+j_{m+1}} \cdots j_k \partial_{x_\mu}\), there is a constant \(C_{\mu,\nu}\) not depending on \(q \in U \cap N_+\) such that we have

\[ |D\dot{\varphi}^p(f(x_\mu)\partial_{x_\mu})(u)| \leq C_{\mu,\nu} |f(x_\mu)\partial_{x_\mu}||_{V_{\mu,\nu}} \]

for \(u \in U_q\), where \(t = l(h+r+1)-1\). Putting \(C = \sum \mu C_{\mu,\nu}\), we obtain

\[ |D\dot{\varphi}^p(X)(u)| \leq C \sum |X||_{V_{\mu,\nu}} \]

for \(u \in U \cap (\cup N_+^\nu)\). Since \(\cup N_+^\nu = N^+\) and \(\varphi(X) \equiv 0\) on \(N-N^+\), it follows that the above inequality holds for all \(u \in U\). By Theorem 1 we have \(t \leq a\nu + b\), which completes the proof of Lemma 9.

Next, we consider the case where \(M\) is not compact. Let \(q\) be a point of \(N = N^+\) with \(\varphi(q) = \{p_1, \ldots, p_t\}\). Then by ii) of Proposition 1 there are neighborhoods \(U\) and \(U_{\nu}\) of \(q\) and \(p_\nu\) respectively such that \(\varphi(q') = \{p'_1, \ldots, p'_m\} \subset \cup U_{\nu}\) for any \(q' \in U\). We may assume that there is an open set \(V\) which is diffeomorphic to the unit disk and contains \(\cup U_{\nu}\). By the similar argument as above, we can show that \(|\varphi(X)|_{V_{\nu,\nu}} \leq C |X||_{V_{\nu,\nu+b}}\) and hence \(\varphi\) is continuous.

Thus we have proved

**Theorem 5.** Any homomorphism \(\varphi: \mathcal{A}(M) \rightarrow \mathcal{A}(N)\) is continuous in the \(C^\infty\)-topology.

By Corollary 1 to Proposition 1 in § 2 and Theorem 1.3.2 in [4] we have

**Corollary.** If \(N\) is compact then \(\varphi\) induces a local homomorphism \(\text{Diff}(M) \rightarrow \text{Diff}(N)\).

**Proof of Lemma 10.** By the definition of \(\Phi\) we have

\[ \Phi(f(x)) = - \sum_{\mathcal{I}} \sum_{|\mathcal{I}| = h} Z_{\mathcal{I}} D\dot{\varphi} \left[ f(y) - f(x_1) - \sum_{k=1}^{k(h+r+1)-1} \sum_{\mathcal{J}} \int_0^1 \cdots \int_0^1 \partial_{j_1} \cdots \partial_{j_k} f(x(k)) \, dt(k) \right] \]
\[ + \sum_{m=0}^k (-1)^m \sum_{\mathcal{I}} x_{i_1} \cdots x_{i_m} \sum_{j_{m+1}^{+j_{m+1}} \cdots j_k} \partial_{x_\mu}(u) \big|_{x_{\nu} = x_{\nu}}. \]
where \( s = l(h+1) - 1 \). Let \( S_k \) be the symmetrization operator with respect to \( j_1, \ldots, j_k \). Then we have easily

\[
S_k \sum_{m=0}^{h} (-1)^m \sum_{1 \leq t_1 < \cdots < t_m \leq k} x^{j_1}_{t_1} \cdots x^{j_m}_{t_m} y^{j_{m+1}+\cdots+j_k} = S_k \prod_{\nu=1}^{h} (y^{j_\nu} - x^{j_\nu}_v).
\]

The interior of \([ \cdot ]\) in (6) is equal to

\[
f(y) - f(x_i) - \sum_{k=1}^{l} \sum_{r=0}^{s} \int_{0}^{1} \cdots \int_{0}^{1} \partial_{j_1} \cdots \partial_{j_k} f(x(k)) dt(k) S_k \prod_{\nu=1}^{l} (y^{j_\nu} - x^{j_\nu})
\]

\[
= \int_{0}^{1} \frac{d}{dt_1} f(x_1 + t_1(y - x_1)) dt_1 - \sum_{k=1}^{l} \cdots
\]

\[
= \sum_{j_1} \int_{0}^{1} \partial_{j_1} f((1-t_1)x_1 + t_1 y) dt_1 (y^{j_1} - x^{j_1})
\]

\[
- \sum_{j_1} \int_{0}^{1} \partial_{j_1} f((1-t_1)x_1 + t_1 x_2) dt_1 (y^{j_1} - x^{j_1}) - \sum_{k=2}^{s} \cdots
\]

\[
= \sum_{j_1,j_2} \int_{0}^{1} \int_{0}^{1} \partial_{j_1} \partial_{j_2} f((1-t_1)x_1 + (1-t_2)t_1 x_2 + t_2 y) dt_1 dt_2 (y^{j_1} - x^{j_1})(y^{j_2} - x^{j_2})
\]

\[
- \sum_{k=1}^{s} \cdots
\]

\[
= \sum_{j_1,\ldots,j_s} \int_{0}^{1} \cdots \int_{0}^{1} \partial_{j_1} \cdots \partial_{j_s} f((1-t_1)x_1 + \cdots + (1-t_{s+1})t_s \cdots t_1 x_{s+1}
\]

\[
+ t_{s+1} t_s \cdots t_1 y) dt(s+1) S_{s+1} \prod_{\nu=1}^{l} (y^{j_\nu} - x^{j_\nu})
\]

Since \( s+1 = l(h+1) \) and \( |\alpha| \leq h \), we have

\[
D_y \left[ \begin{array}{c} y-x_v \\ \nu_1=\ldots=\nu_l \end{array} \right] \equiv 0
\]

which completes the proof of Lemma 10.

References


Kazuo Masuda
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku
Tokyo, Japan