On unramified abelian extensions of a complete field
under a discrete valuation with arbitrary residue
field of characteristic $p \neq 0$ and its application
to wildly ramified $\mathbb{Z}_p$-extensions

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Introduction.

Let $k$ be a complete field under a discrete valuation with residue field $\bar{k}$
of characteristic $p \neq 0$. In this paper we shall state a theory of unramified
abelian extensions of $k$ (see the main theorem below) and apply this result to
fully ramified $\mathbb{Z}_p$-extensions of $k$ (see § 4, Theorem 4, Remarks 1 and 2).

The main result of this paper is as follows.

Fix a fully ramified cyclic extension $k'$ of $k$ of degree $m$, and for a finite
unramified extension $K$ of $k$, put

$$G^*(K) = N_{K'/K}(U_{K'}) \cap k/N_{k'/k}(U_k),$$

where $K' = Kk'$ and $U_k$ is the group of units of $k$. Put $W(k'/k) = \bigcup G^*(K)$,
where the union is taken in $U_k/N_{k'/k}(U_k)$ over all finite unramified extensions
$K$ of $k$. Let $\mathcal{F}_m$ be the set of all finite abelian unramified extensions $K$ of $k$
such that $\sigma^m = 1$ for all $\sigma \in G(K/k)$, where $G(K/k)$ is the Galois group of $K/k$,
and let $\hat{W}(k'/k)$ be the set of all finite subgroups of $W(k'/k)$. Then we have the following

Main Theorem.\footnote{We found this theorem to simplify the proof of [5], § 6, Theorem and its
Corollary 2, which is the original form of Theorem 4 in this paper. Our first motiva-
tion of [5] was to consider the problem of finding the class field theory of $\mathbb{Q}(t)_p$ (see
Ihara [2]).} Under the above assumptions, the following statements
(1) and (2) are valid:

1. If $K \in \mathcal{F}_m$, then $G^*(K)$ is canonically isomorphic to the character group
   of $G(K/k)$.
2. $\mathcal{F}_m$ corresponds bijectively to $\hat{W}(k'/k)$ by $K \rightarrow G^*(K)$. Moreover, we have
   $G^*(K_1) \subset G^*(K_2)$ if and only if $K_1 \subset K_2$ for $K_1, K_2 \in \mathcal{F}_m$. 

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This theorem can be regarded as an analogue of the theory of Kummer extensions and Witt theory \([10]\) and it contains both of them essentially. When \(m \equiv 0 \pmod{p}\), this is equivalent to Kummer theory; when \(m\) is a power of \(p\), it is equivalent to Witt theory \([10]\) essentially. However, our formulation is more useful for our application. For \(W(k'/k)\), see the Remarks at the end of §3.

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Notations.

(1) \((\text{For a complete field } k \text{ under a discrete valuation})\) \(\text{ord}_k\): the normalized additive valuation of \(k\). \(\mathcal{O}_k\): the ring of integers of \(k\). \(U_k\): the group of units of \(k\). \(U_k^i = \{u \in U_k \mid \text{ord}_k(u-1) \geq i\}\) for \(i \geq 1\). \(\bar{k}\): the residue field of \(k\). \(\bar{a}\) (for \(a \in \mathcal{O}_k\)): the image of \(a\) by the canonical homomorphism of \(\mathcal{O}_k\) to \(\bar{k}\).

(2) \(\mathbb{Z}\): the ring of rational integers. \(\mathbb{Z}_p\): the ring of \(p\)-adic integers. \(\mathbb{Q}_p\): the field of \(p\)-adic numbers. \(N = \{z \in \mathbb{Z} \mid z \geq 1\}\). \(m|n\): \(m\) divides \(n\) for \(m, n \in \mathbb{N}\).

(3) \(K^*\): the multiplicative group of a field \(K\). \(G(K/k)\): the Galois group of a Galois extension \(K/k\). \(\text{Hom}(G_1, G_2)\): the group of homomorphisms of a group \(G_1\) to an abelian group \(G_2\). \(N_{K/k}\): the norm map of \(K\) to \(k\) for a finite Galois extension \(K\) of \(k\). \([G, G]\): the commutator group of a group \(G\). \(\langle u \rangle\) or \(\langle u \mid u \in S \rangle\): the subgroup of a group \(G\), generated by \(u \in G\) or by a subset \(S\) of \(G\) respectively. \#(\(S\)): the number of elements of a finite set \(S\). \(\text{Ker } F\) (for a homomorphism \(F\) of a group \(G\) to a group \(G'\)): the kernel of \(F\). \(\text{Im } F\): the image of \(F\).

§ 1. Norm groups.

In this section we shall prove the following Theorem 1, which will be used for the proof of Theorem 2. When \(\bar{k}\) is finite, Theorem 1 is well known (e.g. Artin-Tate \([1]\), Chap. XI, §4 and Iyanaga \([3]\), Chap. V, §2). However, its proof is not valid for arbitrary residue field \(\bar{k}\). We use Sen \([7]\), Lemma 1 and Serre \([8]\), Chap. V.

**Theorem 1.** Let \(k\) be a complete field under a discrete valuation with residue field of characteristic \(p \neq 0\) and let \(k'\) be a finite fully ramified cyclic extension of \(k\). Then we have \(N_{k'/k}(U_k') = N_{k'/k}(U_k') \cap U_k\) for each \(i, j \in \mathbb{N}\) such that \(\phi(i-1) < j \leq \phi(i)\), where \(\phi\) is the Hasse function of \(k'/k\).
We need also the following

**Lemma 1.** Let \( p \) and \( k \) be as in Theorem 1 and let \( k_n \) be a fully ramified cyclic extension of \( k \) of degree \( p^n \). Let \( t_1 < t_2 < \cdots < t_n \) be the sequence of all the ramification numbers of \( k_n/k \). Let \( \phi : k_n/k \) be the Hasse function of \( k_n/k \). Put \( S_1 = \{ N \in \mathbb{N} | N \neq \phi(m) \text{ for all } m \in \mathbb{N} \text{ and } N < t_\alpha \} \) and \( S_2 = \{ N \in \mathbb{N} | N = t_\beta + mp^{t_\beta-1} \text{ with } 1 \leq j < n, m \equiv 0 \pmod{p}, m \in \mathbb{N} \text{ and } N < t_\beta \} \). Then \( S_1 = S_2 \).

**Proof.** Let \( s_i \) be such that \( \phi(s_i) = t_i \) for \( i = 1, 2, \ldots, n \) and let \( t_0 = s_0 = 0 \). By Hasse-Arf’s theorem, \( s_i \in \mathbb{Z} \). Then we have easily \( S_1 = \{ N \in \mathbb{N} | N \neq s_i + (m_i - s_i)p^i \text{ for } s_i \leq m_i \in \mathbb{Z} < s_{i+1} \text{ and } i = 0, 1, \ldots, n-1 \} \). Now let \( N \in S_2 \). Then \( N = t_\beta + mp^{t_\beta-1} \) \( \text{with } 1 \leq j < n, m \equiv 0 \pmod{p} \text{ and } m \in \mathbb{N} \). Let \( i \in \mathbb{N} \) be such that \( t_i \leq N < t_{i+1} \).

Since \( N > t_j \), we have \( j \leq i \leq n-1 \). If \( N \in S_1 \), then \( N = t_i + s_j p^j \) with \( 0 \leq s_i < s_{i+1} - s_i \text{ and } s \in \mathbb{Z} \). Since \( t_i - t_j \equiv 0 \pmod{p} \text{ and } i \geq j \), this implies that \( mp^{t_\beta-1} \equiv 0 \pmod{p} \text{ hence } m \equiv 0 \pmod{p} \text{, which is a contradiction, hence } N \in S_2 \). Hence \( S_2 \subseteq S_1 \).

Conversely let \( N \in S_2 \). If \( N \in S_2 \), then \( N = t_j + m_j p^j \) \( \text{with } 1 \leq j \leq n-1, m_j \in \mathbb{Z} \text{ and } m_j \geq 0 \). Let \( j_0 \) be the maximum of such \( j \), then we have easily \( t_{j_0} \leq N < t_{j_0+1} \).

This implies that \( N \in S_2 \), which is a contradiction, hence \( N \in S_1 \). Hence \( S_1 \subseteq S_2 \).

Therefore \( S_1 = S_2 \).

**Lemma 2.** Let notations be as in Lemma 1 and let \( \sigma \) be a generator of \( G(k_n/k) \). Let \( N \in \mathbb{N} \) be such that \( N \neq \phi(m) \text{ for all } m \in \mathbb{N} \text{ and } N < t_\alpha \) and let \( A \in k_n \) be such that \( \text{ord}_{k_n}(A) = N \). Then there exists \( x \in U_{k_n}^{(1)} \) such that \( x^{p^{t_\beta-1}+1} = 1 + A \pmod{\pi_n^{t_\beta+1}} \), where \( \pi_n \) is a prime element of \( k_n \).

**Proof.** By Lemma 1, \( N = t_i + mp^{t_\beta-1} \) \( \text{with } 1 \leq j < n, m \equiv 0 \pmod{p} \text{ and } m \in \mathbb{N} \). By Sen [7], Lemma 1, there exists \( y \in k_n^* \) such that \( \text{ord}_{k_n}(y^p - y) = N \). For \( \lambda \in U_n \), put \( z_\lambda = 1 + \lambda y \) and \( B = y^\sigma - y \), then \( z_\lambda^\sigma - z_\lambda = \lambda B \), hence \( (z_\lambda)^{p^{t_\beta-1}} - 1 + \lambda B \pmod{\pi_n^{t_\beta+1}} \). There exists \( \lambda \in U_n \) such that \( \text{A} = \lambda B \pmod{\pi_n^{t_\beta+1}} \).

For this \( \lambda \in U_n \), put \( x = z_\lambda \), then the assertion follows.

Now we can prove Theorem 1.

**Proof of Theorem 1.** It is easily verified that it is enough to prove the theorem when \( k' = k_n \) where \( k_n \) is as in Lemma 1. By Serre [8], Chap. V, § 6, Proposition 8, \( N_{k_n/k}(U_{k_n}^{(1)}) \subseteq N_{k_n/k}(U_{k_n}^{(j)}) \cap U_{k_n}^{(0)} \). By Serre [8], Chap. V, § 6, Corollary 3, we may suppose \( \phi(t) \leq t_\alpha \). Now conversely let \( N_{k_n/k}(z) \subseteq N_{k_n/k}(U_{k_n}^{(j)}) \cap U_{k_n}^{(0)} \) with \( z \in U_{k_n}^{(j)} \). Then by Lemma 2 and Serre [8], Chap. V, § 6, Proposition 9, there exists \( z_i \in k_n^* \) such that \( z \cdot z_i^{t_\alpha} \in U_{k_n}^{(j)} \), hence \( N_{k_n/k}(z) = N_{k_n/k}(z \cdot z_i^{t_\alpha}) \subseteq N_{k_n/k}(U_{k_n}^{(j)}) \).

§ 2. Canonical isomorphism.

In this section we shall prove the following Theorem 2 and Corollaries to Theorem 2, which will be used for the proof of the main theorem. The statement (1) of the main theorem is an immediate consequence of Theorem 2 (see
Corollary 1 to Theorem 2).

**Theorem 2.** Let \( k \) be a complete field under a discrete valuation with residue field of characteristic \( p \neq 0 \) and let \( k'/k \) be a finite fully ramified cyclic extension. Let \( K/k \) be a finite unramified Galois extension and put \( K'=Kk' \), \( T_K=\{y^{s-1}|y\in K'\} \), \( V_K=\{y^{s-1}|y\in U_K\} \), \( G^*(K)=N_{K'/K}(U_K)\cap k/N_{K'/k}(U_k) \) and \( G=G(K/k) \), where \( s \) is a generator of \( G(K'/K) \). Then there exists a canonical isomorphism \( F_K: G^*(K)\to \text{Hom}(G, T_K/V_K) \).

**2.1. Proof of Theorem 2.**

For the proof of Theorem 2 we need Theorem 1 and the following two lemmas.

**Lemma 3.** Let \( k \) and \( K \) be two complete fields under a discrete valuation and let \( k'/k \) be a finite fully ramified cyclic extension. Suppose that \( K \) is an extension of \( k \) with ramification index 1. Put \( K'=Kk' \). Let \( T_K, V_K, T_{K'} \) and \( V_{K'} \) be as in Theorem 2. Then the following (1), (2), (3) are valid:

1. (Serre [8], p. 104, Exercise.) \( G(k'/k)\cong T_{k'}/V_{k'} \) by \( \sigma\to (\pi'\sigma^{-1} \mod V_{k'}) \), where \( \pi' \) is a prime element of \( k' \).
2. \( T_{k'}/V_{k'} \cong T_K/V_K \) by \( (x \mod V_{k'})\to (x \mod V_K) \), where \( x\in T_K \).
3. \( V_{x'}\cap T_{k'}=V_{k'} \).

**Proof.** Since \( \pi' \) is also a prime element of \( K' \), it follows from the statement (1) that \( (\pi'\sigma^{-1} \mod V_{k'}) \) generates \( T_{k'}/V_{k'} \), where \( s \) is a generator of \( G(K'/K) \). Therefore the given homomorphism in the statement (2) is surjective, hence bijective by (1). The statement (2) implies the statement (3).

**Lemma 4.** Let \( k, k', K, K', V_K \) and \( G \) be as in Theorem 2. Let \( u\in U_k\cap N_{K'/K}(U_{K'}) \) and \( A\in U_{k'} \) be such that \( N_{K'/K}(A)=u \). Suppose that \( A^{\sigma-1}\in V_K \) for all \( \sigma\in G \), identifying \( G \) and \( G(K'/k) \). Then \( u\in N_{k'/k}(U_{k'}) \).

**Proof.** Since \( V_K\subset U_{k'} \), we have \( \bar{\bar{A}}^\sigma=\bar{\bar{A}} \) for all \( \sigma\in G \), hence \( A=aA_1 \) with \( a\in U_k \) and \( A_1\in U_{k'} \), since \( K'/k' \) is unramified. Therefore we may suppose that \( A\in U_{k'}^{[m]} \) from the beginning. Suppose that \( u\in U_{k'}^{[m]} \) with some \( m\geq 1 \). By applying Theorem 1 to \( K'/K \), we may suppose that \( A=1+\lambda \pi\phi(m) \mod \pi\phi(m)+1 \), where \( \pi' \) is a prime element of \( k' \), \( \phi \) is the Hasse function of \( K'/K \) and \( \lambda\in \mathcal{O}_{K'} \). Then \( A^{\sigma-1}=1+(\lambda^s-\lambda)\pi\phi(m) \mod \pi\phi(m)+1 \). Since \( V_K\cap U_{k'}^{[m]}\subset U_{k'}^{[m+1]} \) (see Serre [7], p. 104, Ex. a)), we have \( \bar{\bar{A}}^\sigma=\bar{\bar{A}} \) for all \( \sigma\in G \), hence we can take \( \lambda \) in \( \mathcal{O}_{k'} \). Put \( B=(1-\lambda \pi\phi(m))A \). Then \( B\in U_{k'}^{(m+1)} \), \( A^{\sigma-1}=B^{\sigma-1}\in V_K \), and \( N_{K'/K}(B)\in U_{k'}^{[m+1]} \) by Serre [8], Chap. V, Proposition 8. Applying the above procedure to \( B \), we have \( u\in N_{k'/k}(U_{k'}) \) by induction on \( m \).

**Proof of Theorem 2.** Identify \( G \) with the Galois group \( G(K'/k') \). For \( u\in N_{K'/K}(U_K)\cap k \) and \( \sigma\in G \), put \( f_u(\sigma)=A^{\sigma-1} \mod V_K \), where \( A\in U_K \) is such that \( N_{K'/K}(A)=u \). It is easily verified that \( f_u(\sigma)\in T_{K'}/V_{K'} \) and that \( f_u(\sigma) \) is independent of the choice of \( A \) and that \( f_u\in \text{Hom}(G, T_{K'}/V_{K'}) \). Put \( F_K(u)=f_u \), then it is easily verified that \( F_K \) is a homomorphism of \( N_{K'/K}(U_K)\cap k \) to
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Horn (G, TK/VK). By Lemma 4, Ker $F_K = N_{k'/k}(U_K)$. Now we shall show that $F_K$ is surjective. Let $x \in \text{Hom}(G, TK/VK)$. Let $L'$ be the subfield of $K'$ fixed by Ker $x$ and put $L = L' \cap K$. Let $\sigma_i \in G$ be such that $x(\sigma_i)$ generates $\text{Im} x$. By (2) of Lemma 3, $x(\sigma_i)^d = x^{d^{-1}} \mod V_K$, with some $x \in k''$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$. If $d = [L' : k']$, then $x(\sigma_i)^d = 1$, hence $(x^{d^{-1}})^{m'} = 1$.

2.2. Corollaries to Theorem 2.

In this section we shall state the Corollaries to Theorem 2. The Corollary 1 is the statement (1) of the main theorem in the introduction. Corollaries 2 and 3 will be used for the proof of (2) of the main theorem.

**Corollary 1.** Let notations and assumptions be as in Theorem 2. Put $m = [k' : k]$. Suppose moreover that $\sigma^m = 1$ for all $\sigma \in G$. Let $\tau(G)$ be the character group of $G$. Then $G^*(K)$ is isomorphic to $\tau(G)$.

**Proof.** Since $TK/VK$ is a cyclic group of order $m$ by (1) of Lemma 3, by assumption $\text{Hom}(G, TK/VK) \cong \tau(G)$. Hence the assertion follows from Theorem 2.

**Corollary 2.** Let notations and assumptions be as in Theorem 2. Put $m = [k' : k]$. Let $L$ be the maximal abelian extension of $k$ in $K$ such that $\sigma^m = 1$ for all $\sigma \in G$. Then $G^*(L) = G^*(K)$.

**Proof.** It is trivial that $G^*(K) \subseteq G^*(L)$. Let $H = [G, G] \langle g^m | g \in G \rangle$, then $L$ is the subfield of $K$ fixed by $H$. It is clear that $G^*(K) \cong \text{Hom}(G, TK/VK)$. Hence by Theorem 2, $\#(G^*(K)) = \#(G^*(L))$, so $G^*(K) = G^*(L)$.

**Corollary 3.** Let $K_1, K_2$ be two finite unramified Galois extensions of $k$ such that $K_1 \supseteq K_2$ and put $G_1 = G(K_1/k)$. Let $G^*(K_1), T_{K_1}$, and $V_{K_1}$ be as in Theorem 2, where $K'_1 = K_1 k'$, and let $F_{K_1} : G^*(K_1) \rightarrow \text{Hom}(G_1, T_{K_1}/V_{K_1})$ be the canonical isomorphism defined in Theorem 2. Then $F_{K_1}(G^*(K_1)) = G(K_1/k_2)$. Let $F_{K_2} : G^*(K_2) \rightarrow \text{Hom}(G_1, T_{K_2}/V_{K_2})$ be the canonical isomorphism defined in Theorem 2. Then $F_{K_2}(G^*(K_2)) = G(K_1/k_2)$. Hence by Theorem 2, $\#(F_{K_1}(G^*(K_1))) = \#(G(K_1/k_2))$. Therefore we have the assertion.
§ 3. Proof of the main theorem.

Noting the similarity of Theorem 2 to Kummer theory, we shall prove the statement (2) of the main theorem in the introduction. For the proof we use Theorem 2, Corollaries 1, 2 and 3 to Theorem 2 and the duality of finite abelian groups.

**Proof of the main theorem.** The statement (1) of the main theorem is already proved in Corollary 1 to Theorem 2. By Theorem 2, if $K \in \mathcal{F}_n$, then $G^*(K) \subseteq \mathcal{W}(k'/k)$.

**Existence:** Let $M \in \mathcal{W}(k'/k)$. Then by the definition of $\mathcal{W}(k'/k)$, $G^*(K) \supset M$ for some finite unramified extension $K$ of $k$. By taking the Galois closure of $K$ over $k$, we may suppose that $K/k$ is Galois. Moreover by Corollary 2 to Theorem 2, we may suppose that $K \in \mathcal{F}_n$ from the beginning. Since $K \in \mathcal{F}_n$ by Corollary 1 to Theorem 2, we can regard $\text{Hom}(G(K,k), T_{K_i}/V_{K_i})$ as the character group of $G(K/k)$. Put $H^*=F_{K_1}(M)$, where $F_{K_1}$ is the canonical isomorphism of $G^*(K)$ to $\text{Hom}(G(K,k), T_{K_i}/V_{K_i})$, defined in Theorem 2. Let $H$ be the subgroup of $G(K,k)$ corresponding to $H^*$ by the duality of finite abelian groups. Then $H^* = \{ f \in \text{Hom}(G(K,k), T_{K_i}/V_{K_i}) | f=1 \text{ on } H \}$. Let $K$ be the subfield of $K_1$ fixed by $H$, then $K \in \mathcal{F}_n$ and $F_{K_1}(M)=F_{K_1}(G^*(K))$ by Corollary 3 to Theorem 2, hence $M=G^*(K)$ by Theorem 2.

**Uniqueness:** Let $K_1, K_2 \in \mathcal{F}_n$ be such that $G^*(K_1) \supset G^*(K_2)$. Put $K=K_1 \cdot K_2$, $G=G(K/k)$ and $G_i=G(K/K_i)$ for $i=1,2$. Let $F_K: G^*(K) \rightarrow \text{Hom}(G, T_{K_i}/V_{K_i})$ be the canonical isomorphism defined by Theorem 2. By Corollary 3 to Theorem 2, $F_K(G^*(K)) = \{ f \in \text{Hom}(G, T_{K_i}/V_{K_i}) | f=1 \text{ on } G_i \}$ for $i=1,2$. Since $K \in \mathcal{F}_n$, by Corollary 1 to Theorem 2 $\text{Hom}(G, T_{K_i}/V_{K_i})$ is isomorphic to the character group of $G$. Then by the duality of finite abelian groups, $G^*(K_1) \supset G^*(K_2)$ implies $G_1 \subseteq G_2$, so $K_1 \subseteq K_2$. In particular, $G^*(K_1)=G^*(K_2)$ implies $K_1=K_2$.

**Remark 1.** Let $k$ be a complete field under a discrete valuation $v$ with arbitrary residue field $\overline{k}$ of characteristic $p \neq 0$ and assume that $p$ is a prime element of $k$. Let $k_0$ be the subfield of $k$ satisfying the conditions: (i) $k_0$ is complete with respect to the restriction of $v$ to $k$; (ii) the residue field $\overline{k}_0$ is the maximum perfect subfield of $\overline{k}$, i.e., $\overline{k}_0 = \bigcap_{n=1}^{\infty} (\overline{k})^{p^n}$. By MacLane [4], such a $k_0$ really exists. Let $k_0^n/k_0$ be a fully ramified cyclic extension of degree $p^n$ and put $k_n = k_0^n \cdot k$. Then it can be proved that $W(k_0^n/k) = H_n(k)/N_{k_0^n/k}(U_{k_0^n}),$ where $H_n(k) = \{ x \in U_k | x = \sum_{i=0}^{n-1} \lambda_i \cdot k_0^{i} \cdot p^i \text{ (mod } p^{n+1}) \}$ with $\lambda_i \in \mathcal{O}_k$.

**Remark 2.** If $k$ is perfect, then $W(k'/k) = U_k/N_{k'/k}(U_k)$. Hence the main theorem in the introduction gives an interpretation of a quotient group $U_k/N_{k'/k}(U_k)$; it can be regarded as the character group of the Galois group $G(K_m/k)$, where $K_m$ is the composite field of all fields in $\mathcal{F}_m$. 


In this section, we shall apply the main theorem to fully ramified cyclic extensions and $\mathbb{Z}_p$-extensions of $k$.

**Lemma 5.** Let $k$ be a complete field under a discrete valuation. Let $k_1, k_2$ be two finite fully ramified abelian extensions of $k$ such that $k_1L=k_2L$ with an extension $L/k$ of ramification index 1 (i.e., a prime element of $k$ is a prime element of $L$). Suppose that $N_{k_1/k}(k_1)\cap N_{k_2/k}(k_2)$ contains a prime element of $k$. Then $k_1=k_2$.

**Proof.** We may suppose that $k_i/k$ is cyclic and that $L$ is a Galois extension of $k$, by taking the Galois closure of $L$ over $k$. Since $k_i(L)/k$ is cyclic, we may suppose $L\subset k, k_2$. Put $Lk_1=Lk_2=L_1$ and let $s$ be a generator of $G(L_1/L)$. By assumption, there exist prime elements $\pi_i$ of $k_i$ such that $N_{k_1/k}(\pi_1)=N_{k_2/k}(\pi_2)$. Put $u=\pi_1/\pi_2$, then $u\in U_{L_1}$ and $N_{L_1/k}(u)=1$. Hence $y^{s-1}=u$ with $y\in L_1^*$. Now suppose $k_1\neq k_2$. Then there exists $\sigma\in G(L_1/k_1)$ such that $\sigma|k_1\neq 1$. By the statement (1) of Lemma 3, $\pi_1^{s-1}\in V_{k_1}$, hence by the statement (3) of Lemma 3, $\pi_2^{s-1}\in V_{k_2}$. On the other hand, $\pi_2^{s-1}=\sigma^{s-1}=(y^{s-1})^{s-1}\in V_{L_1}$, which is a contradiction. Therefore $k_1=k_2$.

**Lemma 6.** Let $k$ be as in Lemma 5 and let $k_1, k_2$ be two finite fully ramified Galois extensions of $k$ such that $k_iL=k_iL$ with a finite unramified extension $L/k$. Then $N_{k_1/k}(U_{k_1})=N_{k_2/k}(U_{k_2})$.

**Proof.** By taking the Galois closure of $L$ over $k$, we may suppose that $L$ is a Galois extension of $k$. Put $L'=Lk_1=Lk_2$. Since $L'/k_i$ is unramified, we have $N_{L'/k}(U_{L'})=U_{L'}$. Since $k_i/k$ is fully ramified and $[k_1:k]=[k_2:k]$, we have the assertion.

**Theorem 3.** Let $k, k'$ and $W(k'/k)$ be as in the main theorem in the introduction. Let $\mathcal{F}=\mathcal{F}(k')=\{k^*| k^* is a fully ramified cyclic extension of $k$ such that $k^*L=k^*L$ with an unramified extension $L$ of $k$\}. Let $F_k: \mathcal{F}=W(k'/k)$ be a map defined by $k^*/k^*=\langle \pi^* \rangle$ of $k^*$, where $\pi'$ and $\pi''$ are prime elements of $k'$ and $k''$ respectively. Then $F_k$ is bijective and independent of the choice of $\pi'$ and $\pi''$.

**Proof.** By Lemma 6, $F_k$ is independent of the choice of $\pi'$ and $\pi''$.

$F_k$ is injective: Let $k_i\in \mathcal{F}$ with $i=1, 2$. By assumption, $Lk_1=Lk_2=Lk'$ with an unramified extension $L$ of $k$. Suppose that $F_k(k_1)=F_k(k_2)$. Then by the definition of $F_k$ and by Lemma 6, $N_{k_1/k}(k_1)=N_{k_2/k}(k_2)$. Hence by Lemma 5, $k_1=k_2$. Hence $F_k$ is injective.

$F_k$ is surjective: Let $u\in W(k'/k)$ and let $m'$ be the order of $\langle u \rangle$. Then $m'|m$. By the main theorem, there exists an unramified cyclic extension $K/k$ of degree $m'$ such that $G^*(K)=\langle u \rangle$. Put $K'=Kk'$. By Galois theory, there exist $m'$ cyclic extensions $k_1, \ldots, k_{m'}$ of degree $m$ such that $k^i\neq k_l$ and $k_i\subset K'$.
for $i=1, 2, \ldots, m'$. Clearly $F_{k'}(k_i) \equiv \langle u \rangle$. Since $F_{k'}$ is injective, $F_{k'}(k_i) = u$ with some $i$. Hence $F_{k'}$ is surjective. This completes the proof.

Now we apply Theorem 3 to $\mathbb{Z}_p$-extensions of $k$. Fix a fully ramified $\mathbb{Z}_p$-extension $k_\infty$ of $k$, and let $k_n/k$ be the sub-extension of $k_\infty/k$ of degree $p^n$. For $m \geq n \geq 1$, let $\rho_{mn}^\varphi: W(k_m/k) \rightarrow W(k_n/k)$ be a homomorphism defined by $x \mod N_{k_m/k}(U_{k_m}) \rightarrow x \mod N_{k_n/k}(U_{k_n})$ with $x \in N_{k_m/k}(U_{k_m}) \cap k$, where $\hat{k}_{ur}$ is the completion of the maximum unramified extension of $k$ and $k_m = \hat{k}_{ur}k_m$. Then $\{W(k_n/k), \rho_{mn}^\varphi\}$ is a projective system. Let $W(k_\infty)$ be the projective limit of this system. Then we have directly the following Theorem 4 by Theorem 3.

**Theorem 4.** Let $k$, $p$, $k_\infty$ and $W(k_\infty)$ be as above. Let $\mathcal{F}(k_\infty) = \{k'_\infty \mid k'_\infty$ is a fully ramified $\mathbb{Z}_p$-extension of $k$ such that $k_\infty L = k'_\infty L$ with an unramified extension $L$ of $k\}$. Let $F_{\infty}: \mathcal{F}(k_\infty) \rightarrow W(k_\infty)$ be a map defined by $k' \rightarrow (N_{k'_n/k}(\pi'_n))/N_{k_n/k}(\pi_n) \mod N_{k_n/k}(U_{k_n})$, where $k'_n/k$ and $k_n/k$ are the sub-extensions of $k'_\infty/k$ and $k_\infty/k$ of degree $p^n$ respectively, and where $\pi'_n$ and $\pi_n$ are prime elements of $k'_n$ and $k_n$ respectively. Then $F_{\infty}$ is independent of the choice of prime elements and $F_{\infty}$ is bijective.

**Remark 1.** Suppose the conditions: (i) $p$ is a prime element of $k$, (ii) the finite field $F_p$ with $p$ elements is the maximum perfect subfield of $k$, i.e., $F_p = \bigcap_{n=1}^{\infty} (k)^{p^n}$. As typical examples, we have $k$ such that $\overline{k} = F_p(t)$ (the rational function field over $F_p$ in one variable $t$) or $F_p(t)$ (the field of power series over $F_p$ in one variable $t$). In this case, it is easily verified by [6], Theorem that $\mathcal{F}(k_\infty)$ is the set of all fully ramified $\mathbb{Z}_p$-extensions of $k$.

**Remark 2.** It can be proved that $W(k_\infty) = \varprojlim H_n(k)/N_{k_n/k}(U_{k_n})$ under the above conditions (i), (ii), where $H_n(k)$ is as in the Remark 1 in § 3 and the projective limit is taken with respect to a homomorphism induced by the natural injection of $H_n(k)$ into $H_n(k)$ for $n' \geq n$. Therefore under the above conditions (i), (ii), as a Corollary to Theorem 4, it can be proved that $\bigcap_{n=1}^{\infty} N_{k'_n/k}(k'_n)$ contains a prime element of $k$ if and only if there exists a $\mathbb{Z}_p$-extension $k_c$ of $\mathbb{Q}_p$ such that $k_\infty = k_c k_c(3)$. Note that $W(k_\infty) = U_k^{(1)}$ if $k = \mathbb{Q}_p$ and that in this case Theorem 4 follows from local class field theory.

**References**


(2) This can be regarded as a generalization of [5], § 6, Corollary 2 to Theorem.

(3) This is [5], § 6, Corollary 3 to Theorem.
Unramified abelian extensions of a complete field


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