Compact minimal submanifolds of a sphere with positive Ricci curvature

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(Received June 7, 1977)
(Revised Oct. 20, 1977)

1. Introduction.

Let $M$ be an $n$-dimensional simply connected compact orientable submanifold minimally immersed in an $(n+p)$-dimensional sphere of constant curvature 1. The pinching problem with respect to the scalar curvature of $M$ [6] [1] and the sectional curvature of $M$ [4] [8] have been studied. In this note, we shall prove a pinching theorem with respect to the Ricci curvature of $M$. Some examples are:

**EXAMPLE 1.** In general, let $S^q(r)$ denote a $q$-dimensional sphere in $R^{n+1}$ with radius $r$. Let $m$ and $n$ be positive integers such that $m < n$ and let $M_{m,n-m} = S^m(n/m) \times S^{n-m}(n-m/n)$. We imbed $M_{m,n-m}$ into $S^{n+1} = S^{n+1}(1)$ as follows. Let $(u, v)$ be a point of $M_{m,n-m}$, where $u$ (resp. $v$) is a vector in $R^{m+1}$ (resp. $R^{n-m+1}$) of length $\sqrt{m/n}$ (resp. $\sqrt{n-m/n}$). We can consider $(u, v)$ as a unit vector in $R^{m+1} \times R^{n-m+1}$. It is easily shown that $M_{m,n-m}$ is a minimal submanifold. Furthermore from the fact the first eigenvalue of the Laplacian of $M_{m,n-m}$ is $n$ and the dimension of the eigenspace is $n+2$, we can prove the following.

Let $X$ be a minimal immersion of $M_{m,n-m}$ into $S^{n+p}$ such that the immersion is full, i.e. $X(M_{m,n-m})$ is not contained in a linear subspace of $R^{n+p+1}$. Then $p=1$ and the immersion is rigid. The Ricci curvature of $M_{m,n-m}$ varies between $n(m-1)/m$ and $n(n-m-1)/(n-m)$.

**EXAMPLE 2.** We can define a minimal immersion of an $n$-dimensional complex projective space $P_2^m/(n+1)$ with holomorphic sectional curvature $2n/(n+1)$ into $S^{n(n+2)-1}$ such that the usual coordinate functions of $R^{n(n+2)}$ are all independent hermitian harmonic functions of degree 1 on $P_2^m/(n+1)$. 

N. R. Wallach proved in [7] that if \( X \) is a minimal immersion of \( P_{2n/(n+1)} \) into \( S^{n+p} \) such that the immersion is full, then \( p=n^2-1 \) and the immersion is rigid. The Ricci curvature of \( P_{2n/(n+1)} \) is equal to \( n \).

**Theorem.** Let \( M \) be an \( n \)-dimensional simply connected compact orientable minimal submanifold immersed in \( S^{n+p} \) such that the immersion is full. If \( n \geq 4 \) and the Ricci curvature of \( M \geq n-2 \), then \( M \) is either \( S^n \) (totally geodesic), \( M_{m,m} \) in \( S^n \) (\( n=2m \)) or \( P_{4/3} \) in \( S^7 \).

The author would like to express his sincere thanks to Professor K. Ogiue for his many valuable suggestions.

### 2. Preliminaries.

Let \( M \) be an \( n \)-dimensional Riemannian manifold isometrically immersed in an \( (n+p) \)-dimensional space form \( \tilde{M} \) of constant curvature \( \tilde{c} \). We denote by \( \nabla \) (resp. \( \tilde{\nabla} \)) the covariant differentiation of \( M \) (resp. \( \tilde{M} \)). Then the second fundamental form \( \sigma \) of the immersion is given by

\[
\sigma(X, Y) = \tilde{\nabla}_XY - \nabla_XY
\]

and it satisfies \( \sigma(X, Y) = \sigma(Y, X) \). We choose a local field of orthonormal frames \( e_1, \ldots, e_n, \tilde{e}_1, \ldots, \tilde{e}_p \) in \( \tilde{M} \) in such a way that, restricted to \( M \), \( e_1, \ldots, e_n \) are tangent to \( M \). With respect to the frame field of \( M \) chosen above, let \( \omega^1, \ldots, \omega^n, \tilde{\omega}^1, \ldots, \tilde{\omega}^p \) be the field of dual frames. Then the structure equations of \( M \) are given by\(^{(*)}\)

\[
\begin{align*}
(2.1) \quad & d\omega^a = -\sum \omega^b \wedge \omega^b, \quad \omega^b_0 + \omega^b_0 = 0, \\
(2.2) \quad & d\omega^a_0 = -\sum \omega^b_0 \wedge \omega^b_0 + \tilde{c} \omega^a \wedge \omega^b.
\end{align*}
\]

Restricting these forms to \( M \), we have the structure equations of the immersion

\[
\begin{align*}
(2.3) \quad & \omega^a = 0 \\
(2.4) \quad & \omega_i^a = \sum h^a_j \omega^j, \quad h_j^a = h_j^a_i \\
(2.5) \quad & d\omega^t = -\sum \omega^j_0 \wedge \omega^i, \quad \omega^t_0 + \omega^t_0 = 0 \\
(2.6) \quad & d\omega^0_j = -\sum \omega^i \wedge \omega^j_0 + \Omega^0_j, \quad \Omega^0_j = \frac{1}{2} \sum R^0_j \omega^k \wedge \omega^k \\
(2.7) \quad & R^a_{jkl} = \tilde{c}(\delta^a_j \delta^l_k - \delta^a_k \delta^l_j) + \sum (h^a_i h^a_j - h^a_i h^a_j).
\end{align*}
\]

\(^{(*)}\) We use the following convention on the ranges of indices unless otherwise stated: \( A, B, C = 1, \ldots, n, \tilde{A}, \ldots, \tilde{p} \); \( i, j, k, t = 1, \ldots, n \); \( a, \gamma = 1, \ldots, \tilde{p} \).
The second fundamental form $\sigma$ and $h_{ij}^\alpha$ are related by

$$\sigma(e_i, e_j) = \sum h_{ij}^\alpha e_\alpha.$$  

Define $h_{ik}^\alpha$ by

$$\sum h_{ik}^\alpha \omega^k = d h_{ik}^\alpha - \sum h_{ik}^\alpha \omega^j - \sum h_{ik}^\alpha \omega^k + \sum h_{ik}^\alpha \omega^l.$$  

Then from (2.2), (2.3) and (2.4) we have

$$h_{ik}^\alpha = h_{ik}^\alpha.$$  

Then second fundamental form $\sigma$ is said to be parallel if $h_{ik}^\alpha = 0$ for all $i, j, k, \alpha$. The second fundamental form $\sigma$ satisfies a differential equation. In fact we have the following.

**Lemma 2.1 ([6]).**

$$\frac{1}{2} \Delta (\sum h_{ik}^\alpha h_{ij}^\alpha) = \sum h_{ik}^\alpha h_{ik}^\alpha - \sum (\sum(h_{ik}^\alpha h_{jk}^\alpha - h_{jk}^\alpha h_{ik}^\alpha))^2 - \sum h_{ij}^\alpha h_{ik}^\alpha h_{ik}^\alpha + n \sum h_{ij}^\alpha h_{ij}^\alpha,$$

where $\Delta$ denotes the Laplacian.

### 3. Lemmas.

In general, for a matrix $A=(a_{ij})$ we denote by $N(A)$ the square of the norm of $A$, i.e. $N(A) = \sum a_{ij}^2$. Clearly, $N(A)=N(T^{-1}AT)$ for any orthogonal matrix $T$. Now we have

$$\sum (\sum (h_{ik}^\alpha h_{jk}^\alpha - h_{jk}^\alpha h_{ik}^\alpha))^2 = \sum N(A_{ij}A_{ij} - A_{ij}A_{ij}),$$

where $A_{ij}=(h_{ij}^\alpha)$.

**Lemma 3.1.** $(n \times n)$-symmetric matrix $(h_{ij}^\alpha)$ is positive semi definite. In particular

1. $1 - \sum h_{ij}^\alpha h_{ij}^\alpha \geq 0$ for each $j$,

2. $n \geq \|\sigma\|^2,$

where $\|\sigma\|^2 = \sum h_{ij}^\alpha h_{ij}^\alpha.$

**Proof.** From Gauss equation (2.7) and the fact the immersion is minimal, we obtain

$$S(e_j, e_l) = (n-1)\delta_{jl} - \sum h_{ij}^\alpha h_{ij}^\alpha,$$

where $S$ denotes the Ricci tensor of $M$. From the assumption of the theorem, $S(e_j, e_l) = (n-2)\delta_{jl} - \sum h_{ij}^\alpha h_{ij}^\alpha$ is the $(j, l)$ entry of a positive semi definite symmetric matrix. Q. E. D.

**Lemma 3.2.** For each $\alpha$
In particular, we have
\[ \sum_{\beta} N(A_{\gamma} A_{\beta} - A_{\beta} A_{\gamma}) \leq 4N(A_{\alpha}) - 4N(A_{\alpha}^2). \]

**Proof.** Let \( \lambda_1^a, \ldots, \lambda_n^a \) be the eigenvalues of \( A_a \). By a simple calculation, we obtain
\[ \sum_{\beta} N(A_{\gamma} A_{\beta} - A_{\beta} A_{\gamma}) = \sum_{\beta \neq \alpha, i, j} (h^i_{\alpha \beta})^2 (\lambda_i^a - \lambda_j^a)^2 = \sum_{\beta \neq \alpha, i, j} (h^i_{\alpha \beta})(\lambda_i^a - \lambda_j^a)^2. \]

Since \((\lambda_i^a - \lambda_j^a)^2 \leq 2((\lambda_i^a)^2 + (\lambda_j^a)^2)\), we obtain
\[ \sum_{\beta} N(A_{\gamma} A_{\beta} - A_{\beta} A_{\gamma}) \leq \sum_{\beta \neq \alpha, i, j} 2(h^i_{\alpha \beta})(\lambda_i^a)^2 + (\lambda_j^a)^2) = 4 \sum_{\beta \neq \alpha, i, j} (h^i_{\alpha \beta})(\lambda_i^a)^2. \]

From Lemma 3.1 (1)
\[ 1 - (\lambda_i^a)^2 \geq \sum_{\beta \neq i, j} (h^i_{\alpha \beta})^2 \text{ for each } i. \]

Hence we obtain
\[ \sum_{\beta} N(A_{\gamma} A_{\beta} - A_{\beta} A_{\gamma}) \leq 4 \sum_{i} (1 - (\lambda_i^a)^2)(\lambda_i^a)^2 = 4N(A_{\alpha}) - 4N(A_{\alpha}^2). \]

**Lemma 3.3.**
\[ N(A_{\alpha}^2) \geq \frac{N(A_{\alpha})}{n} \text{ for each } \alpha. \]

The equality holds if and only if \( A_{\alpha}^2 \) is proportional to the identity.

**Proof.** Let \( \lambda_1^a, \ldots, \lambda_n^a \) be the eigenvalues of \( A_a \). Then
\[ nN(A_{\alpha}^2) - (N(A_{\alpha}))^2 = n \sum_{i} (\lambda_i^a)^4 - (\sum_{i} (\lambda_i^a)^2)^2 = \sum_{i,j} ((\lambda_i^a)^2 - (\lambda_j^a)^2)^2. \]

The equality holds if and only if \( (\lambda_i^a)^2 = \ldots = (\lambda_n^a)^2 \).

**4. Proof of theorem.**

We set \( S_{\alpha\beta} = \sum_{i,j} h^i_{\alpha \beta} h^j_{\alpha \beta} \). Then \((p \times p)\)-matrix \((S_{\alpha\beta})\) is symmetric and can be diagonalized for a suitable choice of a basis \( e_{\gamma}, \ldots, e_{\beta} \) at each point so that
\[ \sum_{i,j} h^i_{\alpha \beta} h^j_{\alpha \beta} h^k_{\alpha \beta} = \sum_{\alpha} N(A_{\alpha})^p. \]

From Lemma 2.1, 3.1 (2) and 3.3, we obtain
\[ \frac{1}{2} (\Delta \| \sigma \|^2) \geq \sum_{i,j} h^i_{\alpha \beta} h^j_{\alpha \beta} + n \| \sigma \|^2 - 4\| \sigma \|^2 + 4 \sum_{\alpha} N(A_{\alpha})^2 - \sum_{\alpha} N(A_{\alpha})^2 \]
\[ \geq \sum_{i,j} h^i_{\alpha \beta} h^j_{\alpha \beta} + (n - 4) \| \sigma \|^2 + \frac{4}{n} \sum_{\alpha} N(A_{\alpha})^2 - \sum_{\alpha} N(A_{\alpha})^2. \]
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\[ \begin{align*}
&= \sum h_{ijk} h_{ijk} + (n-4)\|\sigma\|^2 - \frac{(n-4)}{n} \sum N(A_a)^2 \\
& \geq \sum h_{ijk} h_{ijk} + \frac{(n-4)}{n} \|\sigma\|^2(n-\|\sigma\|^2) \geq 0, \quad \text{for } n \geq 5
\end{align*} \]

at each point. Since \( M \) compact and orientable, we obtain that \( \sum h_{ijk} h_{ijk} = 0 \). Furthermore if \( n \geq 5 \), we obtain that \( \|\sigma\|^2(n-\|\sigma\|^2) = 0 \). If \( M \) is not totally geodesic, then \( \|\sigma\|^2 = n \). Hereafter we consider the case where \( M \) is not totally geodesic. Since the second fundamental form \( \sigma \) is parallel, \( M \) is locally symmetric. Since the equality of Lemma 3.3 holds, the eigenvalues of each \( A_a \) can be written as \( \lambda^a, \ldots, \lambda^n \). If \( A_a = 0 \) for some \( a \), then from the fact that the second fundamental form \( \sigma \) is parallel and a result of J. Erbacher, the image of \( M \) is contained in some hypersphere of \( S^{n+p} \). This contradicts the assumption that the immersion is full. From the above and the equality of Lemma 3.2 holds, we have the equality of Lemma 3.1 (1). This proves that

\[ S = (n-2)g, \]

where \( g \) denotes the metric tensor of \( M \).

**CASE** \( n \geq 5 \). Since \( \sum N(A_a)^2 = \|\sigma\|^2(\sum N(A_a))^2 \), we obtain that \( (p-1) A_a \) must be zero so that \( p = 1 \). Since \( p = 1 \) and \( \|\sigma\|^2 = n \), a result of [1] implies that \( M \) must be \( M_{m,n-m} \). Furthermore \( S = (n-2)g \) shows that \( M = M_{m,m} \).

**CASE** \( n = 4 \). Since \( M \) is simply connected and locally symmetric with \( S = 2g \), from [5], \( M \) must be \( S^2(\sqrt{\frac{1}{2}}) \times S^2(\sqrt{\frac{1}{2}}), P^4_{1,0} \) or \( S^4(\sqrt{\frac{3}{2}}) \).

From [2], if \( S^s(r) \) is minimally immersed in \( S^{s+p} \), \( r = \sqrt{\frac{s(s+3)}{4}} \) for some positive integer \( s \). \( S^4(\sqrt{\frac{3}{2}}) \) can not be immersed in \( S^{s+p} \). Q. E. D.

**REMARK.** Although we can prove the theorem without use of the result of [5], it is somewhat more complicated. Furthermore we can prove the following.

Let \( M \) be an \( n \)-dimensional minimal submanifold immersed in \( S^{n+p} \) such that the immersion is full. If \( n \geq 4 \), the Ricci curvature of \( M \geq n-2 \) and the scalar curvature of \( M \) is constant, then \( M \) is locally either \( S^n \) (totally geodesic), \( M_{m,m} \) in \( S^{n+1} (n=2m) \) or \( P^s_{2/s} \) in \( S^s \).

**References**


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