Comparison theorems for
Banach spaces of solutions of $\Delta u = Pu$
on Riemann surfaces

By Takeyoshi SATÔ

(Received Aug. 26, 1977)

§ 1. Introduction.

Let $R$ be an open Riemann surface and $P$ a density on $R$, that is, a non-negative Hölder continuous function on $R$ which depends on the local parameter $z = x + iy$ in such a way that the partial differential equation

$$(1.1) \quad \Delta u = Pu, \quad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2,$$

is invariantly defined on $R$. A real valued function $f$ is said to be a $P$-harmonic function in an open set $U$ of $R$, if $f$ has continuous partial derivatives up to the order 2 and satisfies the equation (1.1) on $U$. The totality of bounded $P$-harmonic functions on $R$ is denoted by $PB(R)$. Then, $PB(R)$ is a Banach space with the uniform norm

$$(1.2) \quad \|f\| = \sup_{z \in R} |f(z)|.$$

H. L. Royden [1] studied the comparison problem of Banach space structures of $PB(R)$ for different choices of densities $P$ on a hyperbolic Riemann surface $R$ and proved the following comparison theorem: If $P$ and $Q$ are non-negative densities on $R$ such that there is a constant $c \geq 1$ with

$$(1.3) \quad c^{-1}Q \leq P \leq cQ$$

outside some compact subset of $R$, then the Banach spaces $PB(R)$ and $QB(R)$ are isomorphic. On the other hand, concerning this comparison problem M. Nakai [1] gave a different criterion for $PB(R)$ and $QB(R)$ to be isomorphic and proved the following theorem: If two densities $P$ and $Q$ on $R$ satisfy the condition

$$(1.4) \quad \int_R |P(z) - Q(z)| \{G^p(z, w_1) + G^q(z, w_0)\} \, dx \, dy < +\infty$$

for some points $w_0$ and $w_1$ in $R$, where $G^p(z, w)$ and $G^q(z, w)$ are Green’s functions of $R$ associated with (1.1) and the equation $\Delta u = Qu$ respectively, then Banach spaces $PB(R)$ and $QB(R)$ are isomorphic.
A. Lahtinen [1] considered the equation (1.1) for densities $P$ which he called acceptable densities. Acceptable densities can also have negative values, and so, $P$-harmonic functions do not obey the usual maximum principle. Lahtinen gave generalizations of Nakai's comparison theorem for acceptable densities and also showed, in Lahtinen [2], that for non-negative densities Royden's condition (1.3) is a special case of Nakai's condition (1.4). Recently, M. Nakai [4] and M. Glasner [1] gave, simultaneously, a necessary and sufficient condition for the existence of an isomorphism $T$ between $PB(R)$ and $QB(R)$ such that $|f-T(f)|$ is bounded by a potential on $R$.

$PX(R)$ is the space consisting of $P$-harmonic functions $f$ on $R$ with a certain boundedness property $X$. As for $X$ we can take $D$ to mean the finiteness of the Dirichlet integral

$$D(f) = \int_{R} \left\{ \left( \frac{\partial f}{\partial x} \right)^{2} + \left( \frac{\partial f}{\partial y} \right)^{2} \right\} dxdy < +\infty,$$

$E$ the finiteness of the energy integral

$$E(f) = D(f) + \int_{R} f^{2}(z)P(z)dxdy < +\infty,$$

$B$ the finiteness of the supremum norm (1.2), and their non-trivial combinations $BD$ and $BE$. In the connection with Royden's comparison theorem, Nakai [3] discussed whether the condition (1.3) is also sufficient for $PX(R)$ and $QX(R)$ to be isomorphic for $X=D, E, BD$ and $BE$, and he actually showed that the answer to this question is affirmative.

In this paper we consider the equation (1.1) with $P \not\equiv 0$ on $R$, and give a new boundedness property $H_{p}$ ($1 \leq p < +\infty$) to $P$-harmonic functions so that the space $PH_{p}(R)$, which consists of $P$-harmonic functions with this boundedness property, may have the comparison theorem. Hardy spaces on Riemann surfaces have been studied by M. Parreau [1], and in the general context of harmonic spaces by L. L. Naim [1]. The Hardy space for the equation (1.1), which is denoted by $PH_{p}(R)$ in this paper, falls within the framework of Naim [1]. By Naim, a $P$-harmonic function $f$ belongs to the Hardy space $PH_{p}(R)$ for the equation (1.1), if and only if $|f|^{p}$ has a $P$-harmonic majorant on $R$. We denote by $\rho_{p}$ the smallest $P$-harmonic majorant of $|f|^{p}$ on $R$, and take $H_{p}$ to mean the finiteness of the expression

$$\|f\|_{p}^{p} = \sup_{w \in R} \left\{ \frac{1}{2\pi} \int_{R} \rho_{p}(z)G^{p}(z, w)P(z)dxdy \right\}^{1/p},$$

where $G^{p}(z, w)$ is the Green function of the equation (1.1) on $R$. Then, we have that, for $1 \leq p < +\infty$,

$$PB(R) \subset PH_{p}(R) \subset PH_{p}(R).$$

In § 2, we show that, for $1 \leq p < +\infty$, $PH_{p}(R)$ is a Banach space under the
norm (1.5), and, in § 3, that $PH_p^J(R)$ is determined by the behavior of the density $P$ near the ideal boundary of $R$. In § 4, it is proved that the condition (1.3) is also sufficient for $PH_p^J(R)$ and $QH_p^J(R)$ to be isomorphic.

For the properties of $P$-harmonic functions we refer to Myrberg's fundamental works (Myrberg [1], [2]), and for the theory of Green potentials with kernel $G^p(z, w)$ to Nakai [2].

§ 2. Definition of the Banach spaces $PH_p^J(R)$.

Let $R$ be a connected Riemann surface and let $N$ be the set $\{0, 1, 2, \ldots\}$. By $\{R_n\}_{n \in N}$ we denote an exhaustion of $R$, which has the following properties:

(1) $R_n$ is a regular region, that is, an open set whose closure $\overline{R_n}$ is compact and whose relative boundary $\partial R_n$ consists of a finite number of closed analytic curves, (2) $\overline{R_n} \subset R_{n+1}$ for $n \in N$, (3) $R = \bigcup_{n=0}^{\infty} R_n$. By the solvability of Dirichlet problem on the regular region $R_n$ with continuous boundary values, for any continuous function $f$ on $\partial R_n$ there exists a unique continuous function $P^*_f$ on $\overline{R_n}$ such that $P^*_f = f$ on $\partial R_n$ and $P^*_f$ is a $P$-harmonic function on $R_n$. Let $z_0$ be a fixed point on $R_n$. Since the mapping $f \mapsto P^*_f(z_0)$ of the space of all finitely continuous functions $f$ on $\partial R_n$ is a non-negative linear functional on this space of functions on $\partial R_n$, there exists a non-negative Radon measure $\mu_{n, z_0}$ on $\partial R_n$ such that

$$\int f d\mu_{n, z_0} = P^*_f(z_0)$$

for all finitely continuous functions $f$ on $\partial R_n$. This measure is the $P$-harmonic measure on $\partial R_n$ relative to $z_0 \in R_n$ and $R_n$.

**Definition 2.1.** A $P$-harmonic function $f$ on $R$ belongs to the space $PH_p(R)$, $1 \leq p < +\infty$, if and only if there exists a constant $m(z_0)$ independent of $n \in N$ such that

$$\|f\|_{p, n, z_0} \leq m(z_0)$$

for all $n \in N$, where $z_0 \in R$ and

$$\|f\|_{p, n, z_0} = \left\{ \int |f|^p d\mu_{n, z_0} \right\}^{1/p}.$$

This space $PH_p(R)$ has been studied in the general context of harmonic spaces by Lumer-Naim [1]. Hence the results contained therein may be applicable to our studies of the space $PH_p(R)$. For convenience, some results of Naim [1] are quoted in the following. A $P$-harmonic function $f$ belongs to the space $PH_p(R)$, $1 \leq p < +\infty$, if and only if $|f|^p$ has a $P$-harmonic majorant on $R$. By this proposition the definition of $PH_p(R)$ is independent of the choice of $z_0 \in R$ and the particular exhaustion $\{R_n\}$ of $R$. Any $P$-harmonic function $f \in PH_p(R)$ is the difference of two positive $P$-harmonic functions in $PH_p(R)$,
For $1 \leq p < +\infty$, and conversely. For $1 \leq p < +\infty$, $PH_p(R)$ is a Banach space under the norm
\[ \|f\|_p^p = \sup_{n \in \mathbb{N}} \|f\|_p^n(z_0). \]
This norm equals $\{p f(z_0)\}^{1/p}$, where $pf$ denotes the smallest $P$-harmonic majorant of $|f|^p$ in $R$.

In the theory of $PH_p(R)$ we admit the case $P=0$, but we assume $P \neq 0$ on $R$ in the following. The $P$-Green function for $R_n$ is an extended real valued function $G^p(R_n, z, w)$ on $R_n \times R_n$ such that for each $w \in R_n$, (1) $G^p(R_n, z, w)$ is $P$-harmonic on $R_n \setminus \{w\}$; (2) $G^p(R_n, z, w) + \log |w-z|$ is bounded in a neighborhood of $w$; (3) $\lim_{z \to w} G^p(R_n, z, w)=0$ for every $b \in \partial R_n$. The increasing sequence $\{G^p(R_n, z, w)\}$ converges uniformly on every compact subset of $R$ to a function $G^p(z, w)$ which we call the $P$-Green function on $R$. $G^p(z, w)$ is the smallest function of $u(z, w)$ such that (1) $u(z, w)$ is a non-negative $P$-harmonic function on $R \setminus \{w\}$; (2) $u(z, w) + \log |z-w|$ is bounded in a neighborhood of $w$. For these and other properties of the $P$-Green function we refer to Myrberg [1] and [2]. An inequality which is a result of Myrberg [2] is quoted here as it is useful in the following:

\[ (2.1) \quad \int_R G^p(z, w)P(z)dx dy \leq 2\pi \]
for every $w \in R$.

Now, we make some preliminaries on $P$-superharmonic functions. For any disk $V$ on $R$ we have the $P$-harmonic measure $\mu^p_{\partial V}$ on the boundary $\partial V$ of $V$ with respect to $z \in V$ satisfying
\[ P^p_\partial(z) = \int f d\mu^p_{z, \partial V} \]
for any continuous function $f$ on $\partial V$, where $P^p_\partial$ is a continuous function on the closure $\overline{V}$ of $V$ such that $P^p_\partial = f$ on $\partial V$ and $P^p_\partial$ is $P$-harmonic on $V$. A $P$-superharmonic function $s$ on an open set of $R$ is then defined as a function with the following properties:

a) $s(z) > -\infty$ at each $z \in S$, $s(\neq plus \infty$ on any component of $S$;

b) $s$ is lower semi-continuous on $S$;

c) For any disk $V$ such that $\overline{V} \subset S$,

\[ s(z) \geq \int sd\mu^p_{z, \partial V} \]
for all $z \in V$.

If $s$ and $-s$ are $P$-superharmonic on an open set $S$ of $R$, then $s$ is $P$-harmonic on $S$.

If $-s$ is $P$-superharmonic on $S$, then $s$ is said to be $P$-subharmonic on $S$. For example, if $f$ is $P$-harmonic on an open set $S$ of $R$, then $|f|^p$, $1 \leq p < +\infty$, ...
is $P$-subharmonic on $S$, and $\max(f, 0), -\min(f, 0)$ are $P$-subharmonic on $S$. The following well-known fact is called the maximum principle and used repeatedly in proofs in this paper. Let $u$ be a $P$-subharmonic function on $G$, and $f$ a $P$-harmonic function on $G$ with continuous boundary values. If $G$ is a relative compact set of $R$ and

$$\limsup_{z \to b} u(z) \leq \limsup_{z \to b} f(z)$$

for all $b \in \partial G$, then $u < f$ on $G$ or $u \equiv f$ on $G$. This principle is a consequence of the general theory on harmonic space. In the case of a continuous $P$-subharmonic function it is given in Myrberg [3].

**Definition 2.2.** A $P$-harmonic function $f$ on a connected Riemann surface $R$ belongs to the space $\text{PH}^p(R)$, $1 \leq p < +\infty$, if and only if there exists a constant $M$ independent of $n \in \mathbb{N}$ such that

$$\int_{R_n} \| f \|^p_{\mu_n(z)} p G^p(R_n, z, w) P(z) dx dy \leq M, \quad w \in R_n,$$

for all $n \in \mathbb{N}$.

We shall see that this space $\text{PH}^p(R)$ is independent of the exhaustion $\{R_n\}$ of $R$.

From now on in this section we shall give properties of our space $\text{PH}^p(R)$ of $P$-harmonic functions on a connected Riemann surface $R$.

**Theorem 2.1.** A $P$-harmonic function $f$ on $R$ belongs to the space $\text{PH}^p(R)$, $1 \leq p < +\infty$, if and only if $|f|^p$ has a $P$-harmonic majorant $u$ on $R$ such that

$$\int_R u(z) G^p(z, w) P(z) dx dy \leq M$$

for every $w \in R$, where $M$ is a positive constant.

**Proof.** If such a majorant $u$ does exist on $R$, then for each $n \in \mathbb{N}$

$$\| f \|^p_{\mu_n(z)} = \left\{ \int |f|^p d\mu_n(x) \right\}^{1/p} \leq \{P_n(z)\}^{1/p} = \{u(z)\}^{1/p}, \quad z \in R_n,$$

that is, $f \in \text{PH}^p(R)$. Furthermore,

$$\int_{R_n} \| f \|^p_{\mu_n(z)} p G^p(R_n, z, w) P(z) dx dy \leq \int_{R_n} u(z) G^p(R_n, z, w) P(z) dx dy \leq M, \quad w \in R_n,$$
for all \( n \in \mathbb{N} \), from which it follows that \( f \) is in the space \( PH_p'(R) \).

Next, let \( f \in PH_p'(R) \). Since the sequence \( \{ \| f \|_{p,n}^p \}_{n \in \mathbb{N}} \) of \( P \)-harmonic functions is increasing, Definition 2.2 and Harnack’s principle imply that

\[
\lim_{n \to +\infty} \| f \|_{p,n}^p(z) = p_H f_p(z), \quad z \in R,
\]

is \( P \)-harmonic by Beppo-Levi’s theorem, which is denoted by \( u \). The maximum principle gives that

\[
|f(z)|^p \leq p_H f_p(z) = \| f \|_{p,n}^p(z),
\]

from which it follows that \( u \) is a \( P \)-harmonic majorant of \( |f|^p \) on \( R \). Since there exists a constant \( M \) independent of \( n \in \mathbb{N} \) such that

\[
\int_{R_n} \| f \|_{p,n}^p(z) G_p^p(R_n, z, w) P(z) dxdy \leq M, \quad w \in R_n,
\]

for all \( n \in \mathbb{N} \), it follows from Beppo-Levi’s theorem, that

\[
\int_R u(z) G_p^p(z, w) P(z) dxdy = \lim_{n \to +\infty} \int_{R_n} \| f \|_{p,n}^p(z) G_p^p(R_n, z, w) P(z) dxdy \leq M, \quad w \in R.
\]

**THEOREM 2.2.** Every \( f \in PH_p'(R) \) is the difference of two positive \( P \)-harmonic functions in \( PH_p'(R) \), and conversely.

**Proof.** Let \( f \in PH_p'(R) \). By Theorem 2.1, there is a \( P \)-harmonic majorant \( u \) of \( |f| \) on \( R \) such that

\[
\int_R u(z) G_p^p(z, w) P(z) dxdy \leq M
\]

for all \( w \in R \). The sequences

\[
\left\{ \int \max(f, 0) d\mu_{p,n}^p \right\}
\]

and

\[
\left\{ \int -\min(f, 0) d\mu_{p,n}^p \right\}
\]

are monotone increasing by the maximum principle and bounded as \( n \) increases. Then, we can define

\[
f_i(z) = \lim_{n \to +\infty} \int \max(f, 0) d\mu_{p,n}^p
\]

and
\[ f_i(z) = \lim_{n \to \infty} - \min(f, 0) d \mu_{n, i}^P \quad z \in R. \]

Here, we have, for \( i = 1, 2, \)
\[
\int_R f_1(z) G^P(z, w) P(z) dxdy \\
\leq \int_R u(z) G^P(z, w) P(z) dxdy \\
\leq M < +\infty, \quad w \in R,
\]
and
\[
f(z) = \lim_{n \to \infty} f d \mu_{n}^P \\
= f_1(z) - f_2(z), \quad z \in R.
\]

Next, we assume that
\[ f(z) = f_1(z) - f_2(z), \]
where \( f_1 \) and \( f_2 \) are positive \( P \)-harmonic functions in \( PH_1'(R) \). Let \( u_i \) be the \( P \)-harmonic majorant of \( f_i \) on \( R \), \( i = 1, 2 \), such that, for \( w \in R \),
\[
\int_R u_i(z) G^P(z, w) P(z) dxdy \leq M_i \\
< +\infty, \quad i = 1, 2.
\]
Then,
\[
|f(z)| \leq f_1(z) + f_2(z) \\
\leq u_1(z) + u_2(z), \quad z \in R,
\]
and
\[
\int_R (u_1(z) + u_2(z)) G^P(z, w) P(z) dxdy \\
\leq M_1 + M_2,
\]
for all \( w \in R \), which implies, by Theorem 2.1, that \( f \in PH_1'(R) \). Q.E.D.

We denote by \( PB(R) \) the space consisting of \( P \)-harmonic functions on \( R \) with finite supremum norms:
\[
\| f \|_R = \sup_{z \in R} |f(z)|.
\]

**Theorem 2.3.** For any finite \( 1 \leq p \leq q \), we have the inclusions
\[
P^*_B(R) \subset PH_1'(R) \subset PH_1''(R) \subset PH_1'(R).
\]

**Proof.** Let \( f \in PB(R) \). Since
\[
\| f \|_{n}^P(z) = \left\{ \int |f|^q d \mu_{n, z}^P \right\}^{\frac{1}{q}}
\]
we have that \( f \in PH_q(R) \). Moreover, the inequality (2.1) implies that, for all \( n \in \mathbb{N} \),

\[
\int_{R_n} (\| f \|_{p,n}(z))^q G^p(R_n, z, w) P(z) dx dy \\
\leq (\| f \|_n)^q \int_{R_n} G^p(R_n, z, w) P(z) dx dy \\
\leq 2\pi (\| f \|_n)^q, \quad w \in R_n.
\]

And so, we have \( f \in PH_q'(R) \), that is, \( PB(R) \subset PH_q'(R) \).

Next, we assume that \( 1 \leq p \leq q \). From the inequality

\[
|a|^p \leq 1 + |a|^q
\]

for a real number \( a \), it follows that

\[
(\| f \|_{p,n}(z))^p = \int_R |f|^p d\mu_{w,z}^{p,n} \\
\leq 1 + (\| f \|_{p,n}(z))^q,
\]

and that

\[
\int_{R_n} (\| f \|_{p,n}(z))^p G^p(R_n, z, w) P(z) dx dy \\
\leq \int_{R_n} G^p(R_n, z, w) P(z) dx dy \\
+ \int_{R_n} (\| f \|_{p,n}(z))^q G^p(R_n, z, w) P(z) dx dy \\
\leq 2\pi + \int_{R_n} (\| f \|_{n,p}(z))^q G^p(R_n, z, w) P(z) dx dy, \quad w \in R. 
\]

Therefore, we have

\[
PH_q'(R) \subset PH'_p(R). \quad \text{Q. E. D.}
\]

**Theorem 2.4.** Any \( f \) in \( PH_q'(R) \) is the difference of two positive \( P \)-harmonic functions in \( PH'_p(R) \), and conversely.

**Proof.** We consider the same functions \( f_1 \) and \( f_2 \) on \( R \) as that in the proof of Theorem 2.2, that is,

\[
f_1(z) = \lim_{n \to \infty} \int \max(f, 0) d\mu_{w,z}^{p,n},
\]

\[
f_2(z) = \lim_{n \to \infty} \int -\min(f, 0) d\mu_{w,z}^{p,n}
\]
for $z \in R$. Since $f \in PH_p'(R)$, there exists a $P$-harmonic majorant $u$ of $|f|^p$ satisfying (2.2) in Theorem 2.1. Then, Hölder’s inequality gives that, for $p$ and $q$ satisfying $1 < p < +\infty$, $1 < q < +\infty$ and $1/p + 1/q = 1$,

$$\int \max(f, 0)d\mu_{n,x} \leq \left[ \int \{\max(f, 0)\}^p d\mu_{n,x} \right]^{1/p} \left( \int d\mu_{n,x} \right)^{1/q} \leq \left( \int \max(f, 0)^p d\mu_{n,x} \right)^{1/p} \leq (\int u d\mu_{n,x})^{1/p} \leq \{u(z)\}^{1/p},$$

that is, $f_1(z)^p \leq u(z)$ on $R$. And, similarly, we have $f_2(z)^p \leq u(z)$ on $R$. Then, we complete the proof of the first assertion.

Let $f = f_1 - f_2$, where $f_1$ and $f_2$ are positive $P$-harmonic functions in $PH_p'(R)$. By Theorem 2.1 there exists $P$-harmonic majorants $u_1$ and $u_2$ of $f_1^p$ and $f_2^p$ on $R$, respectively, which satisfy the condition (2.2) in Theorem 2.1. Then, the inequality

$$(a+b)^p \leq 2^p(a^p+b^p), \quad 1 \leq p < +\infty,$$

gives

$$|f|^p \leq (f_1+f_2)^p \leq 2^p(f_1^p+f_2^p) \leq 2^p(u_1+u_2),$$

and

$$\int_R (u_1(z)+u_2(z))G^p(z,w)P(z)dxdy \leq M + M$$

for all $w \in R$, where $M$ is a constant independent of $w \in R$. Therefore, Theorem 2.1 implies $f \in PH_p'(R)$.

**THEOREM 2.5.** Let $R$ be a connected Riemann surface on which $P \equiv 0$. And, let

$$(2.4) \quad \|f\|_p^p = \sup_{w \in R} \left\{ \lim_{n \to +\infty} \frac{1}{2\pi} \int_{R_n} \|f\|_{p_n(z)}^p G^p(R_n, z, w)P(z)dxdy \right\}^{1/p}$$

for $f \in RH_p'(R)$. Then, for $1 \leq p < +\infty$, $PH_p'(R)$ is a Banach space under the norm $\|f\|_p$, $f \in PH_p'(R)$. This norm equals

$$(2.5) \quad \sup_{w \in R} \left\{ \frac{1}{2\pi} \int_{R^p} f(z)G^p(z, w)P(z)dxdy \right\}^{1/p},$$

where $\_f$ denotes the smallest $P$-harmonic majorant of $|f|^p$ in $R$. 

PROOF. The function $u$ defined by (2.3) in the proof of Theorem 2.1, that is,

$$u(z) = \lim_{n \to +\infty} \left\{ \| f \|^p_{p,n}(z) \right\}^p, \quad z \in R,$$

is the smallest $P$-harmonic majorant of $|f|^p$ in $R$, since, for any $P$-harmonic majorant $s$ of $|f|^p$ in $R$, we have

$$\left\{ \| f \|^p_{p,n}(z) \right\}^p = P^*|f|^p(z) \leq P^*s(z) = s(z), \quad z \in R_n,$$

which gives $u(z) \leq s(z)$ on $R$. By Definition 2.2 and $p_f = u$, Lebesgue's monotone convergence theorem shows that

$$\frac{1}{2\pi} \int_{R^p} f(z) G_p(z, w) P(z) dx dy = \lim_{n \to +\infty} \frac{1}{2\pi} \int_{R^p_n} \left\{ \| f \|^p_{p,n}(z) \right\}^p G_p(R_n, z, w) P(z) dx dy,$$

from which the expression (2.5) of $\| f \|^p_p$ follows.

Next, we have to show that $PH_p(R)$, $1 \leq p < +\infty$, is a vector space with respect to the usual definitions of addition and scalar multiplication of real numbers, and that the non-negative real valued function (2.4) is a norm on $PH_p(R)$. Minkowski's inequality gives that, for $f$ and $g$ in $PH_p(R)$,

$$\left[ \int_{R^p_n} \left\{ \| f \|^p_{p,n}(z) \right\}^p G_p(R_n, z, w) P(z) dx dy \right]^{1/p} \leq \left[ \int_{R^p_n} \left\{ \| f \|^p_{p,n}(z) \right\}^p G_p(R_n, z, w) P(z) dx dy \right]^{1/p} + \left[ \int_{R^p_n} \left\{ \| g \|^p_{p,n}(z) \right\}^p G_p(R_n, z, w) P(z) dx dy \right]^{1/p},$$

which implies that $f + g \in PH_p(R)$ and

$$\| f + g \|^p_p \leq \| f \|^p_p + \| g \|^p_p.$$

It is clear that, for $f \in PH_p(R)$ and a real number $\alpha$, $\alpha f \in PH_p(R)$ and

$$\| \alpha f \|^p_p = |\alpha| \| f \|^p_p.$$

If $f \in PH_p(R)$ satisfies the condition $\| f \|^p_p = 0$, then the smallest $P$-harmonic majorant $p_f$ of $f$ satisfies that $p_f = 0$ everywhere on $R$, since $P^* = 0$ on $R$. So, $f = 0$ everywhere on $R$.

To prove that $PH_p(R)$ is complete with respect to the norm (2.4), let $\{ f_j \}$ be a Cauchy sequence in $PH_p(R)$ with respect to the norm (2.4). Then, we can find a subsequence $\{ f_{j(i)} \}$, $j(1) < j(2) < \cdots$, of $\{ f_j \}$ such that

$$\| f_{j(i+1)} - f_{j(i)} \|^p_p < 1/2^i, \quad i = 1, 2, \ldots.$$
Hölder's inequality and the inequality (2.1) give that, for \( p > 1 \),
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \left\{ p \left( f_{j(t+1)} - f_{j(t)}(z) \right)^{1/p} G^p(z, w) P(z) dxdy \right\} \leq \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ \frac{1}{p} \left( f_{j(t+1)} - f_{j(t)}(z) \right) G^p(z, w) P(z) dxdy \right\}^{1/p} = \| f_{j(t+1)} - f_{j(t)} \|_p^p,
\]
which is evident for \( p = 1 \). Therefore, since
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \sum_{i=1}^{k} \left\{ p \left( f_{j(t+1)} - f_{j(t)}(z) \right)^{1/p} G^p(z, w) P(z) dxdy \right\} \leq \sum_{i=1}^{k} 1/2^i < 1
\]
for every positive integer \( k \), Lebesgue's monotone convergence theorem implies that the series
\[
\sum_{l=1}^{\infty} \{ p \left( f_{j(t+1)} - f_{j(t)}(z) \right)^{1/p} \}
\]
converges almost everywhere on the support of \( P \).

Let \( z_0 \) be a point of the support of the density \( P \) at which (2.6) converges. Then, from the inequality
\[
\| f_{j(t)} - f_{j(k)} \|_p^p = \| \sum_{l=k}^{t-1} \left( f_{j(t+1)} - f_{j(l+1)} \right) \|_p^p \leq \sum_{l=k}^{t-1} \left( f_{j(t+1)} - f_{j(l+1)} \right)^{1/p}
\]
for \( k < l \), it follows that the sequence \( \{ f_{j(t)} \} \) is a Cauchy sequence in \( PH_p(\mathbb{R}) \), for the series (2.6) converges at \( z_0 \). So, there exists a function \( f \) in \( PH_p(\mathbb{R}) \) such that
\[
\lim_{t \to \infty} \| f_{j(t)} - f \|_p^p = 0,
\]
which implies that the sequence \( \{ f_{j(t)} \} \) converges, uniformly on every compact subset of \( \mathbb{R} \), to \( f \) (L. L. Naim [1]).

We now have to prove that \( f \) is contained in \( PH_p(\mathbb{R}) \) and
\[
\lim_{j \to \infty} \| f_j - f \|_p^p = 0.
\]
Since
\[
f_{j(k)} = \sum_{i=1}^{k-1} \left( f_{j(t+1)} - f_{j(t)} \right) + f_{j(t)},
\]
Fatou's lemma gives that
\[ \left[ \frac{1}{2\pi} \int_{R^n} \left\{ \| f - f_{j(t)} \|_p^p(z) \right\}^p G^p(R_n, z, w) P(z) dx dy \right]^{1/p} \leq \liminf_{k \to +\infty} \left[ \frac{1}{2\pi} \int_{R^n} \left\{ \| f_{j(k)} - f_{j(t)} \|_p^p(z) \right\}^p \times G^p(R_n, z, w) P(z) dx dy \right]^{1/p} \leq \lim \inf_{k \to +\infty} \| f_{j(k)} - f_{j(t)} \|_p^p \leq \sum_{i=1}^{\infty} \| f_{j(t+1)} - f_{j(t)} \|_p^p \leq \sum_{i=1}^{\infty} 1/2^i = 1/2^{t-1}, \]

and so,

\[ \| f - f_{j(t)} \|_p^p < 1/2^{t-1}. \]

We can conclude from this inequality that \( f - f_{j(t)} \) is in \( PH_p(R) \). Hence, \( f \) is in \( PH_p(R) \), since

\[ f = (f - f_{j(t)}) + f_{j(t)}. \]

And, furthermore it follows, from (2.7), that

\[ \lim_{t \to +\infty} \| f - f_{j(t)} \|_p^p = 0, \]

which gives that

\[ \lim_{t \to +\infty} \| f - f_j \|_p^p = 0, \]

for \( \{f_j\} \) is a Cauchy sequence in \( PH_p(R) \). Q. E. D.

It will be necessary to consider a disconnected Riemann surface in § 3 and § 4. Let

\[ R = \bigcup_{k=1}^{K} W^k \]

be the decomposition of \( R \) into connected components \( W^k \) of \( R \). We can assume, without loss of generality, that the density \( P \) on \( R \) satisfies \( P \equiv 0 \) on \( W^1, W^2, \ldots, W^L, 1 \leq L \leq K \), and \( P \equiv 0 \) on \( W^{L+1}, W^{L+2}, \ldots, W^K \). Since \( P \equiv 0 \) on \( W^k, L < k \leq K \), \( PH_p(W^k), L < k \leq K \), is the space of harmonic functions on \( W^k \) such that \( |f|^p \) has a harmonic majorant on \( W^k \), that is, \( PH_p(W^k), L < k \leq K \), is the Hardy space of harmonic functions on \( W^k \). This space of harmonic functions on \( W^k \) is denoted by \( H_p(W^k) \). It is a result of Parreau [1] that the space \( H_p(W^k) \) is a Banach space under the norm

\[ \| f \|_p = \left\{ \int f(z_0) \right\}^{1/p}, \quad f \in H_p(W^k), \]

where \( z_0 \) is a point in \( W^k \). Now, we define the space \( PH_p(R) \) for the disconnected Riemann surface \( R \) as follows.
DEFINITION 2.3. A P-harmonic function $f$ on the disconnected Riemann surface $R$ belongs to the space $PH_p^p(R)$, $1 \leq p < +\infty$, if and only if each restriction $f|W^k$ to $W^k$ of $f$ belongs to $PH_p^p(W^k)$ or $H_p(W^k)$ according as $1 \leq k \leq L$ or $L < k \leq K$.

THEOREM 2.6. Let $R$ be the disconnected Riemann surface on which $p \neq 0$. And, let

$$
\|f\|_p^p = \sum_{k=1}^L \|f| W^k\|_p^p + \sum_{k=L+1}^K \|f| W^k\|_p^p
$$

for $f \in PH_p^p(R)$. Then, for $1 \leq p < +\infty$, $PH_p^p(R)$ is a Banach space under the norm (2.8). This norm equals

$$
\sum_{k=1}^L \sup_{w \in W^k} \left\{ \frac{1}{2\pi} \int_{W^k} p(f| W^k)(z) G_p(W^k, z, w) P(z) dxdy \right\}^{1/p} + \sum_{k=L+1}^K \left\{ p(f| W^k)(z^k) \right\}^{1/p},
$$

where $p(f| W^k)$, $1 \leq k \leq K$, denotes the smallest P-harmonic majorant of $|f| W^k|^p$ on $W^k$ and $z^k$, $L < k \leq K$, is a point in $W^k$.

PROOF. This is clear by the preceding lemma. Q.E.D.

In the following of this section we consider the relation between two Banach spaces $PH_p^p(R)$ and $PH_p^q(R)$ under the assumption that the density $P$ vanishes outside a compact subset of the connected Riemann surface $R$.

LEMMA 2.7. If the density $P$ vanishes outside a compact subset of $R$, then $PH_p^p(R) = PH_p^q(R)$ and there exists a positive constant $C$ such that

$$
\|f\|_p^p \leq C \|f\|_q^q
$$

for every $f \in PH_p^p(R)$.

PROOF. We assume that $P$ vanishes outside a compact subset $K$ of $R$. Let $z_0$ be a point of $R$ with $z_0 \in K$. Then, there exists, by Harnack's theorem (Myrberg [1]), a constant $c$ such that

$$
pf(z) \leq c \times pf(z_0)
$$

for every $z \in K$ and every $f \in PH_p^q(R)$. Therefore, the inequality (2.1) gives that

$$
\frac{1}{2\pi} \int_R pf(z) G_p(z, w) P(z) dxdy
$$

$$
= \frac{1}{2\pi} \int_K pf(z) G_p(z, w) P(z) dxdy
$$

$$
\leq \frac{1}{2\pi} c \times pf(z_0) \int_R G_p(z, w) P(z) dxdy
$$

$$
\leq c \times pf(z_0),
$$

and so,
\[ \|f\|_p^p \leq (c)^{1/p} \times \|f\|_p^p, \]
which completes the proof. \( \text{Q. E. D.} \)

**Theorem 2.6.** If the density \( P \) vanishes outside a compact subset of \( R \), then the Banach space \( (PH_p(R), \| \cdot \|_p^p) \) is isomorphic to the Banach space \( (PH_p(R), \| \cdot \|_p) \).

**Proof.** The identity map of \( (PH_p(R), \| \cdot \|_p^p) \) onto \( (PH_p(R), \| \cdot \|_p) \) is a one-to-one continuous linear transformation and so must be an isomorphism by the open mapping theorem. \( \text{Q. E. D.} \)

### §3. The structure of \( PH_p(R) \).

Let \( W \) be a connected or disconnected open subset of \( R \) whose complement is a regular region. Hereafter we always use \( W \) for such a subset of \( R \). To show that the Banach space structure of \( PH_p(R) \) is determined by the behavior of the density \( P \) on a neighborhood of the ideal boundary of \( R \), we define the subset \( PH_p(W; \partial W) \) of \( PH_p(R) \) as follows.

**Definition 3.1.** \( PH_p(W; \partial W), 1 \leq p < +\infty \), is the class of all functions \( f \) in \( PH_p(W) \) such that there exists a continuous extension of \( f \) to the closure \( \overline{W} \) of \( W \) whose restriction to the boundary \( \partial W \) of \( W \) vanishes.

Then, \( PH_p(W; \partial W) \) is a vector space with respect to the usual definitions of addition and scalar multiplication of real numbers. And, \( PH_p(W; \partial W) \) is a subspace of the Banach space \( PH_p(W) \) with the norm (2.8) in Theorem 2.6:

**Theorem 3.1.** \( PH_p(W; \partial W) \) is a closed linear subspace of \( PH_p(W) \).

**Proof.** Let \( f \in PH_p(W; \partial W) \) be the limit of a sequence \( \{f_n\} \) in \( PH_p(W; \partial W) \):

\[ \lim_{n \to \infty} \|f - f_n\|_p^p = 0. \]

It is sufficient to show that \( f|W^k \) has a continuous extension to \( \overline{W}^k \) whose restriction to \( \partial W^k \) vanishes for each connected component \( W^k \) of \( W \). If \( P \neq 0 \) on \( W^k \), then there exists a subsequence \( \{f_{n(i)}\} \) of \( \{f_n\} \) which converges, uniformly on every compact subset of \( W^k \), to \( f \), by the proof of Theorem 2.5. If \( P \equiv 0 \) on \( W^k \), the existence of such a subsequence \( \{f_{n(i)}\} \) follows from the fact

\[ \lim_{n \to \infty} \|f - f_n|W^k\|_p^p = 0. \]

Let \( G^k \) be a regular region which contains the boundary of \( W^k \), and let \( w \) be a continuous function on the closure of \( G^k \cap W^k \) such that \( w \) is \( P \)-harmonic on \( G^k \cap W^k \) and \( w|\partial G^k = m^k \), \( w|\partial W^k = 0 \), where

\[ m^k = \sup_{z \in G^k \cap W^k} |f|W^k(z)| + 1. \]

Then, by the maximum principle we have that

\[ |f_{n(i)}(z)| \leq w(z), \quad z \in G^k \cap W^k \]
for sufficiently large \(i \in \mathbb{N}\), and so,

\[
|f(z)| = \lim_{i \to \infty} |f_{n,i}(z)| \leq w(z), \quad z \in G^k \cap W^k.
\]

This shows that \(\lim_{z \to b} f(z) = 0\) for all \(b \in \partial W^k\), that is, if we extend \(f\) on \(\partial W^k\) so that \(f(b) = 0\) for \(b \in \partial W^k\), then \(f\) belongs to \(PH^p(W; \partial W)\), which completes the proof. Q. E. D.

**Lemma 3.2.** Let \(f\) be in \(PH^p(W; \partial W)\). Then, the smallest P-harmonic majorant \(p_f\) of \(|f|^p\) has a continuous extension to \(\overline{W}\) whose restriction to \(\partial W\) vanishes.

**Proof.** It is sufficient to prove only that \(p_f|W^k\) have this property. The sequence \(\{\|f\|W^k\|p_n(z)\}^{p}\), which is a monotone increasing sequence of P-harmonic functions on \(R_n \cap W^k\), converges to \(p_f|W^k\). Harnack's principle implies that the convergence is locally uniform in \(W^k\). Let \(G^k\) be the same subset of \(R\) as that in Theorem 3.1, and let \(w\) be the P-harmonic function on \(G^k \cap W^k\) which have a continuous extension to the closure of \(G^k \cap W^k\) such that \(w|\partial W^k = 0\) and \(w|\partial G^k = 1\). Then, by the same way as that in the proof of Theorem 3.1, we can show that

\[
\{\|f\|W^k\|p_n(z)\} \leq \beta^k w(z), \quad z \in W^k \cap G^k,
\]

for sufficiently large \(n \in \mathbb{N}\), where

\[
\beta^k = \sup_{z \in \partial W^k \cap W^k} p_f(z).
\]

Therefore,

\[
p_f|W^k(z) \leq \beta^k w(z), \quad z \in W^k \cap G^k,
\]

which implies the conclusion. Q. E. D.

In Rodin and Sario [1] they discussed the problem of finding on a given harmonic space a harmonic function which imitates the behavior of a given harmonic function on a neighborhood of the ideal boundary of the harmonic space. We quote from Chapter VII of Rodin and Sario [1] the method of finding a P-harmonic function which imitates the behavior of a given P-harmonic function on a neighborhood of the ideal boundary of the connected Riemann surface \(R\). This problem of finding such a P-harmonic function on \(R\) can be stated as the following: Given a continuous function \(f\) on the closure \(\overline{W}\) of \(W\) which is P-harmonic on \(W\), find a P-harmonic function \(F\) on \(R\) with

\[
\sup_{z \in \overline{W}} |F(z) - f(z)| < +\infty,
\]

where \(W\) is a neighborhood of the ideal boundary of \(R\): in particular, an open subset of \(R\) whose complement is a regular region of \(R\).

Let \(\{R_n\}\) be an exhaustion of \(R\) with \(\partial R_n \subset (W - \partial W)\). Then, we can find a unique continuous function \(B_n(f)\) on the closure of \(R_n \cap (W - \partial W)\) which is
\[ P\text{-harmonic on } \mathbb{R}^n \cap (W - \partial W) \text{ and which takes the boundary values } f \text{ and } 0 \text{ on the boundaries } \partial W \text{ and } \partial \mathbb{R}^n, \text{ respectively. Since } \lim_{n \to \infty} B_n(f) \text{ exists, an operator } f \mapsto B(f) \text{ from the space of all continuous functions on } \partial W \text{ into the space of continuous functions on } \overline{W} \text{ which is } P\text{-harmonic on } W - \partial W \text{ is defined by} \]
\[ B(f) = \lim_{n \to \infty} B_n(f). \]

The operator \( B \) has the following properties:

(B1) \( B(f + g) = B(f) + B(g), \quad B(cf) = cB(f) \),

(B2) \( B(f)\mid_{\partial W} = f \),

(B3) \( \min(0, \min_{\partial W} f) \leq B(f) \leq \max(0, \max_{\partial W} f) \),

where \( f \) and \( g \) are continuous functions on \( \partial W \) and \( c \) is a real number.

Since the density \( P \) of our equation (1.1) does not vanish constantly, the harmonic space defined by the equation (1.1) is hyperbolic, that is, \( B(1) \not= 1 \) for some choice of \( W \subset \mathbb{R} \), or there is an open set in \( \mathbb{R} \) on which the constant function 1 is not \( P\)-harmonic. Therefore, as a special case of principal function problem solved by Nakai, we have the following existence theorem; Let \( f \) be a continuous function on \( \overline{W} \) which is \( P\)-harmonic on \( W \). Then there always exists a unique \((f, B)\)-principal function, that is, a \( P\)-harmonic function \( F \) on \( \mathbb{R} \) with
\[ B(F)\mid_{\partial W - f} = F\mid_{W - f} \text{ on } W. \]

By reformulation this theorem we obtain the complete solution of the above problem.

To show that the Banach spaces \( PH^p(R) \) and \( PH^p(W; \partial W) \) are isomorphic we define an operator \( \lambda^F \) as follows. Let \( P(R) \) be the space of all \( P\)-harmonic function on \( R \). And, consider the linear space \( P(W; \partial W) \) of continuous functions on \( \overline{W} \) which are \( P\)-harmonic on \( W \) and whose restriction to \( \partial W \) vanish constantly.

**Definition 3.2.** We define an operator \( \lambda^F \) by
\[ \lambda^F(f) = \lim_{n \to \infty} P^f \]
for \( f \in P(W; \partial W) \) which is the difference of two non-negative functions in \( P(W; \partial W) \), where \( P^f \) is the solution of Dirichlet problem of the equation (1.1) with the boundary value \( f \) on \( \partial R_n \).

To see that the operator \( \lambda^F \) is well-defined for such a \( f \) in \( P(W; \partial W) \), let
\[ f = f_1 - f_2, \quad f_i \in P(W; \partial W), \quad f_i \geq 0, \quad i = 1, 2. \]

We can find, by the existence theorem of the principal function problem, \( P\)-harmonic functions \( F_i \), \( F_i \) defined on \( R \) satisfying
\[ \sup_{z \in W} |F_i(x) - f_i(z)| < +\infty, \quad i = 1, 2. \]

These supremums are denoted by \( m_1 \) and \( m_2 \), respectively. Since
Banach spaces of solutions of $\Delta u = Pu$

$F_i + m_i \geq P_{i_1}$ on $R_n$ ($i = 1, 2$)

for every $n \in \mathbb{N}$ and the sequences $\{P_{i_1}\}$ and $\{P_{i_2}\}$ are monotone increasing sequences of $P$-harmonic functions, the $\lim_{n \to \infty} P_{i_1}$ ($i = 1, 2$) is a $P$-harmonic function by Harnak's theorem. Therefore, we have

$$\lim_{n \to \infty} P_i = \lim_{n \to \infty} P_{i_1} - \lim_{n \to \infty} P_{i_2},$$

that is, $\lambda f(f)$ is well-defined for any difference $f = f_1 - f_2$ of two non-negative functions in $P(W; \partial W)$ and is a $P$-harmonic function on $R$.

This operator $\lambda f$ is referred to as the canonical extension, and was defined by Nakai [3] on the smaller domain than that of our definition. The domain in his definition was the class $PB(W; \partial W)$ of bounded continuous functions on $W$ $P$-harmonic on $W$ and vanishing on $\partial W$.

Since the $P$-Green function $G^P(z, W)$ is strictly positive, symmetric and continuous on $R \times R$ and is finite unless $z = w$, $G^P(z, w)$ is taken as a kernel in the sense of potential theory. If $\mu$ is a measure on $R$ and

$$G^P(z, \mu) = \int_R G^P(z, w) d\mu(w)$$

is $P$-superharmonic on $R$, then $G^P(z, \mu)$ is called the $P$-Green potential of $\mu$. The $P$-Green potentials are quite similar to the harmonic Green potentials. Since the potential theoretic method is a powerful tool for the study of the operator $\lambda f$ and is extensively used in this section, we list some important potential theoretic principles in the following. The theory of $P$-Green potentials is developed in Nakai [2].

Frostman's Maximum Principle. If the inequality $G^P(z, \mu) \leq 1$ holds on the compact support $S_\mu$ of $\mu$, then the same inequality holds on the whole space $R$.

Equilibrium Principle. For an arbitrary compact subset $K$ of $R$ there always exists a unique measure called equilibrium measure of $K$ satisfying $S_\mu \subseteq K$ and $G^P(z, \mu) = 1$ on $K$ except for a subset of $\partial K$ of capacity zero and $G^P(z, \mu) \leq 1$ on $R$.

To show that the range $\lambda f(PH^P(W; \partial W))$ of $\lambda f$ is contained in $PH^P(R)$, we shall prepare three lemmas.

Lemma 3.3. Let $S$ and $T$ be open subsets of $R$ and $H$ a non-negative function on $S \times T$. If (a) for each $w \in T$, $H(\cdot, w)$ is continuous on $S$, (b) for each $z \in S$, $H(z, \cdot)$ is $P$-harmonic on $T$ and (c)

$$h(w) = \int_S H(z, w) d\mu(z) < +\infty$$

for each $w \in T$, then $h$ is $P$-harmonic on $T$.

Proof. It can be shown that $H(z, w)$ is a non-negative measurable function on $S \times T$ to which Fubini's theorem can be applied. Then, for any disk
V such that $\overline{V} \subseteq T$

$$\int_{\partial V} h d \mu_w^V = \int_{\partial V} \left\{ \int_{\partial V} H(z, \cdot) d\mu_w^V \right\} d\mu(z),$$

where $\mu_w^V$ is the $P$-harmonic measure with respect to $V$ and $w \in V$. This shows that $h$ is $P$-harmonic on $T$. Q.E.D.

The following lemma gives a relation between $P$-Green's potentials for different regions, when one is a subset of the other. For the harmonic case, this fact is stated in Helmes [1]. So we only restate it for our case.

**Lemma 3.4.** Let $S$ and $T$ be regular regions such that $S \supseteq T$, and let $\mu$ be a measure on $S$ such that $\mu(S - T) = 0$ and $G^P(S, z, \mu)$ is a finite $P$-Green's potential. Then, there is a non-negative $P$-harmonic function $h$ on $T$ which satisfies

$$G^P(S, z, \mu) = G^P(T, z, \mu|T) + h(z)$$

on $T$, where $\mu|T$ is the restriction of $\mu$ on $T$ and $G^P(S, z, w)$ is the $P$-Green's function of $S$.

**Proof.** For $z, w \in T$ with $z \neq w$, let

$$H(z, w) = G^P(S, z, w) - G^P(T, z, w),$$

which is positive. Then, for each $z \in T$, $H(z, w)$ is a $P$-harmonic function on $T$, since $z$ is a removable singular point, and so, $H(z, \cdot)$ is a continuous function for each $z \in T$. Also, $H(\cdot, w)$ is a $P$-harmonic function for each $w \in T$, for $H(z, w)$ is symmetric. Since $G^P(S, z, \mu) \geq G^P(T, z, \mu|T)$ on $T$ by $G^P(S, z, w) \geq G^P(T, z, w)$ on $T \times T$,

$$G^P(S, z, \mu) - G^P(T, z, \mu|T) = \int_{\partial V} H(z, w) d\mu(w) < +\infty,$$

where the last integral is a $P$-harmonic function on $T$ by the preceding lemma. Q.E.D.

Let $W$ be an open subset of $R$ whose complement is a regular region. We assume that $P \equiv 0$ on $W^1, W^2, \ldots, W^L, (1 \leq L \leq K)$ and $P \equiv 0$ on $W^{L+1}, W^{L+2}, \ldots, W^K$, where

$$W = \bigcup_{i=1}^{K} W^i$$

is the decomposition of $W$ into connected components $W^1, W^2, \ldots, W^K$.

**Lemma 3.5.** If a nonnegative $P$-harmonic function $f$ in $P(W; \partial W)$ satisfies that, for every $i, 1 \leq i \leq L$, 

$$\sup_{w \in W^i} \int_{W^i(z)} f(z) d\mu(z) = +\infty,$$

then
PROOF. Let \( \{ R_n \} \) be an exhaustion of \( R \) such that \( \partial R_0 \subseteq W \). Then, since the sequence \( \{ P_f \} \) converges increasingly to \( \lambda^w_P(f) \) on \( R \), the maximum principle gives that

\[
P_f \leq \max_{\partial W} \lambda^w_P(f) + f
\]
on \( R_n \cap W^i \) for each \( n \in \mathbb{N} \). Therefore, for \( 1 \leq i \leq L \),

\[
\int_{R_n \cap W^i} P_f(z) G^P(R_n \cap W^i, z, w) P(z) dx dy 
\leq \max_{\partial W} \lambda^w_P(f) \times \int_{R_n \cap W^i} G^P(R_n \cap W^i, z, w) P(z) dx dy 
+ \int_{R_n \cap W^i} f(z) G^P(R_n \cap W^i, z, w) P(z) dx dy
\]

\[
\leq 2\pi \times \max_{\partial W} \lambda^w_P(f) + \sup_{w \in W^i} \int_{W^i} f(z) G^P(W^i, z, w) P(z) dx dy
\]

\[
< +\infty .
\]

Let

\[
M^i = \sup_{w \in W^i} \int_{W^i} \lambda^w_P(f) | W^i(z) G^P(W^i, z, w) P(z) dx dy .
\]

Then, Lebesgue's monotone convergence theorem gives that

\[
\int_{W^i} \lambda^w_P(f) | W^i(z) G^P(W^i, z, w) P(z) dx dy = \lim_{n \to \infty} \int_{R_n \cap W^i} P_f(z) G^P(R_n \cap W^i, z, w) P(z) dx dy
\]

\[
\leq 2\pi \times \max_{\partial W} \lambda^w_P(f) + \sup_{w \in W^i} \int_{W^i} f(z) G^P(W^i, z, w) P(z) dx dy ,
\]

from which it follows that \( M^i < +\infty , \ 1 \leq i \leq L \).

To show that the integral

\[
\int_R \lambda^w_P(f)(z) G^P(z, w) P(z) dx dy
\]
is a \( P \)-Green's potential, that is, \( \equiv +\infty \), let \( \alpha \) be a number such that

\[
\sup_{z \in \partial R_0} G^P(z, w_0) < \alpha ,
\]

and let

\[
\beta^i = \inf_{z \in \partial R_0 \cap W^i} G^P(W^i, z, w_0) ,
\]

where \( w_0 \) is a fixed point in \( (W^i - \partial W^i) \cap R_0 \). Since the sequence \( \{ G^P(R_n, z, w) \} \) converges increasingly to \( G^P(z, w) \) on \( R \), we have
for every \( n \in \mathbb{N} \). Then, the maximum principle gives that
\[
G^p(R_n, z, w_0) \leq \delta^i G^p(W^i, z, w_0)
\]
on \((R_n - R_0) \cap W^i\), where \( \delta = \alpha / \beta^i \). So, we have
\[
G^p(z, w_0) = \lim_{n \to \infty} G^p(R_n, z, w_0)
\leq \delta^i G^p(W^i, z, w_0)
on \((R - R_0) \cap W^i\). Since (3.1) and (3.3) give that
\[
\int_{(R - R_0) \cap W^i} \lambda^p_f \, dz \leq \delta^i \times \sup_{w \in W^i} \int_{(R - R_0) \cap W^i} \lambda^p_f \, dz
\leq \delta^i M^i < +\infty,
\]
which shows that
\[
\int_{(R - R_0) \cap W^i} \lambda^p_f \, dz = \sum_{i=1}^{L} \int_{(R - R_0) \cap W^i} \lambda^p_f \, dz
\]
is a \( P \)-Green potential. Then,
\[
\int_{R - R_0} \lambda^p_f \, dz = \sum_{i=1}^{L} \int_{R - R_0} \lambda^p_f \, dz
\]
is a \( P \)-Green potential. And, since
\[
\int_{R_0} \lambda^p_f \, dz \leq \sup_{R_0} \lambda^p_f \times \int_{R} G^p(z, w) \, dz
\leq 2\pi \times \sup_{R_0} \lambda^p_f \}
< +\infty,
\]
the integral (3.2) is a \( P \)-Green potential.

To show that the \( P \)-Green potential (3.2) is finite everywhere on \( R \), let \( w \) be any point in \( R \), and let \( V \) be a disc with center at \( w \). Then, since the \( P \)-Green potential
\[
\int_{R - V} \lambda^p_f \, dz \leq \sup_{R_0} \lambda^p_f \times \int_{R} G^p(z, w) \, dz
\]
is \( P \)-harmonic on \( V \): continuous on \( V \), the inequality
implies that the $P$-Green potential (3.2) is finite everywhere on $R$.

The integral
\[
\int_{(R_n-R_0)\cap W^t}\lambda^P(f)(z)G^P(R_{n+1}, z, w)P(z)dxdy, \quad 1 \leq i \leq L,
\]
is a finite $P$-Green potential on $R_{n+1}$, for this integral is smaller than the integral (3.2). So, Lemma 3.4 implies that there exists a $P$-harmonic function $u_n^i$ on $W^t\cap R_{n+1}$ such that

\[
(3.4) \int_{(R_n-R_0)\cap W^t}\lambda^P(f)(z)G^P(R_{n+1}, z, w)P(z)dxdy \\
= \int_{(R_n-R_0)\cap W^t}\lambda^P(f)(z)G^P(R_{n+1}\cap W^t, z, w)P(z)dxdy + u_n^i(w)
\]
for $w \in W^t\cap R_{n+1}$. Since $u_n^i|\partial R_{n+1}\cap W^t=0$ and, for any $w_0 \in \partial W^t$,
\[
u_n^i(w_0)=\int_{(R_n-R_0)\cap W^t}\lambda^P(f)(z)G^P(R_{n+1}, z, w_0)P(z)dxdy \\
\leq \sup_{w_0 \in \partial W^t}\int_{R-R_0}\lambda^P(f)(z)G^P(z, w)P(z)dxdy \\
< +\infty,
\]
denoting by $\varepsilon^t$ the above supremum the maximum principle gives
\[
u_n^i \leq \varepsilon^t \quad \text{on} \quad R_{n+1}\cap W^t.
\]
Since, by (3.3) and (3.4), the Lebesgue's monotone convergence theorem implies that
\[
\int_{(R-R_0)\cap W^t}\lambda^P(f)(z)G^P(z, w)P(z)dxdy \\
= \lim_{n \to +\infty}\int_{(R_n-R_0)\cap W^t}\lambda^P(f)(z)G^P(R_{n+1}, z, w)P(z)dxdy \\
= \int_{(R-R_0)\cap W^t}\lambda^P(f)(z)G^P(W^t, z, w)P(z)dxdy + \lim_{n \to +\infty}u_n^i(w) \\
\leq M^t + \varepsilon^t, \quad w \in W^t,
\]
the Frostman's maximum principle shows that the inequality
\[
\int_{(R-R_0)\cap W^i} \lambda^W_p(f(z)G^P(z, w)P(z)dxdy \\
\leq M^i + \varepsilon^i
\]
holds on \( R \), for the support of the measure of the \( P \)-Green potential
\[
\int_{(R-R_0)\cap W^i} \lambda^W_p(f(z)G^P(z, w)P(z)dxdy
\]
is contained in \( W^i \). Therefore, we have
\[
\int_{R} \lambda^W_p(f(z)G^P(z, w)P(z)dxdy = \sum_{i=1}^{L} \int_{(R-R_0)\cap W^i} \lambda^W_p(f(z)G^P(z, w)P(z)dxdy \\
+ \int_{R_0} \lambda^W_p(f(z)G^P(z, w)P(z)dxdy
\]
\[
\leq \sum_{i=1}^{L} (M^i + \varepsilon^i) + 2\pi \times \sup_{R_0} \lambda^W_p(f)
\]
for every \( w \in R \), which completes the proof. Q. E. D.

**THEOREM 3.6.** \( \lambda^W_p(PH^p_p(W; \partial W)) \subset PH^p_p(R) \), \( 1 \leq p < +\infty \).

**PROOF.** Let \( f \) be in \( PH^p_p(W; \partial W) \). Theorem 2.5 states that the smallest \( P \)-harmonic majorant \( \phi(f|W^i) \) of \( |f|W^i|p \) on \( W^i \) satisfies
\[
\sup_{w \in W^i} \int_{W^i} \phi(f|W^i)(z)G^P(W^i, z, w)P(z)dxdy < +\infty,
\]
for \( i, 1 \leq i \leq L \). By Definition 3.2 and Lemma 3.2, the maximum principle shows that
\[
\lambda^W_p(\phi(f)) \geq \phi(f) \text{ on } W.
\]
Then, since \( \{\lambda^W_p(\phi(f))\}^{1/p} \) is a \( P \)-superharmonic function on \( R \) by Hölder's inequality, we have
\[
|P^f| \leq \{\lambda^W_p(\phi(f))\}^{1/p} \text{ on } R_n,
\]
from which it follows that
\[
|\lambda^W_p(f)|^p = |\lim_{n \to +\infty} P^f|^{p} \\
\leq \lambda^W_p(f) \text{ on } R.
\]
That is, \( \lambda^W_p(f) \) is a \( P \)-harmonic majorant of \( |\lambda^W_p(f)|^p \) on \( R \). And, by (3.5), Lemma 3.5 shows that
\[
\sup_{w \in R} \int_{R} \lambda^W_p(f(z)G^P(z, w)P(z)dxdy < +\infty.
\]
Therefore, by Theorem 2.1, \( \lambda^W_p(f) \) belongs to the space \( PH^p_p(R) \). Q. E. D.
Let \( \{R_n\} \) be an exhaustion such that \( R_n \supseteq \partial W \). For a given function \( g \) on \( W \), let \( g_n \) be a function defined on \( \partial R_n \cup \partial W \) such that
\[
g_n|\partial W = 0 \quad \text{and} \quad g_n|\partial R_n = g.
\]
If \( g \) is a non-negative \( P \)-harmonic function on \( R \), the sequence \( \{P_{g_n} \cap W\} \) is a monotone decreasing sequence of \( P \)-harmonic functions. Then,
\[
\lim_{n \to \infty} P_{g_n} \cap W
\]
exists and is a \( P \)-harmonic function on \( R \). Now, if \( g \) is the difference of two non-negative \( P \)-harmonic functions, then we can define an operator \( \mu_W^p \), which was referred to as the canonical restriction by Nakai ([3], [4]), as follows:

**DEFINITION 3.3.** For \( g \in P(\mathcal{R}) \) which is the difference of two non-negative \( P \)-harmonic functions on \( R \),
\[
\mu_W^p(g) = \lim_{n \to \infty} P_{g_n} \cap W.
\]

**THEOREM 3.7.** \( \mu_W^p \circ \lambda_W^p \) is the identity mapping on \( PH_p^p(W; \partial W) \).

**PROOF.** Let \( f \) be in \( PH_p^p(W; \partial W) \), and suppose \( f \geq 0 \) on \( W \). Since
\[
P_{g_n} \cap W = f + P_{g_n} \cap W \quad \text{on} \quad (\lambda_W^p(f))_n,
\]
and
\[
0 \leq \lambda_W^p(f) - f \leq 0 \leq \lambda_W^p(f) - f \quad \text{on} \quad R_n \cap W,
\]
we have, by \( \lambda_W^p(f) = \lim_{n \to \infty} P_{g_n} \), that
\[
(3.6) \quad \mu_W^p \circ \lambda_W^p(f) = \mu_W^p(\lambda_W^p(f)) = \lim_{n \to \infty} P_{g_n} \cap W = f
\]
for every \( f \in PH_p^p(W; \partial W) \) with \( f \geq 0 \) on \( W \). From the linearity of \( \lambda_W^p \) and \( \mu_W^p \), (3.6) follows for any \( f \in PH_p^p(W; \partial W) \).

**LEMMA 3.8.**
\[
\mu_W^p(PH_p^p(\mathcal{R})) \subseteq PH_p^p(W; \partial W).
\]

**PROOF.** It is sufficient to prove this lemma only for a non-negative \( g \) in \( PH_p^p(\mathcal{R}) \). Then, from
\[
g \geq P_{g_n} \cap W
\]
on \( R_n \cap W \), it follows that
\[
p g \geq |g|^p
\]
on $W$, that is, $\mu g|W$ is a $P$-harmonic majorant of $|\mu_P^p(g)|^p$ on $W$. Furthermore, Theorem 2.5 shows that

$$\sup_{w \in R} \int_{R} p g(z) G(z, w) P(z) dx dy < +\infty,$$

which implies, by Theorem 2.1, that $\mu_P^p(g) \in PH_p^p(W)$ for every $g$ in $PH_p^p(R)$. And, it is shown that $\mu_P^p(g)$ has a continuous extension to the closure $\overline{W}$ of $W$ whose restriction to $\partial W$ vanishes. That is, $\mu_P^p(g) \in PH_p^p(W; \partial W)$.

Q. E. D.

A $P$-potential on $R$ is a non-negative $P$-superharmonic function on $R$ whose greatest $P$-harmonic minorant is non-positive. As in the case of classical Green potentials, we can show that any $P$-harmonic minorant of a $P$-Green potential is non-positive. Then, a $P$-Green potential is a $P$-potential. It is useful to modify a terminology and a lemma which was stated in Nakai [3]. A function $f$ on $R$ will be referred to as a quasi $P$-potential if $|f|$ is majorant by a $P$-potential.

**Lemma 3.9.** If $f$ is a continuous quasi $P$-potential such that $-|f|$ is $P$-superharmonic on $R$, then $f \equiv 0$ on $R$.

**Proof.** Assume that $|f|$ is majorant by a $P$-potential $p$. Since

$$0 \leq |f| \leq P_{|f|} \leq P_{p},$$

from

$$\lim_{n \to \infty} P_{p^n} = 0$$

it follows that $f \equiv 0$ on $R$. Q. E. D.

**Theorem 3.10.** $\lambda_P^{\text{w}} \mu_P^p$ is the identity mapping on $PH_p^p(R)$.

**Proof.** For $f \in PH_p^p(R)$, let $f_n$ and $f'_n$ be functions on $\partial R_n \cup \partial W$ such that

$$f_n |\partial R_n = f, \quad f_n |\partial W = 0$$

and

$$f'_n |\partial R_n = 0, \quad f'_n |\partial W = f.$$ 

If $f \geq 0$ on $R$, by the equilibrium principle, there exists a $P$-Green potential $G^p(z, \mu)$ such that

$$G^p(z, \mu) \leq \sup_{R-W} f, \quad z \in R,$$

$$G^p(z, \mu) = \sup_{R-W} f, \quad z \in R-W,$$

and the support of $\mu$ is contained in $R-W$. Since

$$0 \leq f(z) - P_{\mu}^{p^n}(z)$$
for every \( n \in \mathbb{N} \), it follows that

\[
0 \leq f(z) - \mu_p(f)(z) = f(z) - \lim_{n \to \infty} P_{\mathbb{B}^n \cap W}(z) \leq G_p(z, \mu), \quad z \in R \cap W,
\]

which shows that the function \( f - \mu_p(f) \) is a quasi \((-\mu_p, \mu_p)\)-potential on \( W \).

Next, let \( g = \lambda_p \ast \mu_p(f) \), which is contained in \( PH_p(R) \). By

\[
\mu_p(f) - \lambda_p \ast \mu_p(f) = \mu_p(g) - g,
\]

the above discussion shows that the function

\[
\mu_p(f) - \lambda_p \ast \mu_p(f)
\]

is also a quasi \((-\mu_p, \mu_p)\)-potential for a non-negative function \( f \) in \( PH_p(R) \). Therefore,

\[
f - \lambda_p \ast \mu_p(f) \leq |f - \mu_p(f)| + |\mu_p(f) - \lambda_p \ast \mu_p(f)|,
\]

the \( P \)-harmonic function \( f - \lambda_p \ast \mu_p(f) \) is a quasi \( P \)-potential on \( W \), which shows that \( f = \lambda_p \ast \mu_p(f) \) by Lemma 3.9. And, it is evident that this equality holds for any \( f \) in \( PH_p(R) \), since \( \lambda_p \) and \( \mu_p \) are linear.

**Q. E. D.**

**Corollary 3.11.** \( \mu_p \) is a one-to-one map of \( PH_p(R) \) onto \( PH_p(W; \partial W) \), and

\[
\lambda_p : PH_p(W; \partial W) \to PH_p(R)
\]

is the inverse of \( \mu_p \).

**Proof.** This corollary follows easily from Theorem 3.7 and Theorem 3.10. **Q. E. D.**

**Theorem 3.12.** The mapping

\[
\mu_p : PH_p(R) \to PH_p(W; \partial W)
\]

is an isomorphism, that is, \( PH_p(R) \) and \( PH_p(W; \partial W) \) are isomorphic.

**Proof.** It is clear that \( \mu_p \) is linear on \( PH_p(R) \). Since

\[
|P_{\mathbb{B}^n \cap W}|^p \leq \sup_{(x, y) \in C_{\mathbb{B}^n \cap W}} \leq P_{\mathbb{B}^n \cap W} \leq g |R \cap W|, \quad n \in \mathbb{N},
\]

for \( g \in PH_p(R) \), as \( n \to +\infty \) it is shown that \( g \) is a \( P \)-harmonic majorant of \( |\mu_p(g)|^p \) on \( W \) for \( g \in PH_p(R) \). So,

\[
g \leq \sup \mu_p(g),
\]

by which Theorem 2.5 and Definition 2.6 imply that
Therefore, \( \mu_w \) is a continuous mapping of \( PH_p(R) \).

Since \( \mu_w \) is a continuous linear one-to-one mapping of the Banach space \( PH_p(R) \) onto the Banach space \( PH_p(W; \partial W) \), the open mapping theorem gives that \( \mu_w \) is an open mapping, that is,

\[
\mu_w : PH_p(R) \rightarrow PH_p(W; \partial W)
\]

is an isomorphism. 

**Corollary 3.13.** If \( P \) and \( Q \) are two densities on \( R \) such that \( P=Q \) outside a compact subset of \( R \), then \( PH_p(R) \) and \( QH_p(W; \partial W) \) are isomorphic.

**Proof.** Assume that \( P=Q \) on \( W \subset R \). The Banach spaces \( PH_p(R) \), \( QH_p(W; \partial W) \) are isomorphic with the Banach space \( PH_p(W; \partial W) \). Q. E. D.

**§ 4. The comparison theorem.**

In the first part of this section we assume \( R \) to be connected, and let \( P \) and \( Q \) be two densities on \( R \). We shall prove that the spaces \( PH_p(R) \) and \( QH_p(W; \partial W) \) \((1 \leq p < \infty)\) are isomorphic providing the existence of a constant \( c \geq 1 \) such that

\[
c^{-1}Q \leq P \leq cQ
\]
on \( R \).

**Lemma 4.1.** Let \( P \) and \( Q \) be densities on \( R \) which are not identically zero. If there exists a constant \( c \geq 1 \) such that

\[
c^{-1}Q \leq P \leq cQ
\]
on \( R \), then we have

\[
G^Q(z,w) = G^P(z,w) + \frac{1}{2\pi} \int_R (P(\zeta) - Q(\zeta))G^Q(\zeta, w)G^P(\zeta, z) d\zeta d\eta
\]

for every \( z, w \in R \) with \( z \neq w \), where \( \zeta = \xi + i\eta \).

**Proof.** The Green’s formula implies that, for \( z, w \in R \) with \( z \neq w \),

\[
G^Q(R_n, z, w) = G^P(R_n, z, w) + \frac{1}{2\pi} \int_{R_n} (P(\zeta) - Q(\zeta))G^Q(R_n, \zeta, w)G^P(R_n, \zeta, z) d\zeta d\eta,
\]

where \( \zeta = \xi + i\eta \).

Let

\[
F(z, w, \zeta) = |P(\zeta) - Q(\zeta)| G^Q(\zeta, w) G^P(\zeta, z).
\]

To prove (4.2), we show that, if \( z \neq w \), the integral

\[
\int_R F(z, w, \zeta) d\zeta d\eta
\]
is finite. Let $U$ and $V$ be disks with centers $z$ and $w$, respectively, such that $V \cap U = \emptyset$. Then, since (4.1) implies that
\[ |P - Q| \leq cP, \quad |P - Q| \leq cQ \]
on $R$, and the maximum principle gives that
\[ \sup_{\zeta \in \bar{V}} G^P(\zeta, z) \geq G^P(\zeta, z), \quad \zeta \in \bar{V}, \]
and
\[ \sup_{\zeta \in \bar{V}} G^Q(\zeta, w) \geq G^Q(\zeta, w), \quad \zeta \in R - V, \]
we have
\[
\int_{R} F(z, w, \zeta) d\xi d\eta \leq \sup_{\zeta \in \bar{V}} G^P(\zeta, z) \times \int_{R} |P(\zeta) - Q(\zeta)| G^Q(\zeta, w) d\xi d\eta \\
\leq \sup_{\zeta \in \bar{V}} G^P(\zeta, z) \times \int_{R} G^Q(\zeta, w) Q(\zeta) d\xi d\eta \\
\leq 2\pi c \times \sup_{\zeta \in \bar{V}} G^P(\zeta, z) < +\infty
\]
and
\[
\int_{R - V} F(z, w, \zeta) d\xi d\eta \leq \sup_{\zeta \in \bar{V}} G^Q(\zeta, w) \times \int_{R} P(\zeta) G^P(\zeta, z) d\xi d\eta \\
\leq 2\pi c \times \sup_{\zeta \in \bar{V}} G^Q(\zeta, w) < +\infty.
\]
Therefore,
\[
\int_{R} F(z, w, \zeta) d\xi d\eta = \int_{V} F(z, w, \zeta) d\xi d\eta + \int_{R - V} F(z, w, \zeta) d\xi d\eta < +\infty
\]
for $z \neq w$ in $R$.

Since the sequences $\{G^Q(R_n, z, w)\}$ and $\{G^P(R_n, z, w)\}$ converge increasingly to $G^Q(z, w)$ and $G^P(z, w)$, respectively, we have
\[
\lim_{n \to +\infty} (P(\zeta) - Q(\zeta)) G^Q(R_n, \zeta, w) G^P(R_n, \zeta, z) \\
= (P(\zeta) - Q(\zeta)) G^Q(\zeta, w) G^P(\zeta, z)
\]
and
\[
|P(\zeta) - Q(\zeta)| G^Q(R_n, \zeta, w) G^P(R_n, \zeta, z) \leq F(z, w, \zeta)
\]
for each $n \in \mathbb{N}$. The Lebesgue's theorem of dominated convergence implies that, if $z \neq w$,
\[
\lim_{n \to +\infty} \int_{R_n} (P(\zeta) - Q(\zeta)) G^Q(R_n, \zeta, w) G^P(R_n, \zeta, z) d\xi d\eta \\
= \int_{R} (P(\zeta) - Q(\zeta)) G^Q(\zeta, w) G^P(\zeta, z) d\xi d\eta.
\]
Therefore, (4.2) follows from (4.3).

**Lemma 4.2.** Let $P$ and $Q$ be densities on $R$ which are not identically zero
on $\mathbb{R}$ and which satisfies (4.1) on $\mathbb{R}$. If a continuous function $f$ on $\mathbb{R}$ satisfies the condition

$$\text{sup}_{w \in \mathbb{R}} \int_{\mathbb{R}} |f(z)| G^p(z, w) Q(z) \, dx \, dy < +\infty,$$

then $f$ also satisfies

$$\text{sup}_{w \in \mathbb{R}} \int_{\mathbb{R}} |P(z) - Q(z)| G^q(z, w) |f(z)| \, dx \, dy < +\infty.$$

And, in this case we have

$$\text{sup}_{w \in \mathbb{R}} \int_{\mathbb{R}} |P(z) - Q(z)| G^q(z, w) |f(z)| \, dx \, dy \leq c(c+1) \times \text{sup}_{w \in \mathbb{R}} \int_{\mathbb{R}} |f(z)| G^p(z, w) Q(z) \, dx \, dy.$$

PROOF. Since the inequality (4.1) gives

$$|P - Q| \leq cP, \ cQ \text{ on } \mathbb{R},$$

from Lemma 4.1 it follows that

$$G^q(z, w) \leq G^p(z, w) + \frac{c}{2\pi} \int_{\mathbb{R}} Q(\zeta) G^q(\zeta, w) G^p(\zeta, z) \, d\xi \, d\eta.$$

Then, by the inequalities (2.1) and (4.6),

$$\int_{\mathbb{R}} |P(z) - Q(z)| G^q(z, w) |f(z)| \, dx \, dy \leq c \int_{\mathbb{R}} Q(z) G^q(z, w) |f(z)| \, dx \, dy \leq c \int_{\mathbb{R}} Q(z) G^p(z, w) |f(z)| \, dx \, dy$$

$$+ \frac{c}{2\pi} \int_{\mathbb{R}} Q(z) |f(z)| \left\{ \int_{\mathbb{R}} Q(\zeta) G^q(\zeta, w) G^p(\zeta, z) \, d\xi \, d\eta \right\} \, dx \, dy \leq c(c+1) \times \text{sup}_{w \in \mathbb{R}} \int_{\mathbb{R}} |f(z)| G^p(z, w) Q(z) \, dx \, dy.$$

This inequality completes our proof. Q.E.D.

We define an auxiliary transformation $T_{pq}^f$ of real valued continuous functions $f$ defined on the closure $\overline{R}_n$ of $R_n$ as follows:

$$T_{pq}^f(w) = f(w) + \frac{1}{2\pi} \int_{\overline{R}_n} (P(z) - Q(z)) G^q(R_n, z, w) f(z) \, dx \, dy.$$

Lemma 4.3. If $f$ is continuous on $\overline{R}_n$ and $P$-harmonic on $R_n$, then $T_{pq}^f(f)$ is $Q$-harmonic on $R_n$ and is a continuous function on $\overline{R}_n$ such that

$$T_{pq}^f(f) \mid_{\partial R_n} = f \mid_{\partial R_n}.$$
Banach spaces of solutions of $\Delta u = Pu$

**Proof.** The Green's formula and the properties of Green's function $G^Q(R_n, z, w)$ imply that $T^{PQ}_f(f)$ is the solution of Dirichlet problem with respect to the equation $\Delta u = Q u$ and the domain $R_n$ with the boundary value $f$ on $\partial R_n$ (see, for example, Nakai [1]).

**Definition 4.1.** For a real-valued continuous function $f$ defined on the connected Riemann surface $R$ satisfying the condition (4.4) in Lemma 4.2, we define a transformation $T^{PQ}_f(f)$ as follows:

$$T^{PQ}_f(f)(w) = f(w) + \frac{1}{2\pi} \int_{\partial R} (P(z) - Q(z))G^Q(z, w)f(z)dxdy,$$

which is well defined by Lemma 4.2.

**Lemma 4.4.** Let $P$ and $Q$ be densities on $R$ which are not identically zero, and assume that there is a constant $c$ satisfying (4.1). If a continuous function $f$ on $R$ satisfies the condition (4.4) in Lemma 4.2, then

$$T^{PQ}_f(f) = \lim_{n \to \infty} T^{PQ}_f(f).$$

**Proof.** Let $\alpha$ be the function

$$z \mapsto c\left\{Q(z)G^P(z, w)|f(z)| + \frac{c}{2\pi} \times Q(z)|f(z)| \times \int_{\partial R} Q(\zeta)G^Q(\zeta, w)G^P(\zeta, z)d\zeta d\eta\right\},$$

which satisfies that

$$(4.8) \int_R \alpha(z)dxdy \leq c(c+1)\sup_{w \in R} \int_R |f(z)|G^P(z, w)Q(z)dxdy < +\infty.$$  

Since

$$\lim_{n \to \infty} \int_{\partial R_n} (P(z) - Q(z))G^Q(R_n, z, w)f(z)(z) = (P(z) - Q(z))G^Q(z, w)f(z)$$

and, by Lemma 4.1 and the inequality (4.6),

$$|P(z) - Q(z)|G^Q(R_n, z, w)|f(z)| \leq cQ(z)G^Q(z, w)|f(z)| \leq \alpha(z),$$

Lebesgue's theorem on dominated convergence implies, by (4.8), that

$$\lim_{n \to \infty} \int_{\partial R_n} (P(z) - Q(z))G^Q(R_n, z, w)f(z)dxdy = \int_{\partial R} (P(z) - Q(z))G^Q(z, w)f(z)dxdy,$$

from which it follows that

$$\lim_{n \to \infty} T^{PQ}_f(f)(w) = T^{PQ}_f(f)(w), \quad w \in R.$$

**Q.E.D.**

**Lemma 4.5.** Under the assumption of Lemma 4.4, $T^{PQ}_f(f)$ is a $Q$-harmonic function on $R$.

**Proof.** Since a sequence $\{f_n\}$ of $Q$-harmonic functions on a domain $U$ of $R$ such that $|f_n| \leq M < +\infty$ has a subsequence which converges uniformly on
each compact subset of $U$ to a $Q$-harmonic function on $R$ (refer to Myrberg [1]), it is sufficient to show that the sequence $\{T_{PQ}(f)\}$ of $Q$-harmonic functions is uniformly bounded on a neighborhood $V$ of any $w \in R$. Lemma 4.2 shows that

$$|T_{PQ}(f)(w)| \leq \sup_{w \in V} \left\{ |f| + \frac{1}{2\pi} \int_{R^n} |P(z) - Q(z)| G^Q(R_n, z, w) |f(z)| \, dx \, dy \right\}$$

$$\leq \sup_{w \in V} |f| + \sup_{w \in R} \frac{1}{2\pi} \int_{R} |P(z) - Q(z)| G^Q(z, w) |f(z)| \, dx \, dy$$

$$\leq \sup_{w \in V} |f| + c(c+1)/2\pi \times \sup_{w \in R} \int_{R} |f(z)| G^P(z, w) Q(z) \, dx \, dy$$

$$< +\infty, \quad w \in V.$$

Q.E.D.

**Lemma 4.6.** Let $P$ and $Q$ be densities on $R$ which are not identically zero, and assume that there exists a constant $c \geq 1$ satisfying the inequality (4.1) on $R$. If $f$ is in $PH_p^p(R)$ $(1 \leq p < +\infty)$, then $T_{PQ}(f)$ is contained in the space $QH_p^p(R)$.

**Proof.** From Theorem 2.3, it follows that a function $f$ in $PH_p^p(R)$ satisfies the condition in Theorem 4.2, that is, $T_{PQ}(f)$ is defined for $f$ in $PH_p^p(R)$. Also, $T_{PQ}(f)$ is defined by Theorem 2.5.

Since it is evident that

$$|T_{PQ}(f)|^p = |f|^p \leq_T T_{PQ}(f)$$

on $\partial R_n$ for every $n \in \mathbb{N}$, the $Q$-subharmonic function $|T_{PQ}(f)|^p$ is dominated by the $Q$-harmonic function $T_{PQ}(f)$ on $R_n$ for each $n \in \mathbb{N}$. Thus, Lemma 4.4 shows that

$$|T_{PQ}(f)|^p \leq_T T_{PQ}(f)$$

on $R$, that is, $T_{PQ}(f)$ is a $Q$-harmonic majorant of $|T_{PQ}(f)|^p$ on $R$.

To prove $T_{PQ}(f) \in QH_p^p(R)$, it is sufficient, by Theorem 2.1, to show that

$$\sup_{w \in R} \int_{R} T_{PQ}(f)(z) G^Q(z, w) Q(z) \, dx \, dy < +\infty.$$  

By Definition 4.1, this integral equals to

$$\left(4.9\right) \int_{R} f(z) G^Q(z, w) Q(z) \, dx \, dy + \int_{R} \left\{ \frac{1}{2\pi} \int_{R} (P(\zeta) - Q(\zeta)) G^Q(\zeta, z) \right\} \times_{p} f(\zeta) \, d\zeta \, dy \right\} G^Q(z, w) Q(z) \, dx \, dy.$$  

The first term of (4.9) is dominated by

$$\int_{R} f(z) G^P(z, w) Q(z) \, dx \, dy$$

$$+ \int_{R} f(z) \left\{ \frac{1}{2\pi} \int_{R} |P(\zeta) - Q(\zeta)| G^Q(\zeta, w) \right\} \times_{p} f(\zeta) \, d\zeta \, dy \right\} G^Q(z, w) Q(z) \, dx \, dy.$$  

\begin{align*}
&\times G^p(\zeta, z)d\xi d\eta \right)Q(z)dxdy \\
&\leq \left\{1 + \frac{1}{2\pi} \int_R \left| P(\zeta) - Q(\zeta) \right| G^q(\zeta, w)d\xi d\eta \right\} \\
&\times \sup_{w \in \mathcal{R}} \int_R p f(z) G^p(z, w)Q(z)dxdy \\
&\leq c(1 + c) \times \sup_{w \in \mathcal{R}} \int_R p f(z) G^p(z, w)P(z)dxdy,
\end{align*}

where the inequality \(|P - Q| \leq cQ\) on \(R\) and Lemma 4.1 were used. The inequality (4.5) in Lemma 4.2 shows that the second term of (4.9) is dominated by

\[c(c+1) \times \sup_{w \in \mathcal{R}} \int_R p f(z) G^p(z, w)P(z)dxdy.\]

Therefore, we have

\[
\sup_{w \in \mathcal{R}} \int_R T_{PQ}(pf)(z)G^q(z, w)Q(z)dxdy \\
\leq c(c+1)^2 \times \sup_{w \in \mathcal{R}} \int_R p f(z) G^p(z, w)P(z)dxdy < +\infty.
\]

Q.E.D.

**Lemma 4.7.** Let \(P\) and \(Q\) be densities which are not identically zero on the connected Riemann surface \(R\). If there exists a constant \(c \geq 1\) satisfying the inequality (4.1) on \(R\), then \(T_{PQ}\) is a bounded linear transformation from \(PH^p_p(R)\) into \(QH^q_p(R)\), and \(T_{QP}\) is a bounded linear transformation from \(QH^q_p(R)\) into \(PH^p_p(R)\).

**Proof.** Since Lemma 4.6 shows that \(T_{PQ}(f)\) is well-defined and is contained in the space \(QH^q_p(R)\) for every \(f \in PH^p_p(R)\), it is clear that \(T_{PQ}\) is a linear mapping of \(PH^p_p(R)\) into \(QH^q_p(R)\).

Since \(T_{PQ}(pf)\) is a \(Q\)-harmonic majorant of \(|T_{PQ}(f)|^p\) on \(R\) (this was shown in the proof of Lemma 4.6), by (4.10) in the proof of Lemma 4.6 and (4.1), we have that

\[
\|T_{PQ}(f)\|_{q}^p = \sup_{w \in \mathcal{R}} \frac{1}{2\pi} \int_R (T_{PQ}(f))(z)G^q(z, w)Q(z)dxdy \\
\leq \sup_{w \in \mathcal{R}} \frac{1}{2\pi} \int_R T_{PQ}(pf)(z)G^q(z, w)Q(z)dxdy \\
\leq c(c+1)^2 \sup_{w \in \mathcal{R}} \frac{1}{2\pi} \int_R p f(z) G^p(z, w)P(z)dxdy.
\]
that is
\begin{equation}
\|TPQ(f)\|_p \leq \{c(c+1)^2\}^{1/p} \times \|f\|_p^p
\end{equation}
for every \( f \in PH_p(R) \). This shows that the mapping \( TPQ \) is a bounded linear transformation from \( PH_p(R) \) into \( QH_p(R) \). By changing the roles of \( P \) and \( Q \) we can see that \( TQP \) is a bounded linear transformation from \( QH_p(R) \) into \( PH_p(R) \).

**LEMMA 4.8.** If \( P \) and \( Q \) satisfy the same assumption as that in Theorem 4.7, then \( TQP \circ TPQ \) is the identity on \( PH_p(R) \), and \( TPQ \circ TQP \) is the identity on \( QH_p(R) \).

**PROOF.** Since \( PH_p(R) \subset PH'_1(R) \) \( (1 \leq p < +\infty) \), any function \( f \) in \( PH_p(R) \) satisfies that
\[
c^{-1} \int R |f(z)|^P(z)Q(z)dxdy
\leq \int R |f(z)|^P(z)P(z)dxdy
\leq \int R |f(z)|G^P(z, w)P(z)dxdy
\leq 2\pi \times \|f\|_p^p < +\infty, \quad w \in R,
\]
which implies, by Lemma 4.2, that
\[
\sup_{w \in R} \int R |P(z) - Q(z)|G^P(z, w)G^Q(z, w)|f(z)|dxdy < +\infty.
\]
Therefore, the last function of the inequality
\[
|(Q(z) - P(z))G^P(R_n, z, w)TPQ(f)(z)|
\leq c \left\{ P(z)G^P(z, w)|f(z)| + \frac{1}{2\pi} P(z)G^P(z, w) \right\}
\times \int_{R_n} \left| P(\zeta) - Q(\zeta) \right| G^Q(R_n, \zeta, z)|f(\zeta)|d\zeta d\eta
\leq c \left\{ P(z)G^P(z, w)|f(z)| + \frac{1}{2\pi} P(z)G^P(z, w) \right\}
\times \int R \left| P(\zeta) - Q(\zeta) \right| G^Q(z, w)|f(\zeta)|d\zeta d\eta
\]
is integrable for any fixed \( w \in R_n \), where this inequality is obtained by the definition of \( TPQ(f) \) and \( |P - Q| \leq cP \) on \( R \). Since
Lebesgue's theorem on bounded convergence gives that
\[
\lim_{n \to +\infty} \int_{R_n} (Q(z) - P(z)) G^P(R_n, z, w) T^P(f)(z)\, dx dy = \int_R (Q(z) - P(z)) G^P(z, w) T^P(f)(z)\, dx dy,
\]
from which it follows that
\[
\lim_{n \to +\infty} T^P \circ T^Q(f) = T^P \circ T^Q(f)
\]
on \(R\) for \(f \in PH^p(R)\). On the other hand, the maximum principle shows, by
Lemma 4.3, that
\[
T^Q \circ T^P(f) = f \text{ on } R_n,
\]
for every \(n \in N\), and so,
\[
T^Q \circ T^P(f) = f \text{ on } R,
\]
for any \(f \in PH^p(R)\).

By changing the roles of \(P\) and \(Q\) we have also that
\[
T^Q \circ T^P(g) = g \text{ on } R,
\]
for \(g \in QH^p_0(R)\).

**THEOREM 4.9.** Under the same assumption as that in Lemma 4.8, \(T^P \circ T^Q\) is an isomorphism between \(PH^p(R)\) and \(QH^p_0(R)\). And, \(T^Q \circ T^P\) is the inverse of \(T^P \circ T^Q\).

**PROOF.** This follows from Lemma 4.7 and 4.8. Q. E. D.

Now, let \(R\) be a disconnected Riemann surface, and let
\[
R = \bigcup_{k=1}^{K} W^k
\]
be the decomposition of \(R\) into connected components \(W^k, k=1, 2, \ldots, K,\) of \(R\). If the densities satisfy the relation
\[
(4.12) \quad c^{-1} Q \leq P \leq c Q \quad \text{on } R \quad (c \geq 1),
\]
then we can assume that \(W^1, W^2, \ldots, W^L \quad (1 \leq L \leq K)\) are connected components of \(R\) on which \(P \equiv 0\) and \(Q \equiv 0\), and that \(W^{L+1}, W^{L+2}, \ldots, W^K\) are connected components of \(R\) on which \(P \equiv 0\) and \(Q \equiv 0\).

**DEFINITION 4.2.** If the relation (4.12) holds on the disconnected Riemann surface \(R\), we define the function \(T^P \circ T^Q(f)\) on \(R\) for \(f \in PH^p_0(R)\) as follows:
\[
T^P \circ T^Q(f) | W^k = T^P(f | W^k), \quad 1 \leq k \leq L,
\]
and
\[
T^P \circ T^Q(f) | W^k = f | W^k, \quad L < k \leq K.
\]
By changing the roles of $P$ and $Q$ we define also $T_{QP}(g)$ for $g \in QH^p_p(R)$.

**Theorem 4.10.** Let $R$ be a Riemann surface which may be disconnected, and assume (4.12). Then, $T_{PQ}$ is an isomorphism between $PH^p_p(R)$ and $QH^p_p(R)$. And, $T_{QP}$ is the inverse of $T_{PQ}$.

**Proof.** Lemma 4.9 gives this theorem. Q.E.D.

Let $R$ be a connected hyperbolic Riemann surface and let $P$ and $Q$ be two densities on $R$. In the following, we prove the order comparison theorem: If there exists a constant $c \geq 1$ such that

$$c^{-1}Q \leq P \leq cQ$$

on $R$ except possibly for a compact subset $K$ of $R$, then $PH^p_p(R)$ and $QH^p_p(R)$ are isomorphic.

Let $W$ be an open subset of $R$ such that $R-W \supset K$ and $R-W$ is a regular region. Then, since (4.13) is valid on the whole $W$, which may be considered a Riemann surface, Lemma 4.10 states that there is the isomorphism between $PH^p_p(W)$ and $QH^p_p(W)$, which is denoted by $T_{PQ}$ in the following.

**Lemma 4.11.** If the inequality (4.13) holds on $W$, then $T_{PQ}$ may be considered an isomorphism of $PH^p_p(W; \partial W)$ onto $QH^p_p(W; \partial W)$.

**Proof.** Since $PH^p_p(W; \partial W)$ and $QH^p_p(W; \partial W)$ are closed subspaces of $PH^p_p(W)$ and $QH^p_p(W)$, respectively, it is necessary only to prove that $T_{PQ}(f) \in QH^p_p(W; \partial W)$ for $f \in PH^p_p(W; \partial W)$.

Let $\{R_n\}$ be an exhaustion of $R$ such that $R_n \supset R-W$, $n=0, 1, 2, \dots$, and let

$$\alpha = \sup_{w \in \partial R_0} |T_{PQ}(f)(w)|.$$

We denotes by $\omega$ the continuous function on $\overline{R_n \cap W}$ such that $\omega$ is $Q$-harmonic on $\overline{R_n \cap W}$ and $\omega|\partial W=0$, $\omega|\partial R_0=1$.

Since Lemma 4.4 states that

$$\lim_{n \to +\infty} T_{PQ}^n(f) = T_{PQ}^n(f) \text{ on } W,$$

where $T_{PQ}^n$ is defined for a continuous function on $\overline{R_n \cap W}$ which is $Q$-harmonic on $W \cap R_n$, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|T_{PQ}^n(f)(w)| \leq (\alpha + \epsilon)\omega(w), \quad w \in W \cap R_0$$

for $n > n_0$. So, as $n \to +\infty$, we have

$$|T_{PQ}^n(f)(w)| \leq (\alpha + \epsilon)\omega(w), \quad w \in W \cap R_0,$$

from which

$$T_{PQ}^n(f)|\partial W=0,$$

that is,

$$T_{PQ}^n(f) \in QH^p_p(W; \partial W)$$

follows. Q.E.D.
THEOREM 4.12 (THE ORDER COMPARISON THEOREM). Let \( P \) and \( Q \) be two densities on a connected Riemann surface. If there exists a constant \( c \geq 1 \) such that
\[
c^{-1}Q \leq P \leq cQ
\]
on \( R \) except possibly for a compact subset \( K \) of \( R \), then \( PH_p(R) \) and \( QH_p(R) \) are isomorphic.

PROOF. Let \( W \) be the same open subset of \( R \) as that defined before Lemma 4.11. Then, by Theorem 3.12 and Lemma 4.11, the mapping
\[
\lambda_q^w \circ T_p^w \circ \mu_p^w : PH_p(R) \rightarrow QH_p(R)
\]
is an isomorphism. Q. E. D.

References

M. Glasner

L. L. Helms

A. Lahtinen

L. Myrberg

L. Lumer-Naim

M. Nakai

M. Parreau

B. Rodin and L. Sario
H.L. Royden

L. Sario and M. Nakai

J.L. Schiff

Takeyoshi SATō
Mathematics Laboratory
Iwamizawa College
Hokkaido University of Education
Iwamizawa, Hokkaido
Japan