On \((x)\)-complexes

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Let \(N\) be a finitely generated module over a noetherian local ring \(R\) with maximal ideal \(m\). It is well known that a maximal \(R\)-sequence and a maximal \(N\)-sequence have connections with a minimal injective resolution of \(N\). For example, the length of a maximal \(R\)-sequence, namely the depth of \(R\), is equal to the length of a minimal injective resolution of \(N\), namely the injective dimension of \(N\), if it is finite, and the length of a maximal \(N\)-sequence, namely the depth of \(N\), is equal to the minimal integer of \(i\) with \(\mu^i(m, N) > 0\) where \(\mu^i(m, N)\) is the dimension of an \(R/m\)-vector space \(\text{Ext}^i_R(R/m, N)\). But we have thought there are more connections between them. In particular we are interested in studying possible connections between the terms of an \(R\)-sequence or of an \(N\)-sequence, and the terms of a minimal injective resolution of \(N\).

First we shall introduce a complex associated to a minimal injective resolution of \(N\) for a sequence of elements in \(m\) [see Definition 1]. This complex characterizes some \(N\)-sequence. We shall study properties of this complex. In particular we shall give, using a term of this complex, a necessary and sufficient condition for the following conjecture of Bass to hold: a noetherian local ring is Cohen-Macaulay if it possesses a finitely generated module of finite injective dimension. Moreover we shall show some property of a minimal injective resolution, applying this complex.

Throughout this note, \(R\) is a noetherian local ring with a unique maximal ideal \(m\). The unlabeled Hom and Ext mean \(\text{Hom}_R\) and \(\text{Ext}_R\), respectively.

We begin by introducing a definition. Let \(x_0, x_1, \ldots, x_r\) be a sequence of elements in \(m\). We denote this sequence by \((x_0, x_1, \ldots, x_r)\) or, for brevity, \((x)\). The ideal generated by \(x_0, x_1, \ldots, x_r\) is also denoted by \((x_0, x_1, \ldots, x_r)\) or \((x)\).

Let \(N\) be an \(R\)-module and \(0 \rightarrow N \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \) be a minimal injective resolution of \(N\).

**DEFINITION 1.** Let \(N_{(x)} = \{e \in E_0 \mid x_0 e \in d_1^{-1}(N)\}\). For any integer \(i\) with \(0 < i \leq r\), we define inductively \(N_i^{(x)}\) as follows; \(N_0^{(x)} = \{e \in \langle 0 : (x_0, x_1, \ldots, x_{i-1}) \rangle_R \mid x_i e \in d_i^{-1}(N_{(x)})\}\). For \(i > r\), \(N_i^{(x)} = \langle 0 : (x_0, x_1, \ldots, x_r) \rangle_R\). Each \(N_i^{(x)}\) is a submodule of \(E_i\). Each \(d_i\) induces an \(R\)-homomorphism \(N_i^{(x)} \to N_i^{(x)}\), denoted again by \(d_i\). In this case we have a complex of \(R\)-modules.
This complex is unique for a sequence \((x)\) and \(N\) up to isomorphism. This complex is also denoted by \(0 \to N \to N^0 \to N^1 \to \cdots\) for simplicity, which is called \((x)\)-complex under \(N\).

As its obvious properties, we obtain the followings;

1) For each \(i\) \(N^i\) can be naturally identified with a submodule of \(\text{Hom}(R/(x_0, x_1, \cdots, x_{i-1}), E^i)\), since \((0 : (x_0, x_1, \cdots, x_{i-1}))_{E^i}\) is naturally isomorphic to \(\text{Hom}(R/(x_0, x_1, \cdots, x_{i-1}), E^i)\). In this case, for each \(i\) with \(i > r\), \(N^i = \text{Hom}(R/(x), E^i)\).

2) For each \(i > r\), \(N^i\) is \(R/(x)\)-injective.

3) If \((x) = (x_0, x_1, \cdots, x_r)\) is an \(R\)-sequence, the sequence \(N^{r+1} \to N^{r+2} \to \cdots\) is exact, because \(\text{Ext}^i(R/(x), N) = 0\) for \(i > r + 1\) follow from proj. dim \(R/(x) = r + 1\).

PROPOSITION 2. If \((x) = (x_0, x_1, \cdots, x_r)\) is an \(R\)-sequence, then an \((x)\)-complex under any \(R\)-module \(N\) is always acyclic.

PROOF. Let \(0 \to N \to E^0 \to E^1 \to E^2 \to \cdots\) be a minimal injective resolution of \(N\). From an exact sequence \(0 \to \text{Hom}(R/(x_0), E^0/d^{-1}(N)) \to \text{Hom}(R/(x_0), E^1) \to \text{Hom}(R/(x_0), E^2)\), we obtain an exact sequence \(0 \to \text{Hom}(R/(x_0), E^0/d^{-1}(N)) \to \text{Hom}(R/(x_0), E^1) \to \text{Hom}(R/(x_0), E^2)\). Since \(\text{Hom}(R/(x_0), E^0/d^{-1}(N)) \cong N^0/d^{-1}(N) \cong d^0(N^0)\), \(0 \to d^0(N^0) \to \text{Hom}(R/(x_0), E^1) \to \text{Hom}(R/(x_0), E^2)\) is exact. So \(N^0 \to N^1 \to N^2\) is exact. Since \(\text{Ext}^i(R/(x_0), N) = 0\) for \(i > 1\), \(\text{Hom}(R/(x_0), \ldots)\) is a functor from \(R\)-modules to \(R/(x_0)\)-modules which preserves minimal injective resolutions [see 1, Lemma 2.1]. Hence we have a minimal injective resolution of the \(R/(x_0)\)-module \(d^i(N^0) : 0 \to d^i(N^0) \to \text{Hom}(R/(x_0), E^i) \to \cdots \to \text{Hom}(R/(x_0), E^i) \to \cdots\). Using the same argument as above, \(N^{i+1} \to N^{i+2} \to \cdots\) is exact, and so on.

COROLLARY 3. Let \(0 \to N \to E^0 \to E^1 \to \cdots \to E^r \to 0\) be a minimal injective resolution of a finitely generated \(R\)-module \(N\) where \(r\) is finite. Then, for any \(R\)-sequence \(x_0, x_1, \cdots, x_s\), there is an element \(y_i\) of \(E^i\) \((i = 1, 2, \cdots, s)\) such that \((x_0, x_1, \cdots, x_{i-1})y_i = 0\) and \(x_i y_i \neq 0\) for \(i = 1, 2, \cdots, s\).

PROOF. Let \(0 \to N \to E^0 \to E^1 \to \cdots \to E^r \to 0\) be the \((x_0, x_1, \cdots, x_i)\)-complex under \(N\). This complex is acyclic and \(N^{i+k} = (0 : (x_0, x_1, \cdots, x_i))_{E^i+k}\) for \(1 \leq k \leq r-i\). Since depth \(R/(x_0, x_1, \cdots, x_i) + \text{Sup} \{j \mid \text{Ext}^j(R/(x_0, x_1, \cdots, x_i), N) \neq 0\} = \text{depth} R\) \((= r)\) [see 5, Théorème (4.15)], we have \(\text{Sup} \{j \mid \text{Ext}^j(R/(x_0, x_1, \cdots, x_i), N) \neq 0\} = i+1\). Hence the sequence \(\text{Hom}(R/(x_0, x_1, \cdots, x_i), E^i) \to \text{Hom}(R/(x_0, x_1, \cdots, x_i), E^{i+1}) \to \cdots\) is exact and so \(N^{i+k} = (0 : (x_0, x_1, \cdots, x_i))_{E^i+k}\) is not empty. Choose any element \(y_i\) in \(N^{i+k} = (0 : (x_0, x_1, \cdots, x_i))_{E^i+k}\). Then we obtain \((x_0, x_1, \cdots, x_{i-1})y_i = 0\) and \(x_i y_i \neq 0\).

THEOREM 4. Let \(x_0, x_1, \cdots, x_r\) be an \(R\)-sequence. Then \(x_0, x_1, \cdots, x_r\) is an \(N\)-sequence if and only if \(x_i\) is a nonzero divisor of \(N^i\) for \(i = 0, 1, \cdots, r\) and
On \((x)\)-complexes

PROOF. First we shall show 'if' part. Since the sequence \(0 \to R \xrightarrow{x_0} R \to R/(x_0) \to 0\) is exact, we have \(x_0 E = E\) for any injective \(R\)-module \(E\).

Let \(0 \to N^d \to E^d \to E^{d+1} \to \cdots\) be a minimal injective resolution of \(N\). It is obvious that \(x_0\) is a nonzero divisor of \(N\) and so is a nonzero divisor of \(E^0\). Hence the multiplication map by \(x_0 : E^0 \to E^0\) is an isomorphism. This isomorphism induces \(E^0/d^0(N) \cong E^0/x_0d^0(N)\). So we obtain \(d^0(N) \cong N^d/x_0d^0(N)\) \(\cong \text{Hom}(R/(x_0), E^0/x_0d^0(N)) \cong \text{Hom}(R/(x_0), E^0/x_0d^0(N)) \cong \{e \in E^0 \mid x_0 e \in x_0d^0(N)\}\). This shows \(x_0\) is a nonzero divisor on \(N/x_0 N\). Since \(0 \to d^0(N) \to \text{Hom}(R/(x_0), E^1) \to \text{Hom}(R/(x_0), E^2) \to \cdots\) is a minimal injective resolution of the \((R/(x_0))\)-module \(d^0(N)^{\oplus N/x_0 N}\), repeating the same reasoning we prove inductively that \(x_0, x_1, \cdots, x_r\) is an \(N\)-sequence. Conversely assume that \(x_0, x_1, \cdots, x_r\) is an \(N\)-sequence. Then it is trivial that \(x_0\) is a nonzero divisor of \(N^0\). Moreover we have \(N/(x_0, x_1, \cdots, x_r)N \cong d^0(N)\), proceeding in the above fashion, and \(d^i(N)^{\oplus N} \cong N/(x_0, x_1, \cdots, x_r)N\) for \(i=0, 1, \cdots, r\).

COROLLARY 5. Let \(x_0, x_1, \cdots, x_r\) be elements of \(m\). Then \(x_0, x_1, \cdots, x_r\) is an \(R\)-sequence if and only if the \((x_0, x_1, \cdots, x_r)\)-complex under \(R : 0 \to R \xrightarrow{x_0} R \to R^2 \to \cdots\) is acyclic and \(x_i\) is a nonzero divisor of \(R^i\) for \(i=0, 1, \cdots, r\).

THEOREM 6. Let \((x) = (x_0, x_1, \cdots, x_r)\) be an \(R\)-sequence and \(N\) a nonzero finitely generated \(R\)-module. If \(x_0, x_1, \cdots, x_s\) \((s \leq r)\) is an \(N\)-sequence, each term \(N^i\) of the \((x)\)-complex under \(N\) is finitely generated \((0 \leq i \leq s)\). Conversely assume each \(N^i\) is finitely generated \((0 \leq i \leq s < r)\). Then \(x_0, x_1, \cdots, x_s\) is an \(N\)-sequence.

PROOF. Assume \(x_0, x_1, \cdots, x_s\) is an \(N\)-sequence \((s \leq r)\). Then we have \(N^0/d^0(N) \cong d^0(N)^{\oplus N}/x_0N\). Since both \(N/x_0 N\) and \(d^0(N)\) are finitely generated, so is \(N^0\). In general we have \(N^i/d^i(N^i) \cong d^i(N^i)^{\oplus N}/(x_0, x_1, \cdots, x_s)N\) \((0 \leq i \leq s)\). It is inductively proved that each \(N^i\) is finitely generated \((0 \leq i \leq s)\). Assume the converse. Since \((0 : x_0) \subseteq N^0, (0 : x_0) \subseteq N^s\) is finitely generated. On the other hand \((0 : x_0) \subseteq N^0\text{ as an }R/(x_0)\text{-module}\). So we have \((0 : x_0) \subseteq 0 = 0\), because depth \(R/(x_0) = 0\). Hence \(x_0\) is a nonzero divisor of \(N/x_0 N\).

Suppose \(x_0, x_1, \cdots, x_s\) is an \(N\)-sequence \((0 \leq i < s)\). Since \((0 : x_0, x_1, \cdots, x_{i+1}) \subseteq N^{i+1}, (0 : x_0, x_1, \cdots, x_{i+1}) \subseteq N^{i+1}\), \(0 : (x_0, x_1, \cdots, x_{i+1}) \subseteq N^{i+1}\) is finitely generated. Hence it is a zero module and so \(x_{i+1}\) is a nonzero divisor of \((0 : (x_0, x_1, \cdots, x_{i+1}) \subseteq N^{i+1}\). Hence \(x_{i+1}\) is a nonzero divisor of \(N^{i+1}\) and so of \(d^i(N^i)\). Since \(d^i(N^i) \cong N/(x_0, x_1, \cdots, x_i)N, x_0, x_1, \cdots, x_{i+1}\) is an \(N\)-sequence. By induction the proof is completed.

COROLLARY 7. Let \((x) = (x_0, x_1, \cdots, x_s)\) be an \(R\)-sequence \((s < \text{depth } R)\) and \(N\) a nonzero finitely generated \(R\)-module. Then \(x_0, x_1, \cdots, x_s\) is an \(N\)-sequence if and only if \(N = N^0, d^0(N^0) \cong N^1, \cdots, d^{i-1}(N^{i-1}) \cong N^i\) for the \((x)\)-complex under \(N\): \(0 \to N^a \to N^b \to N^c \to \cdots\).
PROOF. The 'if' part follows from the theorem. Suppose \( x_0, x_1, \ldots, x_s \) is an \( N \)-sequence. Let \( 0 \rightarrow N \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \) be a minimal injective resolution of \( N \). Since \( x_0 \) is a nonzero divisor of \( R \) and of \( E^0 \), the multiplication map by \( x_0 \) induces an automorphism of \( E^0 \). So the multiplication map by \( x_0 : N^0 \xrightarrow{d^{-1}} (N) (\equiv N) \) is an isomorphism. In general, \( x_i \) is a nonzero divisor of \( d^{i-1}(N^{i-1}) (\equiv N/(x_0, x_1, \ldots, x_{i-1})N) \) and \( \text{Hom}(R/(x_0, x_1, \ldots, x_{i-1}), E^i) \) is an injective envelope of the \( R/(x_0, x_1, \ldots, x_{i-1}) \)-module \( d^{i-1}(N^{i-1}) (1 \leq i \leq s) \). Using the same argument as above, we have \( N^i \equiv d^{i-1}(N^{i-1}) (1 \leq i \leq s) \).

**Corollary 8.** Let \( R \) be a Gorenstein local ring with a maximal ideal \( m \). Then, for any maximal \( R \)-sequence \((x)\), each term of the \((x)\)-complex under \( R \) is finitely generated and its final nonzero term is isomorphic to the injective envelope of an \( R/(x) \)-module \( R/m \).

**Theorem 9.** Assume \( \text{depth } R \leq n+1 \) for a nonnegative integer \( n \). If there exists a sequence \((x) = (x_0, x_1, \ldots, x_n) \) of elements of \( m \) such that the \((x)\)-complex under \( R \) is acyclic with each term finitely generated, then \( \dim R \leq n+1 \).

**Proof.** Let \( 0 \rightarrow R \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \) be a minimal injective resolution of \( R \). Then \((0 : x_0)_{E^0} \) is embedded in the first term \( R^0 \) of the \((x)\)-complex under \( R \) and so is finitely generated. If \((0 : x_0)_{E^0} \neq 0 \) and hence \( \dim R \leq 1 \) [see 4, Corollaire 1.3]. Assume \((0 : x_0)_{E^0} = 0 \). Then \( x_0 \) is a nonzero divisor of \( R \). Since \((0 : (x_0, x_1))_{E^1} \) is embedded in \( R^1 \), it is finitely generated. If \((0 : (x_0, x_1))_{E^1} \neq 0 \), we have depth \( R/(x_0, x_1) = 0 \) and so \( \dim R \leq 2 \). If \((0 : (x_0, x_i))_{E^i} = 0 \), \( x_i \) is a nonzero divisor of \((0 : x_0)_{E^i} \). Since \( d^i(R^i) \equiv R/x_0 R \) is embedded in \((0 : x_0)_{E^i} \), \( x_i \) is a nonzero divisor of \( R/x_0 R \) and \( \dim R = n+1 \) and so \((0 : (x_0, x_1, \ldots, x_n))_{E^{n+1}} \neq 0 \). Since \((0 : (x_0, x_1, \ldots, x_n))_{E^{n+1}} (\equiv R^{n+1}) \) is finitely generated by the hypothesis, we obtain \( \dim R \leq n+1 \).

**Theorem 10.** Suppose a noetherian local ring \( R \) possesses a finitely generated module \( N \) of finite injective dimension such that, for any maximal \( R \)-sequence \((x) = (x_0, x_1, \ldots, x_r) \), some term of the \((x)\)-complex under \( N \) is nonzero and finitely generated. Then \( R \) is Cohen-Macaulay.

**Proof.** Assume \( N^i \) is finitely generated where \( N^i \) is some term of the \((x)\)-complex under \( N : 0 \rightarrow N \rightarrow N^0 \rightarrow N^1 \rightarrow \cdots \rightarrow N^r \rightarrow 0 \). If \( i = r+1 \), our statement obviously holds. When \( i = r \), \( N^{r+1} \) is also finitely generated. Hence we suppose \( i < r \), Let \( 0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{r+1} \rightarrow 0 \) be a minimal injective resolution of \( N \). We have \((0 : (x_0, x_1, \ldots, x_i))_{E^i} = 0 \). For, if \((0 : (x_0, x_1, \ldots, x_i))_{E^i} \neq 0 \), it is finitely generated since it is contained in \( N^i \), and so depth \( R/(x_0, x_1, \ldots, x_i) = 0 \), which is a contradiction. Therefore \( x_i \) is a nonzero divisor on \((0 : (x_0, x_1, \ldots, x_{i-1}))_{E^i} \) and so we have \( d^{i-1}(N^{i-1}) \neq 0 \). Since \( d^{i-1}(N^{i-1}) \) is a submodule of the
On \((x)\)-complexes

finitely generated module \(N^i\) and of finite injective dimension as an \(R/(x_0, x_1, \ldots, x_{i-1})\)-module, its isomorphic module \(N^{i-1}/d^{i-2}(N^{i-2})\) is a finitely generated \(R/(x_0, x_1, \ldots, x_{i-1})\)-module of finite injective dimension. Set \(a = \text{Ann}_{R/(x_0, x_1, \ldots, x_{i-1})}N^{i-1}/d^{i-2}(N^{i-2})\). If \(a = 0\), \(R/(x_0, x_1, \ldots, x_{i-1})\) is Cohen-Macaulay [see 2, Lemma (3.1), (3.3)]. Next assume \(a \neq 0\). Then there is an element \(x'_i\) of \(R\) such that the residue class \(\overline{x'_i}\) belongs to \(a\) and \(x'_i\) is a nonzero divisor on \(R/(x_0, x_1, \ldots, x_{i-1})\) [see 3, Theorem 4.1]. For a maximal \(R\)-sequence \((x') = (x_0, x_1, \ldots, x_{i-1}, x'_i, x'_{i+1}, \ldots, x'_{i+1})\), the \(i\)-th term \(N'_{(x')}\) of the \((x')\)-complex under \(N\) is non finitely generated, for \(N^{i-1}/d^{i-2}(N^{i-2})\) \((\neq 0)\) is embedded in \((0: (x_0, x_1, \ldots, x_{i-1}, x'_i))_{E^i}\) and so \((0: (x_0, x_1, \ldots, x_{i-1}, x'_i))_{E^i} \neq 0\). Thus, if \(i\) is a minimal non negative integer such that \(N^i\) is finitely generated, \(R/(x_0, x_1, \ldots, x_{i-1})\) is Cohen-Macaulay or there is a maximal \(R\)-sequence \((x')\) such that all \(N'_{(x')}\) are non finitely generated for \(0 \leq j \leq i\). When the second statement holds, repeating the above argument, we obtain that \(R/(x_0, x_1, \ldots, x_4)\) is Cohen-Macaulay or there is a maximal \(R\)-sequence \((x'')\) such that \(N''_{(x'')}\) is finitely generated. In either case, \(R\) is Cohen-Macaulay.

References


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