Hölder estimates on higher derivatives of the solution for $\overline{\partial}$-equation with $C^k$-data in strongly pseudoconvex domain

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(Received Aug. 22, 1977)

§ 0. Introduction.

In 1971 Kerzman [4] showed there exists a solution of $\overline{\partial}$-equation with bounded data which is Hölder continuous for any exponent smaller than 1/2. Since then many results have been obtained concerning this problem. Henkin-Romanov [3] and Range-Siu [5] proved the exact 1/2-Hölder estimate. Moreover Siu [6] showed the Hölder continuity of higher derivatives of the solution assuming the data are sufficiently smooth. In this paper we shall improve Siu's result and get a new estimate which is sharper in some tangential directions. We follow the method of Siu [6]; however, various parts of his calculus are ameliorated. I thank Professor H. Tanabe, who encouraged me to write this paper and corrected my manuscript.

0.1. Notations.

Let $Q$ be a bounded strongly pseudoconvex domain in $C^n$ with $C^N$-boundary. We assume that $Q$ is represented as \{ $z \in C^n ; \rho(z) < 0$ \}, where $\rho$ is a function of class $C^N$ and in some neighborhood of $\partial Q$ is strictly plurisubharmonic and satisfies $d\rho \neq 0$. We use the following notations;

$$D_j = \frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \overline{D}_j = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right),$$

$$\|u\|_0 = \sup\{|u(z)| ; z \in Q\},$$

$$\|u\|_e = \sup\{|u(z) - u(\zeta)| / |z - \zeta|^\epsilon ; \zeta, z \in Q, \zeta \neq z\} + \|u\|_0,$$

$$\|u\|_k = \max\{\|D^\alpha \overline{D}^\beta u\|_e ; |\alpha| + |\beta| \leq k\},$$

$$\|u\|_{k+e} = \max\{\|D^\alpha \overline{D}^\beta u\|_e ; |\alpha| + |\beta| \leq k\}$$

where $k \in N$ and $0 < \epsilon < 1$. For a form $f = \sum f_i d\overline{z}_i$, 

$$\|f\|_k = \max\{\|f_i\|_k ; 1 \leq i \leq n\}.$$
DEFINITION 0.1. A vector field \( Y \) on \( \tilde{\Omega} \) is called holomorphic tangential when \( \partial \rho (Y) = 0 \) on \( \partial \Omega \), i.e. if \( Y = \sum a_i(z) \partial /\partial z_i + \sum b_i(z) \partial /\partial \bar{z}_i \), then \( \sum a_i(z) \partial \rho /\partial z_i = 0 \) on \( \partial \Omega \).

When we differentiate forms, we use the Euclid connection, that is, we differentiate them componentwise. In the next section, following Henkin [2] we shall construct an inverse operator for \( \partial \) which we denote by \( T \). Our main results are the followings.

**Theorem 1.** Let \( N \geq k+2 \) and \( f \) be a \( \partial \)-closed \( C^k \)-\((0,1)\) form on \( \tilde{\Omega} \). Then

\[
\| T(f) \|_{k+1/2} \leq C_\delta \| f \|_k.
\]

**Remark.** Our improvement consists in the condition \( N \geq k+2 \). Y. T. Siu required \( N \geq k+4 \) in [6].

**Theorem 2.** Let \( N \geq k+2 \), \( f \) be a \( \partial \)-closed \( C^k \)-\((0,1)\) form on \( \tilde{\Omega} \) and \( Y \) be a \( C^2 \)-holomorphic tangential vector field. Then for \( |\alpha| = k-1 \),

\[
\| Y D^\alpha T(f) \|_\beta \leq C_\beta \| f \|_k, \quad \text{for any } \beta \text{ smaller than } 1.
\]

§ 1. Henkin’s kernel.

In this section we construct an inverse operator for \( \partial \) following Henkin [2].

1.1. Let

\[
F^\ast(\zeta, z) = \sum_{i=1}^N \frac{\partial \rho}{\partial \zeta_i}(\zeta(z_i - \zeta_i)) + \frac{1}{2} \sum_{i,j=1}^N \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta(z_i - \zeta_i)(z_j - \zeta_j)).
\]

Then

\[
-\text{Re} F^\ast(\zeta, z) \geq \rho(\zeta) - \rho(z) + C |\zeta-z|^3,
\]

for \( |\zeta-z| \) and \( |\rho(\zeta)| \) small. Moreover if we set

\[
F(\zeta, z) = \sum \frac{\partial \rho}{\partial \zeta_i}(\zeta(z_i - \zeta_i)) + \sum \phi_i, j(\zeta)(z_i - \zeta_i)(z_j - \zeta_j)
\]

where \( \phi_i, j(\zeta) = \frac{1}{2} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j} \ast \chi_\varepsilon(\zeta) \) (\( \chi_\varepsilon \) a mollifier), then

\[
-\text{Re} F(\zeta, z) \geq C \rho(\zeta) - \rho(z) + |\zeta-z|^3
\]

for \( |\zeta-z|, |\rho(\zeta)| \) and \( \varepsilon \) small. (c.f. [1], [4] and [5].)

1.2. As in Henkin [1] we can prove the existence of a function \( \Phi(\zeta, z) \) defined in \( V \times \tilde{\Omega} = \{ \zeta ; |\rho(\zeta)| < \nu \} \times \{ z ; |\rho(z)| < \nu \}, \nu > 0 \), with the following properties:

1. \( \Phi(\zeta, z) \) belongs to \( C^{N-1}(V \times \tilde{\Omega}) \) and is holomorphic in \( z \).

2. \( \Phi(\zeta, z) \neq 0 \) for \( z \in \tilde{\Omega}, \zeta \in V, \zeta \neq z \) and \( \rho(\zeta) \geq \rho(z) \).
(3) For \( \zeta^0 \in \partial \Omega \), there exist a neighborhood \( U \) of \( \zeta^0 \) in \( V \) and a \( C^{N-1} \) non-vanishing function \( H(\zeta, z) \) on \( U \times U \) holomorphic in \( z \) such that \( \Phi(\zeta, z) = F(\zeta, z)H(\zeta, z) \) on \( U \times U \).

(4) There exist \( n \) \( C^{N-1} \) functions \( P_i(\zeta, z) \), \( 1 \leq i \leq n \), on \( V \times \tilde{\Omega} \) holomorphic in \( z \) such that
\[
\Phi(\zeta, z) = \sum_{i=1}^{n} (z_i - \zeta_i) P_i(\zeta, z) \quad \text{on} \ V \times \tilde{\Omega}.
\]

1.3. **Definition 1.1.**

1) \( C(\zeta, z) = C_\alpha \Phi^{-n}(\zeta, z) \sum_{j=1}^{n} (-1)^j P_j \partial_z P_1 \wedge \cdots \wedge \partial_z P_n \wedge \omega(\zeta), \)

where the notation \( \hat{j} \) means \( \partial_z P_j \) is omitted, \( \omega(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n \) and \( C_\alpha = \frac{(n-1)!}{(2\pi i)^n}. \)

2) Letting
\[
\eta_i = \lambda(\tilde{\zeta}_i - \zeta_i) |z - \zeta|^2 + (1 - \lambda)P_j(\zeta, z) \Phi^{-1}(\zeta, z),
\]
define
\[
K(\zeta, z, \lambda) = C_\alpha \sum_{j=1}^{n} (-1)^j \eta_j \partial_{\zeta_j} \eta_1 \wedge \cdots \wedge \partial_{\zeta_n} \eta_n \wedge \omega(\zeta).
\]

3) Letting \( \tau_j(\tilde{\zeta}) = -(-1)^{j+1} d\zeta_j \wedge \cdots \wedge \partial_{\zeta_n}, \) we set
\[
L(\zeta, z) = -C_\alpha |\zeta - z|^2 \sum_{j=1}^{n} (\tilde{\zeta}_j - \zeta_j) \tau_j(\zeta) \wedge \omega(\zeta).
\]

4) \( K(\zeta, z) \) is given by integrating \( K(\zeta, z, \lambda) \) from 0 to 1 in \( \lambda. \)

5) Moreover we write
\[
C(\zeta, z) = \sum C_j(\zeta, z) \tau_j(\zeta) \wedge \omega(\zeta), \quad L(\zeta, z) = \sum L_j(\zeta, z) \tau_j(\zeta) \wedge \omega(\zeta),
\]
where \( C_j \) and \( L_j \) are defined by these equalities.

**Remark.** The relations \( \sum (z_j - \zeta_j) \eta_j = 1 \) and \( \sum (z_j - \zeta_j) P_j = \Phi \) imply \( d\zeta C(\zeta, z) = 0 \) and \( d\zeta, K(\zeta, z, \lambda) = 0. \)

1.4. **Lemma 1.2.** Let
\[
K(\zeta, z) = \sum_{i,j} K_{i,j}(\zeta, z) d\zeta_1 \wedge \cdots \hat{i} \wedge \cdots \wedge d\zeta_n \wedge \omega(\zeta).
\]

Then \( K_{i,j} \) has the following form:
\[
K_{i,j}(\zeta, z) = \sum_{k=0}^{n-1} \Phi^{-k}(\zeta, z) |\zeta - z|^{2k-2n} \sum_{m=1}^{n} (\tilde{\zeta}_m - \zeta_m) C_{n,1}^{m, k}(\zeta, z)
\]
where \( C_{n,1}^{m, k} \) is \( C^{N-2} \) and holomorphic in \( z. \)

**Proof.** From the definition, the coefficient of \( d\lambda \wedge d\zeta_1 \wedge \cdots \hat{i} \wedge \cdots \wedge d\zeta_n \wedge \omega(\zeta) \) is as follows.
The terms which occur by derivating denominators are represented as some linear combinations of the first two column vectors. So they can be omitted from the above determinant. Hence computing the resulting determinant proves the lemma.

1.5. The following formula is proved in Henkin [2].

\[ u(z) = \int_{\partial \Omega} u(\zeta) C(\zeta, z) + \int_{\partial \Omega} \delta u(\zeta) \wedge K(\zeta, z) - \int_{\partial \Omega} \delta u(\zeta) \wedge L(\zeta, z) \]

for \( u \in C^1(\Omega) \).

**Definition 1.2.** We define the operator \( T \) from the space of continuous \((0, 1)\) forms into the space of continuous functions by

\[ T(f) = \int_{\partial \Omega} f(\zeta) \wedge K(\zeta, z) - \int_{\partial \Omega} f(\zeta) \wedge L(\zeta, z) . \]

One can see \( \delta T(f) = f \), if \( \delta f = 0 \).

§ 2. Estimates of kernels.

2.1. **Definition 2.1.**

1) \( \sigma(\zeta) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j} \tau_j(\zeta) / \omega(\zeta) \),

2) \( S_j = \frac{\partial \rho}{\partial \zeta_j} \left| \frac{\partial \rho}{\partial \zeta_j} \right|^2 \left( \sum_{i=1}^{n} \frac{\partial \rho}{\partial \zeta_i} \frac{\partial}{\partial \zeta_i} + \Delta \rho \right) \)

where \( \Delta \) is the laplacian.

\( S_j \) is a first order differential operator in the neighborhood of \( \partial \Omega \) where \( d \rho \neq 0 \).

**Lemma 2.1.** The following two equalities hold on \( \partial \Omega \).

1) \( |\partial \rho| \omega(\zeta) \wedge \tau_s(\zeta) = (-1)^n \partial \rho / \partial \zeta_s \sigma(\zeta) \),

2) \( |\partial \rho| \omega(\zeta) = \partial \rho / \partial \zeta_s \sigma(\zeta) \).

**Proof.** Since \( \sum \partial \rho / \partial \tau_j \partial \tau_j + \sum \partial \rho / \partial \tau_j \partial \tau_j = 0 \) on \( \partial \Omega \),

\( (1) \) \( \partial \rho / \partial \tau_j \omega(\zeta) \wedge \tau_s(\zeta) = -\partial \rho / \partial \tau_s \partial \xi_1 \wedge \cdots \partial \xi_{j-1} \wedge \partial \xi_j \wedge \partial \xi_{j+1} \cdots \partial \xi_n \wedge \tau_s(\zeta) \)

\[ = (-1)^{n+j+1} \partial \rho / \partial \tau_s \partial \xi_1 \wedge \cdots \wedge \partial \xi_{n-1} \wedge \partial \xi_n \wedge \partial \tau_s(\zeta) \]

\[ = (-1)^n \partial \rho / \partial \tau_s \tau_j(\zeta) / \omega(\zeta) ; \]
Holder estimates

(2) similarly \( \frac{\partial \rho}{\partial \zeta_j} \tau_k(\xi) \wedge \omega(\xi) = \frac{\partial \rho}{\partial \zeta_k} \tau_j(\xi) \wedge \omega(\xi) \).

Multiplying both sides of (1) and (2) by \( \frac{\partial \rho}{\partial \zeta_j} \) and summing up in \( j \), we get 1) and 2).

2.2. Cartan’s formula.

Let \( \omega \) be a form and \( X \) be a vector field. Then the following formula (Cartan’s formula) holds. (c.f. Sternberg [7].)

\[
L_X \omega = X \cdot d \omega + d(X \cdot \omega),
\]

where \( L_X \) is the Lie derivative and \( \cdot \) is the interior product. In the above formula we take \( X = \partial / \partial \zeta_j \). Then the Lie derivative \( L_X \) agrees with the Euclid connection because \( X \) is parallel. So Stoke’s theorem implies the following lemma.

**Lemma 2.2.** If \( \omega \) be a \((2n-1)\) form, then

\[
\int_{\partial \Omega} \frac{\partial}{\partial \zeta_j} \omega = \int_{\partial \Omega} \frac{\partial}{\partial \zeta_j} \omega - d\omega.
\]

**Example 1.** Let \( K(\xi) \) be a \( \delta \)-closed \((n, n-1)\) form and \( u(\xi) \) be a \( C^1 \) function. Then

\[
\int_{\partial \Omega} \frac{\partial}{\partial \zeta_j} \{ u(\xi) K(\xi) \} = \int_{\partial \Omega} \delta u(\xi) \wedge \left\{ \frac{\partial}{\partial \zeta_j} \omega \right\} K(\xi).
\]

We shall use this formula in section 3 for \( u = (\xi - z)^n (\xi - z) \) and \( K = X \cdot C(\xi, z) \), where \( X \) is coming later.

**Example 2.** Let \( f \) be a \( \delta \)-closed \((0, 1)\) form and \( K(\xi) \) be a \( C^1\)-(n, n-2) form. Then

\[
\int_{\partial \Omega} \frac{\partial}{\partial \zeta_j} \{ f(\xi) \wedge K(\xi) \} = - \int_{\partial \Omega} f(\xi) \wedge \left\{ \frac{\partial}{\partial \zeta_j} \omega \right\} dK(\xi).
\]

We shall use this in the proof of Proposition 6.

**Example 3.** If \( a(\xi) \) be a \( C^1 \) function, then

\[
\int_{\partial \Omega} \frac{\partial}{\partial \zeta_j} \{ a(\xi) \sigma(\xi) \} = \int_{\partial \Omega} \{ S_j a(\xi) \} \sigma(\xi).
\]

Indeed on \( \partial \Omega \),

\[
\frac{\partial}{\partial \zeta_j} - d \{ a(\xi) \sigma(\xi) \} = (-1)^n \sum \frac{\partial}{\partial \zeta_i} \{ a(\xi) \rho / \partial \zeta_i \} \omega(\xi) \wedge \tau(\xi)
\]

\[
= \frac{\partial \rho}{\partial \zeta_j} \rho \cdot \left\{ \sum \frac{\partial \rho}{\partial \zeta_i} \partial a / \partial \zeta_i \rho + a(\xi) \Delta \rho \right\} \sigma(\xi).
\]

2.3. Definition 2.2.

1) For \( f \in C^k(\Omega) \),
\[ f^{(k)}(\zeta, z) = f(\zeta) - \sum_{1 \leq i < j \leq k} \frac{(\zeta - z)^{i}(\zeta - \bar{z})^{j}}{i! j!} D^{i}D^{j}f(z). \]

2) For \( f = \sum_{i=1}^{n} f_{i}(\zeta)d\xi_{i} \), \( \delta \)-closed \( (f_{i} \in C_{k}) \),

\[ f^{(k)}(\zeta, z) = \sum_{i=1}^{n} f_{i}^{(k)}(\zeta, z)d\xi_{i}, \]

and for \( |\gamma| \leq k + 1 \), \( f^{(k)}(\zeta) = D^{\delta}f(\zeta) \) where \( D^{\delta} = D^{\gamma}. \)

**Lemma 2.3.** If \( f \in C_{k} \), then

1) \( |f^{(k-1)}(\zeta, z)| \leq C\|f\|_{k} \cdot |\zeta - z|^{k} \),
2) \( |W[f^{(k-1)}(\zeta, z)]| \leq C\|f\|_{k} \cdot |\zeta - z|^{k-1} \),

where \( W \) is \( \partial / \partial z_{j}, \partial / \partial \bar{z}_{j}, \partial / \partial z_{f}, \) or \( \partial / \partial \bar{z}_{f} \).

3) If \( f = \sum_{i} f_{i}d\xi_{i} \) is \( \delta \)-closed \( (f_{i} \in C_{k}) \), then

\[ f^{(k)}(\zeta, z) = f(\zeta) - \sum_{1 \leq i < j \leq k + 1} \frac{(\zeta - z)^{a}}{i! j!} D^{a}f_{i}(z)D^{a}(\zeta - \bar{z}). \]

**Proof.** 1) is a simple consequence of

\[ f^{(k-1)}(\zeta, z) = \sum_{1 \leq i < j \leq k} \frac{(\zeta - z)^{i}(\zeta - \bar{z})^{j}}{i! j!} (1 - \theta)^{k-1} D^{k}f(z + \theta(\zeta - z))d\theta. \]

2) is obvious from 1) and some simple calculations. 3) See Siu [6] p. 174.

**Lemma 2.4.** Let \( c \) be a positive constant. Then

\[ \int_{|x| = 1, x_{1} > 0} \frac{d\sigma}{x_{1}^{2} + c} < \frac{\pi}{2} \log \left( 1 + \frac{1}{c} \right) \]

where \( x = (x_{1}, \ldots, x_{N}) \in \mathbb{R}^{N}, \ N \geq 3 \) and \( d\sigma \) is the canonical measure of the unit sphere in \( \mathbb{R}^{N} \). (See Range-Siu [5] p. 342.)

**Lemma 2.5.**

1) \( \int x_{1} > 0, x_{1}^{2} < \infty \) \( (x_{1} + r^{2})^{-1}r^{2-N}dx_{1} \cdots dx_{N} < C, \)

2) \( \int x_{1} > 0, x_{1}^{2} < \infty \) \( (x_{1} + \delta + r^{2})^{-1}(r^{2} + \delta^{2})^{-1}r^{2-N}dx_{1} \cdots dx_{N} < C(1 + |\log \delta|^{2}), \)

3) \( \int x_{1} > 0, x_{1}^{2} < \infty \) \( (x_{1} + \delta + r^{2})^{-2}r^{2-N}dx_{1} \cdots dx_{N} < C\delta^{1/2}, \)

where \( r^{2} = x_{1}^{2} + \cdots + x_{N}^{2} \) and \( \delta > 0. \)

**Proof.** 1) and 3) See Range-Siu [5].
\[ \int_{|z|<R, x_i>0} (x_1+\delta+r^2)^{-2} (x_2+\delta+r^2)^{-1} r^{3-N} \, dx_1 \cdots dx_N \]

\[ = C \int_0^R \frac{r(r^2+\delta^2)}{r^2+a^2} \, dr \int_{|z|=1, x_i>0} (x_1+r+\delta/r) \, d\sigma \]

\[ < C \int_0^R r(r^2+\delta^2)^{-1} \log (1+r/(r^2+\delta)) \, dr < C (1+|\log \delta|^2). \]

2.5. Later we shall prove that a function \( u \) on \( \Omega \) is \( \alpha \)-Hölder continuous by showing \( |\grad u| \leq C |\rho(z)|^{-\alpha} \), so the following proposition is important.

**Proposition 3.**

1) \( \left| \frac{x-z}{1-\alpha d} \right| < C \left( 1+|\log (-\rho(z))| \right), \)

2) \( \Phi^{-1} \left| \frac{x-z}{1-\alpha d} \right| < C \left( 1+|\log \rho(z)| \right), \)

3) \( \Phi^{-1} \left| \frac{x-z}{1-\alpha d} \right| \leq C |\rho(z)|^{-1/2}, \)

4) \( \Phi^{-1} \left| \frac{x-z}{1-\alpha d} \right| \leq C, \)

where \( d\mu \) is the Lebesgue measure induced on \( \partial \Omega \).

**Proof.** It suffices to show the above inequalities when \( |\rho(z)| \) is small. Let \( \zeta^o \) be the orthogonal projection of \( z \) to \( \partial \Omega \). We calculate the integral near \( \zeta^o \) using some new coordinate system \( (x_1, \ldots, x_{2n-1}) \) with \( x_1=\text{Im} F(\zeta, z) \) (c.f. Henkin [1]).

Let \( S_1 = \{ \zeta \in \partial \Omega; |\zeta^o-\zeta| < \varepsilon \} \). We divide the integral \( \int_{\partial \Omega} \) into \( \int_{s_1} + \int_{s_1^c} \).

Then \( \int_{s_1^c} \leq C \) where \( C \) depends on \( \varepsilon \) but not on \( |\rho(z)| \). Thus we have only to estimate the integrals on \( S_1 \). Let \( \delta = |\rho(z)| \).

1) \( \int_{s_1} \left| \frac{x-z}{1-\alpha \delta} \right| \leq C \int_{|z|<\delta} (r^2+\delta^2)^{-1} r^{3-2n} \, dx_1 \cdots dx_{2n-1} \)

\[ < C \log (1+R^2/\delta^2). \]

2) \( \int_{s_1} \left| \frac{\Phi^{-1}}{1-\alpha \delta} \right| \leq C \int_{|z|<\delta} (r^2+\delta^2)^{-1} r^{3-2n} \, dx_1 \cdots dx_{2n-1} \)

\[ < C (1+|\log \delta|^2). \]
We can show 3) and 4) analogously to the above using Lemma 2.5.

2.6. DEFINITION 2.4. We introduce the notations,

1) \( x;=a_{ia}+a_{ia} \),
2) \( K(z)=1+[\log|\rho(z)|]^{a} \).

LEMMA 2.6. 1) \( |X\Phi(\zeta, z)| \leq C|\zeta-z| \).
   2) Let \( u \) be a function defined in \( \Omega \) such that \( |\text{grad} \, u| < CK(z) \). Then \( u \) is Hölder continuous for any exponent smaller than 1.

PROOF. 1) is an easy consequence of (4) of the properties of \( \Phi(\zeta, z) \).
   2) One can show \( |u(z)-u(z')| \leq A|z-z'|^{a} \) for \( 0<\alpha<1 \) by integrating \( du \) along an appropriate path connecting \( z \) and \( z' \).

§ 3. Estimates in the polynomial case.

Here we shall establish the Hölder estimates of

\[ \int_{\partial\Omega} (\zeta-z)^{\alpha}(\zeta-z)^{a}D_{i}C(\zeta, z). \]

(Recall \( D_{j}=\partial/\partial z_{j} \)).

3.1. LEMMA 3.1. For a function \( a(\zeta, z) \) which is smooth in \( z \)

\[ (\zeta-z)^{\alpha}D_{i}a(\zeta, z)=\sum_{\beta_{1}+\beta_{2}=\beta, \beta_{2}a=0}^{\beta} \binom{\beta}{\beta_{1}}(\zeta-z)^{a-\beta_{2}a}(\zeta, z). \]

PROOF. By induction on \( \beta \), we shall prove

\[ (\zeta-z)^{a}D_{i}a=\sum_{\beta_{1}+\beta_{2}=\beta}^{\beta} \binom{\beta}{\beta_{1}}D_{i}^{\beta}[((\partial/\partial z)^{\beta_{2}}(\zeta-z)^{a}) a], \]

from which the conclusion follows easily. If \( \beta=0 \), it is trivial. Suppose (\( \star \)) is true for \( \beta \). We replace \( a \) by \( D_{j}a \). Combining the equality thus obtained and the following relation

\[ \{(\partial/\partial \zeta)^{\beta_{2}}(\zeta-z)^{a}D_{j}a=D_{j}[\{(\partial/\partial \zeta)^{\beta_{2}}(\zeta-z)^{a} a] + [(\partial/\partial \zeta)^{\beta_{2}}(\partial/\partial \zeta)(\zeta-z)^{a}] a, \]

we get (\( \star \)) with \( D^{\beta'}=D_{j}D^{\beta} \) in the place of \( D^{\beta} \). This proves the lemma.

PROPOSITION 4. Let \( |\gamma|=M \leq N-2 \). Then

\[ \int_{\partial\Omega} (\zeta-z)^{\alpha}(\zeta-z)^{a}D_{i}C(\zeta, z)=\sum_{\gamma_{1}+\gamma_{2}=\gamma, \gamma_{2}a=0}^{\gamma} \binom{\gamma}{\gamma_{1}}D_{j}^{\gamma_{1}}(\zeta-z)^{a-\gamma_{2}a}(\zeta-z)^{a}C(\zeta, z). \]

PROOF. This is an easy conclusion from Lemma 3.1.

3.2. LEMMA 3.2. Let \( |\gamma|=M \leq N-2 \). Then
\[ D^a \int_{\partial \Omega} (\zeta - z)^\beta (\xi - \bar{z})^\gamma C(\zeta, z) \]
\[ = \int_{\partial \Omega} (\zeta - z)^\beta (\xi - \bar{z})^\gamma X^a C(\zeta, z) + \sum_{k=1}^{M-1} \sum_{i=1}^{n+k} \int_{\partial \Omega} (\zeta - z)^{\beta + x} a_i(\zeta, z) / \Phi^{n+k} \sigma(\zeta) \]

where \( a_z \) is \( C^{N+K+i} \) and \( C^\infty \) in \( z \).

**Proof.** We shall prove this by induction on \( \alpha \). If \( \alpha = 0 \), there is nothing to be proved. We assume (*) is true for \( \alpha \). Then we apply \( \partial / \partial z_j \) to (*), regard \( \partial / \partial z_j \) as \( X_j - \partial / \partial z_j \) under the integral sign and convert \( \partial / \partial z_j \) to \( S_j \) using the formula in Example 3 after Lemma 2.2. Then we get

\[ D_j D^a \int_{\partial \Omega} (\zeta - z)^\beta (\xi - \bar{z})^\gamma C(\zeta, z) = \int_{\partial \Omega} (\zeta - z)^\beta (\xi - \bar{z})^\gamma X_j X^a C(\zeta, z) \]
\[ - \int_{\partial \Omega} \partial / \partial z_j \{ (\zeta - z)^\beta (\xi - \bar{z})^\gamma X^a C(\zeta, z) \} \]
\[ + \sum_{k=0}^{M-1} \sum_{i=1}^{n+k} \int_{\partial \Omega} X_j \{ (\zeta - z)^{\beta + x} a_i(\zeta, z) \Phi^{n+k} (\zeta, z) \sigma(\zeta) \} \]
\[ - \int (\zeta - z)^{\beta + x} \{ S_j a_i(\zeta, z) / \Phi^{n+k} (\zeta, z) \} \sigma(\zeta) . \]

Now,

\[ \int_{\partial \Omega} \frac{\partial}{\partial z_j} \{ (\zeta - z)^\beta (\xi - \bar{z})^\gamma X^a C(\zeta, z) \} = \int_{\partial \Omega} (\zeta - z)^\beta (\xi - \bar{z})^\gamma \left[ \frac{\partial}{\partial z_j} - X^a C(\zeta, z) \right] . \]

(See Example 1 after Lemma 2.2.) But

\[ X^a C(\zeta, z) = \sum_{k=0}^{N} \sum_{i=1}^{n+k} \sum_{i=1}^{n+k} (\zeta - z)^i a_{i.1}(\zeta, z) / \Phi^{n+k} (\zeta, z) \tau_i(\zeta) \wedge \omega(\zeta) , \]

where \( a_{i.1} \) is in the class \( C^{N+k-M}, \) so these terms are allowed to appear in the formula for \( \alpha' \). (\( D^\alpha = D_j D^a \))

\[ (\zeta - z)^{\beta + x} S_j [ a_i / \Phi^{n+k} (\zeta, z) ] \]
\[ = (\zeta - z)^{\beta + x} \mathcal{B}_j (\zeta) \left[ \frac{a_i \Delta \rho + S^* a_i}{\Phi^{n+k}} (\zeta, z) - \frac{(n+k) a_i (\zeta, z) \sum (z_i - \zeta_i) S^* P_i (\zeta, z)}{\Phi^{n+k+1} (\zeta, z)} \right] , \]

where \( \mathcal{B}_j (\zeta) = \frac{\partial^2 \rho}{\partial z_j} / | \partial \rho |^2 \) and \( S^* = \sum \partial \rho / \partial \zeta_i \partial / \partial \zeta_i \).

From the hypothesis \( a_i \in C^{N+k-M-1} \), so \( \mathcal{B}_j \Delta \Delta a_i, \mathcal{B}_j a_i \in C^{N+k-M-2} \) and \( \mathcal{B}_j a_i S^* P_i \in C^{N+k-M-1} \). Hence the above terms are allowed to appear in the formula for \( \alpha' \). In a similar way we can decompose the term \( (\zeta - z)^{\beta + x} X_j [ a_i (\zeta, z) / \Phi^{n+k} (\zeta, z) \sigma(\zeta) ] \) into sums of the desired form.
3.3. The following lemma is proved in Siu [6].

**Lemma 3.3.** Let $k$ and $m$ be integers such that $m \leq k \leq N$ and $G$ be an open set of $C^n$. Suppose $\partial \rho / \partial \bar{\zeta}_j \neq 0$ on $U$ where $U$ is the open set appeared in 1.2.

Then for any $h(\zeta, z) \in C^k(U \cap \partial U)$, there exists an $h^*(\zeta, z) \in C^{k-m+1}(U \times G)$ such that $h^*(\zeta, z) = h(\zeta, z)$ for $\zeta \in \partial U \cap U$ and $\partial h^*/\partial \bar{\zeta}_j = \gamma(\zeta, z) \rho^{m-1}(\zeta)$ for some $C^{k-m}$ function $\gamma(\zeta, z)$ on $U \times G$.

**Example.** Suppose $\partial \rho / \partial \bar{\zeta}_j \neq 0$ on $U$. Then applying the lemma to $\partial \rho / \partial \bar{\zeta}_i$ and $\phi_i, k$, we get a $C^1$ function $F^*(\zeta, z)$ such that

\[
F^*(\zeta, z) = F(\zeta, z) \quad \text{on} \quad (U \cap \partial U) \times \bar{G},
\]

\[
|\partial F^*/\partial \bar{\zeta}_j(\zeta, z)| \leq C |\zeta - z| |\rho(\zeta)|^{N-2},
\]

\[-\text{Re } F^*(\zeta, z) \geq C \{\rho(\zeta) - \rho(z) + |\zeta - z|^2\} \quad \text{for } |\rho(\zeta)| \text{ and } |\zeta - z| \text{ small.}
\]

$F^*(\zeta, z)$ will play an important role in the proofs of the following lemmas.

3.4. In what follows in this section all functions are to be $C^\infty$ in $z$.

**Lemma 3.4.** Let $a(\zeta, z)$ be a $C^k$ function and $|\beta| = k < N-2$. Then

\[
\left| \int_{\partial U} (\zeta - z)^\beta a(\zeta, z)/\Phi^{n+k}(\zeta, z)a(\zeta) \right| < C K(z).
\]

**Proof.** If $k=0$, it is already shown in Proposition 3. Let $k \geq 1$. It suffices to show the above inequality when $|\rho(z)|$ is small. As in the proof of Proposition 3, let $\zeta^0$ be the orthogonal projection of $z$ to $\partial U$ and $U$ be the neighborhood of $\zeta^0$ introduced in 1.2. Similarly to Proposition 3 we divide the integral $\int_{\partial U}$ into the sum of $\int_{S_1}$ and $\int_{S_1}$, where $S_i = \{ \zeta \in \partial U; |\zeta - \zeta^0| < \varepsilon \}$. By the same reason as in Proposition 3 we have only to estimate the integral on $S_i$. Since $d\rho \neq 0$ near $\partial U$, we can assume $\partial \rho / \partial \bar{\zeta}_i \neq 0$ near $S_i$. By Lemma 2.1 there exists a $C^k$ function $b(\zeta, z)$ such that on $S_i$

\[
(\zeta - z)^\beta a(\zeta, z)/\Phi^{n+k}(\zeta, z)a(\zeta) = (\zeta - z)^\beta b(\zeta, z)/F^{n+k}(\zeta, z)\sigma(\zeta)\tau_1(\zeta) \wedge a(\zeta).
\]

If we apply Lemma 3.3 to $b(\zeta, z)$, then we get a $C^1$ function $b^*(\zeta, z)$ such that $b(\zeta, z) = b^*(\zeta, z)$ for $\zeta \in S_i$ and

\[
|\partial b^*/\partial \bar{\zeta}_j(\zeta, z)| \leq C |\rho(\zeta)|^{k-1}.
\]

Let $F^*(\zeta, z)$ be the function in the example after Lemma 3.3 and

\[
B = \{ \zeta \in C^n; |\zeta - \zeta^0| < \varepsilon \} \cap \Omega^c \quad \text{and} \quad S_2 = \partial B \cap \Omega^c.
\]

Then we get the following by Stokes’ theorem.
\begin{align*}
  \int_{S_1} (\zeta - z)^{\beta} b(\zeta, z) F^{-n-k}(\zeta, z) \tau_1(\xi) \wedge \omega(\zeta) \\
  = \int_{S_2} d \left( (\zeta - z)^{\beta} b^*(\zeta, z) F^{* -n-k}(\zeta, z) \tau_1(\xi) \wedge \omega(\zeta) \right) \\
  - \int_{S_2} (\zeta - z)^{\beta} b^*(\zeta, z) F^{* -n-k}(\zeta, z) \tau_1(\xi) \wedge \omega(\zeta).
\end{align*}

Since the distance between \( z \) and \( S_z \) is larger than \( \varepsilon \), the integral on \( S_z \) is bounded.

\[ |d(\zeta - z)^{\beta} b^*(\zeta, z) F^{* -n-k}(\zeta, z) \tau_1(\xi) \wedge \omega(\zeta)| < C |\zeta - z|^k | \rho(\zeta) |^{k-1} | F^{* -n-k}(\zeta, z) | + C |\zeta - z|^k+1 | \rho(\zeta) |^{N-2} | F^{* -n-k-1}(\zeta, z) |.
\]

We compute the integral on \( B \) in an appropriate coordinate system \((x_1, \ldots, x_{2n})\) with \( x_1 = \rho(\zeta) \). Then

\[ C |\zeta - z| | F^*(\zeta, z) |^{-n-1} < C(x_1 + \delta + r^2)^{r^2-\delta} (r^2 + \delta)^{-1}, \]

where \( r^2 = x_1^2 + \cdots + x_{2n}^2 \) and \( \delta = |\rho(z)| \). Hence by Lemma 2.5

\[ |\int_B d(\zeta - z)^{\beta} b^*(\zeta, z) F^{* -n-k}(\zeta, z) \tau_1(\xi) \wedge \omega(\zeta)| < C K(z). \]

**Remark.** In the above lemma if \(|\beta| \geq k\), then the bound \( C K(z) \) can be replaced by a constant \( C \).

**Lemma 3.5.** Let \( k \leq N-3 \), \(|\beta| = k \) and \( a(\zeta, z) \in C^{k+1} \). Then

\[ \left| \frac{\text{grad}}{\partial z}\int_{\partial a}(\zeta - z)^{\beta} a(\zeta, z) / \Phi^{* -k}(\zeta, z) \sigma(\zeta) \right| < C K(z). \]

**Proof.** We differentiate the above integral under the integral sign. For \( \partial / \partial \zeta_j \) compute directly and for \( \partial / \partial z_j \) regard it as \( X_j - \partial / \partial \zeta_j \) and convert \( \partial / \partial z_j \) to \( S_j \). Then the conclusion follows from Lemma 3.4.

**Lemma 3.6.** Let \( 1 \leq k \leq N-1 \), \(|\beta| = k \), \(|r| \geq 1 \) and \( a(\zeta, z) \in C^{k-1} \). Then

\[ \left| \int_{\partial a}(\zeta - z)^{\beta} (\zeta - z)^r a(\zeta, z) / \Phi^{* -k}(\zeta, z) \sigma(\zeta) \right| < C K(z). \]

**Proof.** If \( k = 1 \), then

\[ |(\zeta - z)^{\beta} (\zeta - z)^r a(\zeta, z) / \Phi^{* -k}(\zeta, z) | < C |\zeta - z|^{k-1} | \Phi |^{-1}, \]

thus the conclusion follows from Proposition 3. Let \( k \geq 2 \). As in the proof of Lemma 3.4 we get
on $S_1$ for some $b(\zeta, z) \in C^{k-1}$. Next applying Lemma 3.3 we get the extensions $b^*(\zeta, z)$ and $c(\zeta, z)$ of $b(\zeta, z)$ and $(\xi - \bar{z})^\gamma$ such that

$$|\partial b^*/\partial \xi_1| \leq C |\rho(\zeta)|^{k-2} \quad \text{and} \quad |\partial c/\partial \xi_1| \leq C |\rho(\zeta)|^{k-2}.$$ 

Moreover the fact $c(\zeta, z) = (\xi - \bar{z})^\gamma$ on $S_1$ implies

$$|c(\zeta, z)| \leq |\zeta - z| + |(\xi - \bar{z})^\gamma - c(\zeta, z)| \leq C \{ |\zeta - z| + |\rho(\zeta)| \}.$$ 

Hence as in the calculation of Lemma 3.4 we obtain

$$|\partial \{ (\zeta - z)^k b^*(\zeta, z)c(\zeta, z) / F^*(\zeta, z) \} |^+ \leq C |\zeta - z|^k \{ |\zeta - z| + |\rho(\zeta)| \} / |F^*(\zeta, z)|^{k+1}$$

From this the lemma is proved by applying the argument of Lemma 3.4.

**Lemma 3.7.** Let $N-2 \geq k \geq 1$, $|\beta| = k$, $|\gamma| \geq 1$ and $a(\zeta, z) \in C^k$. Then

$$|\text{grad } \int_{S_0} (\zeta - z)^k b^*(\zeta, z)c(\zeta, z) / F^*(\zeta, z) | < C K(z).$$

**Proof.** This is easily shown with the aid of Lemma 3.6 analogously to Lemma 3.5.

**Remark.** If $|\beta| > k$, the above bound can also be replaced by $C$.

**Lemma 3.8.**

$$\int_{S_0} (\zeta - z)^k b^*(\zeta, z)c(\zeta, z) / F^*(\zeta, z) = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}.$$ 

**Proof.** By Henkin's representation (see 1.5) $(z - z')^k = \int_{S_0} (z - z')^k C(\zeta, z)$ holds.

Applying $D^\zeta$ to both sides and taking $z' = z$ we obtain the desired result.

**Lemma 3.9.** Let $Y$ be a $C^2$-holomorphic tangential vector field. Then $|Y_2 \Phi(\zeta, z)| \leq C |\zeta - z|$ for $\zeta \in \partial \Omega$.

**Proof.** This follows easily from the properties of $\Phi(\zeta, z)$.

**Proposition 5.** Let $\alpha, \beta$ be any multi-indices. Then

1) If $|\gamma| \leq N-3$, $\int_{S_0} (\zeta - z)^n b^*(\zeta - \bar{z})^k C(\zeta, z)$ is of class $C^1$.

2) For $|\gamma| = N-2$, $\int_{S_0} (\zeta - z)^n b^*(\zeta - \bar{z})^k C(\zeta, z)$ is $\frac{1}{2}$ Hölder continuous.
3) For \(|\gamma| = N-3\) and \(C^2\)-holomorphic tangential vector field \(Y\),
\[
\int_{\partial\Omega} (\zeta - z)^{\alpha} (\bar{\zeta} - \bar{z})^{\beta} Y \cdot \nabla C(\zeta, z) \, dS(\zeta) \text{ is H"older continuous for any exponent smaller than 1.}
\]

**Proof.** It suffices to show that for any multi-indices \(\alpha, \beta\)

i) for \(|\gamma| \leq N-2\), \(|D^\gamma B(\alpha, \beta, z)|\) is bounded,

ii) for \(|\gamma| = N-1\), \(|D^\gamma B(\alpha, \beta, z)| < C |\rho(z)|^{-1/2}\),

iii) for \(|\gamma| = N-3\), \(|D^\gamma Y D^\gamma B(\alpha, \beta, z)| < C K(z)\)

where \(B(\alpha, \beta, z) = \int_{\partial\Omega} (\zeta - z)^{\alpha} (\bar{\zeta} - \bar{z})^{\beta} C(\zeta, z)\).

If \(\beta = 0\), \(B(\alpha, 0, z)\) is a constant by Lemma 3.8. From Lemma 3.2 for \(|\gamma| = M \leq N-2\),
\[
D^\gamma B(\alpha, \beta, z) = \int_{\partial\Omega} (\zeta - z)^{\alpha} (\bar{\zeta} - \bar{z})^{\beta} X^\gamma C(\zeta, z)
\]

\[
+ \sum_{k=0}^{M-1} \sum_{l=1}^{N} \int_{\partial\Omega} (\zeta - z)^{\alpha + \sigma} a_\gamma(\zeta, z) / (\bar{\zeta} - \bar{z})^{\beta} C(\zeta, z) \cdot \Phi^{n+k} \sigma(\zeta).
\]

By Lemma 3.4, 3.5, 3.6 and 3.7 we have already shown i).

To prove ii), let \(|\gamma| = N-2\). If we write
\[
X^\gamma C(\zeta, z) = \sum_{k=0}^{M-1} \sum_{l=1}^{N} (\zeta - z)^{\alpha} a_\gamma(\zeta, z) / (\bar{\zeta} - \bar{z})^{\beta} C(\zeta, z) \cdot \Phi^{n+k} \sigma(\zeta),
\]

then Lemma 3.7 implies
\[
\left| \text{grad} \int_{\partial\Omega} (\zeta - z)^{\alpha + \sigma} a_\gamma(\zeta, z) / (\bar{\zeta} - \bar{z})^{\beta} C(\zeta, z) \cdot \Phi^{n+k} \sigma(\zeta) \right| < C K(z)
\]

except for \(k=0\). On the other hand, Lemma 3.5 implies
\[
\left| \text{grad} \int_{\partial\Omega} (\zeta - z)^{\alpha + \sigma} a_\gamma(\zeta, z) / (\bar{\zeta} - \bar{z})^{\beta} C(\zeta, z) \cdot \Phi^{n+k} \sigma(\zeta) \right| < C K(z).
\]

In case \(k=0\), we apply \(D_j\) directly under the integral sign to
\[
\int_{\partial\Omega} (\zeta - z)^{\alpha + \sigma} a_\gamma(\zeta, z) / (\bar{\zeta} - \bar{z})^{\beta} C(\zeta, z) \cdot \Phi^{n+k} \sigma(\zeta).
\]

Then the derivation of this term is bounded by \(C |\rho(z)|^{-1/2}\) in view of Proposition 3.

**Remark.** If \(\alpha \neq 0\) or \(|\beta| \geq 2\), the above derivation is bounded by \(C K(z)\).

To prove iii), we show \(|Y D_j D^\gamma B(\alpha, \beta, z)| < CK(z)\).

From the proof of ii) it remains to show
\[
\left| Y \int_{\partial\Omega} (\zeta - z)^{\alpha} (\bar{\zeta} - \bar{z})^{\beta} a_\gamma(\zeta, z) / (\bar{\zeta} - \bar{z})^{\beta} C(\zeta, z) \cdot \Phi^{n+k} \sigma(\zeta) \right| < C K(z).
\]
§ 4. Proofs of theorems.

4.1. Lemma 4.1. The following relations hold:

1) \[ d_\xi \{ X^a K(\zeta, z) \} = X^a d_\xi K(\zeta, z) = X^a \{ C(\zeta, z) - L(\zeta, z) \}. \]

2) \[ \frac{\partial}{\partial \zeta} \xi^a f^{(k)}(\zeta, z) = (D_j f)^{(k-1)}(\zeta, z), \]

\[ \frac{\partial}{\partial \xi} \xi^a f^{(k)}(\zeta, z) = (\bar{D}_j f)^{(k-1)}(\zeta, z), \]

\[ X\xi^a f^{(k)}(\zeta, z) = (D_j f)^{(k)}(\zeta, z) \]

for \( k \geq 0 \) (we set \( f^{(-1)}(\zeta, z) = f(\zeta) \)).

Proof. 2) is proved only by simple calculations.

1) \[ d_\xi K(\zeta, z) = \int_0^1 \frac{d_\xi K(\zeta, z, \lambda)}{d\lambda} \, d\lambda = \int_0^1 d_\lambda K(\zeta, z, \lambda) \]

\[ = K(\zeta, z, 0) - K(\zeta, z, 1) = C(\zeta, z) - L(\zeta, z). \]

(Recall \( d_{\xi, \lambda} K(\zeta, z, \lambda) = 0 \).)

4.2. The following proposition is the main tool for the proofs of the theorems.

Proposition 6. Let \( |\alpha| = k \leq N-2 \) and \( f \) be a \( C^k \) \( \delta \)-closed \((0,1)\) form on \( \bar{Q} \). Then

\[ (°) \quad D^\alpha T(f) = \sum_{1 \leq i_1 \leq k} D^\beta f_i(z) \int_{0}^{\zeta} \frac{C(z - \zeta)}{\beta!} \frac{D^\gamma C(\zeta, z)}{\gamma!} \]

\[ + \sum_{\alpha + \beta = \alpha} \left( \alpha \right) \int_{0}^{\zeta} (D^\alpha f) \cdot (D^\beta f_i) \cdot (D^\gamma f) \cdot (D^\delta f) \cdot \dots \cdot \frac{D^\gamma C(\zeta, z)}{\gamma!} \]

\[ + \sum_{\beta + \gamma = \beta} \left( \beta \right) \int_{0}^{\zeta} (D^\alpha f) \cdot (D^\beta f_i) \cdot (D^\gamma f) \cdot (D^\delta f) \cdot \dots \cdot \frac{D^\gamma C(\zeta, z)}{\gamma!} \]

\[ = (I) + (II) + (III) + (IV) + (V) \]

where \( K^{(k)}(\zeta, z) \) are written in the form

\[ \sum_{|\alpha| + |\beta| = k} a_{\xi, \zeta} (\zeta, z) (D_\xi \xi^a f^{(k)}(\zeta, z')) \cdot \frac{X^a}{X^a} (L_i(\zeta, z) - C_i(\zeta, z)) \quad (a_{\xi, \xi'} \in C^{N-k+k+|\xi|+|\xi'|}). \]

Proof. We shall prove \((°)\) by induction on \( \alpha \). If \( \alpha = 0 \) there is nothing to be proved. Suppose \((°)\) is true for \( \alpha \). We first replace \( f(\zeta) \) by \( g'(\zeta) = f^{(k)}(\zeta, z') \) in \((°) \) \( (\partial_\xi g') = 0 \) by Lemma 2.3) and then set \( z' = z \).
Hölder estimates

Noting

1) \( g_l^0(\zeta, \zeta) = f^l(\zeta, \zeta) \) for \( l \leq k \),

2) \( D^{\nu_1} D^{\nu_2} f^l(\zeta, \zeta) |_{\zeta = 0} = 0 \) for \( |\nu_1| + |\nu_2| \leq k \),

3) \( D^a T(g_{\nu}) = D^a T(f) - \sum_{\beta + 1 \leq k + 1} D^\beta f(z) D^a \left\{ \frac{(z - z')^\beta (\zeta - \overline{z'})^\gamma}{\beta! \gamma!} \right\} - \int_{\partial A} \frac{(\zeta - z')^\beta (\zeta - \overline{z'})^\gamma}{\beta! \gamma!} C(\zeta, z) \}

we get

\[
(*) \quad D^a T(f) = \sum_{\beta + 1 \leq k + 1} D^\beta f(z) \int_{\partial A} \frac{(\zeta - z)^\beta (\zeta - \overline{z})^\gamma}{\beta! \gamma!} D^a C(\zeta, z) \\
+ \sum_{\alpha_1 + \alpha_2 = \alpha} \left( \frac{\alpha_1}{\alpha_2} \right) \int_{\partial A} (D^a f)^{\alpha_1} (D^{\alpha_2}) (\zeta, z) \wedge (\zeta) \\
+ \sum_{k \leq 1} \sum_{\beta + 1 \leq k + 1} \int_{\partial A} (D^a f)^{\alpha_1} (D^{\beta}) (\zeta, z) K_{\beta, \gamma} (\zeta, z) \sigma(\zeta) \\
- \int_{\partial A} f^{(k)} (\zeta, z) \wedge D^a L(\zeta, z) \\
= (i)_a + (ii)_a + (iii)_a + (iv)_a + (v)_a .
\]

Next we compute \( D_j D^a T(f) \) regarding it as

\[
\{ D_j D^a T(f) - D^a T(D_j f) \} + D^a T(D_l f) .
\]

If we apply (*) to the inside of the bracket and (\( *) \) to the last term, we get

\[
D_j (i)_a = (i)_a + (ii)_a + (iii)_a + (iv)_a + (v)_a \\
= D_j \sum_{\beta + 1 \leq k + 1} D^\beta f(z) \int_{\partial A} \frac{(\zeta - z)^\beta (\zeta - \overline{z})^\gamma}{\beta! \gamma!} D^a C(\zeta, z) \\
- \sum_{\beta + 1 \leq k + 1} D^\beta (D_j f) \int_{\partial A} \frac{(\zeta - z)^\beta (\zeta - \overline{z})^\gamma}{\beta! \gamma!} D^a C(\zeta, z) \\
+ \sum_{\beta + 1 \leq k + 1} D^\beta (D_j f) \int_{\partial A} \frac{(\zeta - z)^\beta (\zeta - \overline{z})^\gamma}{\beta! \gamma!} D^a C(\zeta, z) \\
= \sum_{\beta + 1 \leq k + 1} D_j D^\beta f(z) \int_{\partial A} \frac{(\zeta - z)^\beta (\zeta - \overline{z})^\gamma}{\beta! \gamma!} D^a C(\zeta, z) \\
+ \sum_{\beta + 1 \leq k + 1} D_j D^\beta f(z) \int_{\partial A} \frac{(\zeta - z)^\beta (\zeta - \overline{z})^\gamma}{\beta! \gamma!} D^a C(\zeta, z) \\
= \sum_{\beta + 1 \leq k + 1} D_j D^\beta f(z) \int_{\partial A} \frac{(\zeta - z)^\beta (\zeta - \overline{z})^\gamma}{\beta! \gamma!} D^a C(\zeta, z) . \quad (\bar{D}^a = D_j D^a) .
\]
$$D_{\alpha}(ii)_{\alpha,f} - (ii)_{\alpha,D_f} + (\text{II})_{\alpha,D_f}$$

$$= \sum_{a_1 + a_2 = a} \left( \frac{\alpha}{\alpha_1} \right) \left[ (X_j - \partial / \partial \zeta_j) \{ (D^{a_1} f)^{(a_2)} \wedge X^{a_2} K(\zeta, z) \} - (D_j D^{a_1} f)^{(a_2)} \wedge X^{a_2} K(\zeta, z) \} + (D_j D^{a_1} f)^{(a_2)} \wedge X^{a_2} K(\zeta, z) \} \right]$$

$$= \sum_{a_1 + a_2 = a} \left( \frac{\alpha}{\alpha_1} \right) \left[ (D^{a_1} f)^{(a_2)} \wedge X_j X^{a_2} K + (D_j D^{a_1} f)^{(a_2)} \wedge X^{a_2} K \right.$$

$$\left. + (D^{a_1} f)^{(a_2)} \wedge (\partial / \partial \zeta_j \wedge X^{a_2} \{ C(\zeta, z) - L(\zeta, z) \}) \right]$$

$$= \sum_{a_1 + a_2 = a} \left( \frac{\alpha}{\alpha_1} \right) \left[ (D^{a_1} f)^{(a_2)} \wedge X^{a_2} K(\zeta, z) \right.$$

$$\left. + (-1)^n \sum_{i=1}^n \sum_{a_1 + a_2 = a} \left( \frac{\alpha}{\alpha_1} \right) \left[ (D^{a_1} f_i)^{(a_2)} X^{a_2} \{ C(\zeta, z) - L(\zeta, z) \} \right. \right.$$}

where we used Example 2 after Lemma 2.2. The terms in the last sum are to go into (III)$_{\alpha,f}$; in fact we have

$$(-1)^n \sum_{i=1}^n \sum_{a_1 + a_2 = a} \left( \frac{\alpha}{\alpha_1} \right) \left[ (D^{a_1} f_i)^{(a_2)} X^{a_2} \{ C(\zeta, z) - L(\zeta, z) \} \right.$$}

and $a_{\alpha, i} = \partial / \partial \zeta_j | \partial / \partial \zeta_j | \wedge X^{a_2}$ actually belongs to $C^{N-(k+1)+|a_2|}$. Now we compute $D_{\alpha}(\text{III})_{\alpha,f} - (\text{III})_{\alpha,D_f} + (\text{III})_{\alpha,D_f}$ termwisely.

$$\left( X_j - \partial / \partial \zeta_j \right) \{ g^{(b)} K_m^i \sigma(\zeta) \} - (D_j g)^{(b)} K_m^i \sigma(\zeta) + g^{(c)} \left( K_m^i \sigma(\zeta) \right.$$

$$\left. = (D_j g)^{(c-1)} K_m^i \sigma(\zeta) \right.$$}

where $g = D^b D^i f_i$ and $l = k - |b| - |i|$. One can see that the hypothesis for $K_m^i$ are satisfied for these terms.

$$D_{\alpha}(\text{IV})_{\alpha,f} - (\text{IV})_{\alpha,D_f} + (\text{IV})_{\alpha,D_f}$$

$$= - \int \{ X_j - \partial / \partial \zeta_j \left[ f^{(c)}(\zeta, z) \wedge D^c L(\zeta, z) \right] \} \right.$$}

$$\text{Thus the proof of the proposition is complete.}$$
4.3. Proof of Theorem 1.

Let \( f \) and \( k \) be as in Theorem 1. We prove that, for each term in the decomposition (*) of \( D^aT(f) \), the 1/2 Hölder norm is estimated by the right side of the desired inequality. We have already established the estimates for (I) in section 3. It is well-known that (IV) is Hölder continuous for any exponent smaller than 1. Proposition 6 assures that each term of (III) is once more differentiable in \( \zeta \). So we compute its gradient.

\[
D_j \left( \int g^{(1)}(\zeta, z) \hat{R}(\zeta, z) \sigma(\zeta) \right)
= \int X_j \{ g^{(1)}(\zeta, z) \hat{R}(\zeta, z) \sigma(\zeta) \} - S_j \int g^{(1)}(\zeta, z) \hat{R}(\zeta, z) \sigma(\zeta),
\]

where \( l = k - |\beta| - |\gamma| - 1 \), \( g^{(1)}(\zeta, z) = (D^\delta \hat{D}^\beta f)(k-1) \hat{R}(\zeta, z) = K_{\beta, \gamma}(\zeta, z) \).

By Proposition 6, \( |W\hat{R}| < C(|\zeta - z|^{1-\frac{n}{2}} + |\zeta - z|^{1}|\Phi|^{-\frac{n}{2}}) \) where \( W = X_j \) or \( S_j \). Hence

\[
\text{grad} \left( \int (D^\delta \hat{D}^\beta f)(k-1) \hat{R}(\zeta, z) \sigma(\zeta) \right) < C K(z).
\]

Next we observe each term of (II). If \( \alpha_i \) is neither 0 nor \( \alpha \), by Lemma 1.2 \( (D^a f)^{1/a-1} \wedge X^a K(\zeta, z) \) is once more differentiable in \( \zeta \). Thus by the previous method they are shown to be Hölder continuous for any exponent smaller than 1.

Hence the essential parts for the exact 1/2 Hölder estimate are

\begin{itemize}
  \item[(1)] \( \int f^{(k-1)}(\zeta, z) \wedge X^a K(\zeta, z) \)
  \item[(2)] \( \int D^a f(\zeta) \wedge K(\zeta, z) \).
\end{itemize}

We apply \( \partial / \partial z_j \) to (1) and (2) directly under the integral sign. Then Proposition 3 implies that their gradients are dominated by \( |\rho(z)|^{-1/2} \). The proof is complete.

4.4. Proof of Theorem 2.

Let \( k, \alpha, \beta, f \) and \( Y \) be as in Theorem 2. If we write \( Y = \sum a_I(z) \partial / \partial z_I + \sum b_i(z) \partial / \partial z_i \), then

\[
YD^aT(f) = Y_1D^aT(f) + \sum b_i(z)D^a f_i(z),
\]

where \( Y_1 = \sum a_I(z) \partial / \partial z_I \). Hence we can consider \( Y = Y_1 \). By the argument in the proof of Theorem 1 and Proposition 5 it remains to show that

\begin{itemize}
  \item[(1)] \( \sum_{i=1}^5 a_i(z) \int D_i D^a f(\zeta) \wedge K(\zeta, z) \)
  \item[(2)] \( \sum_{i=1}^5 a_i(z) \int D_i D^a f(\zeta) \wedge K(\zeta, z) \).
\end{itemize}
are both \( \beta \)-Hölder continuous.

(I) Set \( g(\zeta) = D^s f(\zeta) \). It suffices to show

\[ \sum a_i(z) \int D_i g(\zeta) \wedge \partial/\partial z_i K(\zeta, z) < C K(z). \]

Then

\[ D_i g(\zeta) \wedge \partial/\partial z_i K(\zeta, z) = D_i g(\zeta) \wedge X_i X^s K(\zeta, z) - D_i g(\zeta) \wedge D_j K(\zeta, z). \]

In the right hand side the first term is bounded by \( C K(z) \). Now

\[ D_i g \wedge D_j K - D_i g \wedge D_i K = D_i \{ D_i g \wedge K \} - D_i \{ D_i g \wedge K \}. \]

So

\[ \int D_i g(\zeta) \wedge D_j K(\zeta, z) - D_j g(\zeta) \wedge D_i K(\zeta, z) = \int D_i g \wedge [\partial/\partial \zeta_i - (C(\zeta, z) - L(\zeta, z))] - D_j g \wedge [\partial/\partial \zeta_i - (C(\zeta, z) - L(\zeta, z))]. \]

Hence the difference is dominated by \( K(z) \).

REMARK. In the above calculus we need approximate \( g(\zeta) \) by smooth form since it is only continuously differentiable. In Kerzman [4] he proved that for \( \delta \)-closed (in distribution sense) \( C^1 \) form \( f \) on \( \Omega \), there exist \( C^\infty \) \( \delta \)-closed forms \( f \in C \) which converge to \( f \) in \( C^1 \) topology.

Thus to estimate \( \sum a_i(z) \int D_i g(\zeta) \wedge D_j K(\zeta, z) \), it suffices to estimate

\[ \sum a_i(z) \int D_j g(\zeta) \wedge D_i K(\zeta, z) = \sum a_i(z) \int D_j g(\zeta) \wedge X_i X^s K(\zeta, z) - D_j g(\zeta) \wedge Y_i K(\zeta, z). \]

But these terms are bounded by \( K(z) \) from Lemma 2.6 and 3.9.

(II) \( f^{(k-1)}(\zeta, z) \wedge X_i X^s K(\zeta, z) = [f^{(k-1)}(\zeta, z) \wedge \partial/\partial z_i X^s K(\zeta, z) \]

\[ + \partial/\partial \zeta_i \{ f^{(k-1)}(\zeta, z) \wedge X^s K(\zeta, z) \} - (D_i f)^{(k-2)}(\zeta, z) \wedge X^s K(\zeta, z)] \]

\[ = [f^{(k-1)}(\zeta, z) \wedge \partial/\partial z_i X^s K(\zeta, z) + f^{(k-1)}(\zeta, z) \wedge \{ \partial/\partial \zeta_i - X^s C(\zeta, z) \} \]

\[ - (D_i f)^{(k-2)}(\zeta, z) \wedge X^s K(\zeta, z)]. \]

The last two terms in the integrand are differentiable in \( \zeta \), so by the preceding method we can show that the last two terms are \( \beta \)-Hölder continuous. Hence in order to prove (II) is \( \beta \)-Hölder continuous, we have only
to show
\[(\#) \quad \left| \sum a_i(x) D_j \int f^{(k-1)}(\zeta, z) \wedge \partial/\partial z_i X^a K(\zeta, z) \right| < CK(z). \]

But
\[D_j \int f^{(k-1)}(\zeta, z) \wedge \partial/\partial z_i X^a K(\zeta, z) = \left[ X_j \{ f^{(k-1)}(\zeta, z) \wedge \partial/\partial z_i X^a K(\zeta, z) \} - f^{(k-1)}(\zeta, z) \wedge \partial/\partial z_i X^a \{ \partial/\partial \zeta_j \wedge C(\zeta, z) \} \right]. \]

Hence
\[\sum a_i(x) D_j f^{(k-1)}(\zeta, z) \wedge \partial/\partial z_i X^a K(\zeta, z) = \left[ \int (D_j f)^{(k-1)}(\zeta, z) \wedge Y_z X^a K(\zeta, z) + f^{(k-1)}(\zeta, z) \wedge Y_z X^a X^a K(\zeta, z) - f^{(k-1)}(\zeta, z) \wedge Y_z X^a \{ \partial/\partial \zeta_j \wedge C(\zeta, z) \} \right]. \]

Hence Lemma 3.9 implies (\#).

References


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