Minimal length of Liouville chain for solutions
of an algebraic differential equation

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§ 0. Introduction.

The author [5] proved that the order of a liouvillian element in Liouville's sense is at most 3 if it satisfies an algebraic differential equation of the first order. Here, we shall generalize his theorem as follows: The order of a liouvillian element in Liouville's sense is at most $3n$ if it satisfies an algebraic differential equation of order $n$.

Let $k$ be an ordinary differential field of characteristic 0, and $\mathcal{Q}$ be a universal extension of $k$. We assume that the field of constants $k_0$ of $k$ is algebraically closed. A finite chain of extending differential subfields $L_0 \subset L_1 \subset \cdots \subset L_n$ in $\mathcal{Q}$ is called a Liouville chain over $k$ if the following three conditions are satisfied:

(i) $L_0$ is an algebraic extension of $k$ of finite degree;
(ii) The field of constants of $L_n$ is $k_0$;
(iii) For each $i (1 \leq i \leq n)$ there exists a finite system of elements $w_1, \ldots, w_r$ of $L_i$ which satisfies the following two conditions: either $w_j \in L_{i-1}$ or $w_j/w_j$ is the derivative of an element of $L_{i-1}$ for each $j (1 \leq j \leq r)$; $L_i$ is an algebraic extension of $L_{i-1}(w_1, \ldots, w_r)$ of finite degree.

A subfield $L$ of $\mathcal{Q}$ is called a liouvillian extension of $k$ if there exists a Liouville chain over $k$ which ends with $L$. Let $z$ be an element of $\mathcal{Q}$. Then, $z$ is called a liouvillian element over $k$ if there exists a Liouville chain over $k$ such that its end contains $z$. In particular, if $k=k_0(x)$ with $x'=1$, then a liouvillian element over $k$ is called an elementary transcendental function of $x$ over $k_0$ (cf. Watson [9, p. 111]). The following definition is due to Liouville [3]: A liouvillian element $z$ over $k$ is said to be of order $m$ if $m$ is the minimum of those $n$ such that the end of a Liouville chain $L_0 \subset \cdots \subset L_n$ over $k$ contains $z$.

Theorem. The order of a liouvillian element over $k$ satisfying an algebraic differential equation over $k$ of order $n$ is at most $3n$.

It follows from the following:

Lemma. Let $k^*$ be a finitely generated differential extension field of $k$ in $\mathcal{Q}$
whose field of constants is \( k_0 \), and \( L^* \) be a differential extension field of \( k^* \) in \( Q \). Suppose that \( L^* \) is contained in a liouvillian extension \( K^* \) of \( k^* \). Then, there exists such an extending chain \( L_0 \subset \cdots \subset L_n \) of differential subfields of \( Q \) that satisfies the following conditions: \( L_0 \) is an algebraic extension of \( k^* \) of finite degree; \( n = \text{tr. deg}_{k^*} L^*; \) \( L^* \subset L_n \); for each \( i \) \((0 < i \leq n)\) there exist an element \( y_i \) of \( L_i \) and elements \( \alpha_i, \beta_i \) of \( L_{i-1} \) such that \( y_i = \alpha_i y_i + \beta_i \) and \( L_i \) is an algebraic extension of \( L_{i-1}(y_i) \) of finite degree.

In § 1 we shall show that Theorem follows from Lemma. In § 3 we shall prove Lemma. In the last § 4 an example of liouvillian element in Theorem whose order attains \( 3n \) will be given. In § 2 we shall show the following:

**Proposition.** Let \( A \) be a differential extension field of \( k \) in \( Q \). Suppose that two elements \( t_1, t_2 \) of \( Q \) are algebraically independent over \( A \) and satisfy \( t_i = a_i t_1 + b_i \) \((i=1, 2)\); here we assume that each of \( a_1, b_1 \) is algebraic over \( A \) and each of \( a_2, b_2 \) is algebraic over \( A(t_1) \). Let \( B \) be a differential extension field of \( A \) in \( Q \). Suppose that \( t_1 \) is transcendental over \( B \), \( t_2 \) is algebraic over \( B(t_1) \) and the field of constants of \( B(t_1) \) is \( k_0 \). Then, there exist an element \( t \) of \( B_1 \) and elements \( a, b \) of \( A_1 \) such that \( t \) is transcendental over \( A \) and \( t' = at + b \), where \( A_1 \) and \( B_1 \) are the algebraic closures of \( A \) and \( B \) respectively.

**Remark 1.** If we replace \( \frac{w_i}{w_j} \in L_{i-1} \) in the definition of a liouvillian element by \( \frac{w_i}{w_j} = a' / a, a \in L_{i-1} \), then we have an “elementary” liouvillian element. For such an element let us modify the definition of “order” by the above replacement. Then the order of an elementary liouvillian element satisfying an algebraic differential equation of order \( n \) is at most \( 2n \). This theorem is due to Singer [8] (cf. Rosenlicht and Singer [7, Theorem 1]). In the special case where \( n = 1 \) and \( k = C(x) \) with \( x' = 1 \) it is due to Mordukhai-Boltovskoi [4] (cf. Ritt [6, p. 86]).

**Remark 2.** Our definition of “liouvillian extension” is slightly stronger than the ordinary one. If we replace \( \frac{w_i}{w_j} = a' / a, a \in L_{i-1} \) by \( \frac{w_i}{w_j} \in L_{i-1} \), then we have the ordinary definition. The difference is not essential (cf. § 1). If we modify our definition of “order” by this replacement, then \( 3n \) in our result is replaced by \( 2n \).

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§ 1. Kolchin’s existence theorem.

Kolchin [2] obtained the following theorem: Let \( A \) be a proper prime differential ideal in the differential polynomial algebra \( k \{u_1, \ldots, u_n\} \) over \( k \), and let \( J \) be an element of \( k \{u_1, \ldots, u_n\} \) which is not in \( A \). Then \( A \) has a solution \((\eta_1, \ldots, \eta_n)\) such that \( J(\eta_1, \ldots, \eta_n) \neq 0 \) and the field of constants of \( k \langle \eta_1, \ldots, \eta_n \rangle \)
is \( k_0 \).

By this theorem we shall show that our Theorem follows from Lemma: Suppose that \( y \) is a liouvillian element over \( k \) satisfying an algebraic differential equation over \( k \) of order \( m \). Let \( L \) denote \( k\langle y \rangle \). Then, \( L \) is contained in a liouvillian extension of \( k \). The transcendence degree \( n \) of \( L \) over \( k \) is at most \( m \). If we set \( k=k^* \) and \( L=L^* \), then the assumption of Lemma is satisfied. Hence there exists such an extending chain \( L_0 \subset \cdots \subset L_n \) of differential subfields in \( \Omega \) as stated in Lemma. There exist elements \( v_0, \ldots, v_{on}, w_{01}, \ldots, w_{0n}, z_{01}, \ldots, z_{on} \) of \( \Omega \) such that

\[
v_{0i}=\alpha_i; \quad w_{0i}=v_{0i}w_{0i}; \quad w_{0i} \neq 0; \quad w_{0i}z_{0i}=\beta_i
\]

for each \( i (1 \leq i \leq n) \). We set \( K=L_n \) for simplicity. Let \( \Sigma \) be a prime differential ideal in the differential polynomial algebra

\[ K\{V_1, \ldots, V_n, W_1, \ldots, W_n, Z_1, \ldots, Z_n\} \]

over \( K \) whose generic point is \( (v_{01}, \ldots, v_{0n}, w_{01}, \ldots, w_{0n}, z_{01}, \ldots, z_{0n}) \), and \( T \) be \( \Pi W_i \ (1 \leq i \leq n) \). Then, \( T \in \Sigma \). The field of constants of \( K \) is \( k_0 \), since \( K \) is an algebraic extension of \( L \). By the existence theorem of Kolchin there exists a zero \( (v_1, \ldots, v_n, w_1, \ldots, w_n, z_1, \ldots, z_n) \) of \( \Sigma \) in \( \Omega \) such that \( T(w_1, \ldots, w_n) \neq 0 \) and the field of constants of

\[ K\langle v_1, \ldots, v_n, w_1, \ldots, w_n, z_1, \ldots, z_n \rangle \]

is \( k_0 \). We have

\[ (y_i/w_i)y'=z_i'=0, \quad 1 \leq i \leq n \]

by \( y_i'=\alpha_i y_i+\beta_i \). Then, there exists a constant \( c_i \) such that \( y_i=w_i(z_i+c_i) \) for each \( i (1 \leq i \leq n) \). We have \( c_i \in k_0 \). Since \( K \) is an algebraic extension of \( k(y_1, \ldots, y_n) \) of finite degree, there exists an element \( t \) of \( K \) such that \( K=k(y_1, \ldots, y_n, t) \). We define a chain \( M_0 \subset M_1 \subset \cdots \subset M_{sn} \) by

\[
M_0=k(\alpha_1, \beta_1),
M_{si-2}=M_{si-3}(v_i) \quad (1 \leq i \leq n),
M_{si-1}=M_{si-2}(w_i) \quad (1 \leq i \leq n),
M_{si}=M_{si-1}(z_i, \alpha_{i+1}, \beta_{i+1}) \quad (1 \leq i < n),
M_{sn}=M_{sn-1}(z_n, t).
\]

This is a Liouville chain over \( k \) and \( M_{sn} \) contains \( y \).
§ 2. Proof of Proposition.

We shall prove that there exist an element \( t_3 \) of \( B_1(t_1, a_2, b_2) \) and elements \( a_3, b_3 \) of \( A_1(t_1, a_2, b_2) \) such that \( t_3 \) is transcendental over \( A_1(t_1) \) and \( t_3 = a_3 t_3 + b_3 \).

Let \( C \) and \( D \) denote \( A_1(t_1, a_2, b_2) \) and \( B_1(t_1, a_2, b_2) \) respectively, and \( G \) be the minimal polynomial of \( t_2 \) over \( D \):

\[
G(T) = T^g + v_1 T^{g-1} + \cdots + v_g, \quad v_i \in D \quad (1 \leq i \leq g).
\]

Then, differentiating \( G(t_2) = 0 \) we have

\[
t_2' \left( g t_2^{g-1} + (g-1) v_1 t_2^{g-2} + \cdots + v_{g-1} \right) + v_1' t_2^{g-1} + \cdots + v_g' = 0.
\]

Hence

\[
v_1' = v_1 a_2 - g b_2
\]

by \( G(t_2) = 0 \) and \( t_2 = a_2 t_2 + b_2 \). Thus

\[
(g t_2 + v_1)' = a_2 g t_2 + v_1.
\]

Suppose that \( v_1 \) is algebraic over \( C \). Then \( g t_2 + v_1 \) is transcendental over \( C \) and algebraic over \( D \). Since \( k_0 \) is algebraically closed, the field of constants of \( D(t_2) \) is \( k_0 \). Hence there exists a positive integer \( q \) such that \( (g t_2 + v_1)^q \in D \). As \( t_3 \) we can take \( (g t_2 + v_1)^q : t_3 = q a_3 t_3 \). If \( v_1 \) is transcendental over \( C \), we can take \( v_1 \) as \( t_3 \). Thus the existence of \( t_3, a_3 \) and \( b_3 \) is proved.

We consider \( C \) and \( D \) as one-dimensional algebraic function fields over \( A_1 \) and \( B_1 \) respectively. There exists a prime divisor \( P \) of \( C \) such that \( v_P( t_1) < 0 \), where \( v_P \) is the normalized valuation belonging to \( P \). Let \( \tau \) be prime element in \( P \) such that \( \tau \in C \). Then \( v_P(\tau') > 0 \) by \( t_1' = a_1 t_1 + b_1 : \) For, \( \tau = \sigma^e \) with some prime element \( \sigma \) in \( P \). We have \( -e \sigma' = a_1 \sigma + b_1 \sigma^{e+1} \) and \( v_P(\sigma') > 0 \). There exists uniquely a prime divisor \( Q \) of \( D \) such that the restriction of \( v_Q^* \) to \( C \) is \( v_P \), where \( v_Q^* \) is the normalized valuation belonging to \( Q \). In this \( Q, \tau \) is a prime element. The completion \( C_P \) of \( C \) with respect to \( P \) is a differential extension of \( C \), and the completion \( D_Q \) of \( D \) with respect to \( Q \) is a differential extension of \( D \); the differentiation is continuous in each completion (cf. Chevalley [1, p. 114]). The latter \( D_Q \) is a differential extension of the former \( C_P \). In \( D_Q \) we have

\[
\tau' = \sum f_i \tau^{i+d}, \quad f_i \neq 0, f_i \in A_1, \quad d > 0,
\]

\[
a_3 = \sum a_i^* \tau^{i+r}, \quad a_i^* \neq 0, a_i^* \in A_1,
\]

\[
b_3 = \sum b_i^* \tau^{i+r}, \quad b_i^* \neq 0, b_i^* \in A_1,
\]

\[
t_3 = \sum i \gamma_i \tau^{i+p}, \quad \gamma_i \neq 0, \gamma_i \in B_1, \quad 0 \leq i < \infty;
\]

here we assume that \( f_i = a_i^* = b_i^* = 0 \) if \( i < 0 \). We shall prove that \( \gamma_i \in A_1 \) for some \( j \). To the contrary suppose that each of \( \gamma_i \) is in \( A_1 \). Since \( t_1 \) and \( t_3 \) are alg-
braically dependent over \( B_1 \), we have

\[ \sum e_{ij} t_i t_j^i = 0 \quad (1 \leq i, j \leq \lambda), \quad e_{ij} \in B_1, \]

where some \( e_{ij} \) is not 0. Let \( \{ \omega_1, \cdots, \omega_p \} \) be a basis of the linear space spanned by all \( e_{ij} \) over \( A_1 \). Then for each \( i, j \) \((1 \leq i, j \leq \lambda)\)

\[ e_{ij} = \sum \delta_{ijh} \omega_h \quad (1 \leq h \leq \mu), \quad \delta_{ijh} \in A_1. \]

We have

\[ 0 = \sum (\sum \delta_{ijh} t_i t_j^i) \omega_h \quad (1 \leq h \leq \mu, 1 \leq i, j \leq \lambda). \]

By our assumption \( t_1, t_3 \in A_1((\tau)) \). Hence we have

(1)

\[ \sum \delta_{ijh} t_i t_j^i \in A_1((\tau)) \quad (1 \leq i, j \leq \lambda) \]

for each \( h (1 \leq h \leq \mu) \). Since \( \omega_1, \cdots, \omega_p \) are linearly independent over \( A_1 \), each of (1) is 0. Since \( t_1, t_3 \) are algebraically independent over \( A_1 \), we have

\[ \delta_{ijh} = 0, \quad 1 \leq i, j \leq \lambda; 1 \leq h \leq \mu. \]

Hence, each of \( e_{ij} \) is 0. This is a contradiction. Thus we may suppose that \( \gamma_j \in A_1 \) and \( \gamma_i \in A_1 \) \((0 \leq i < j)\) for some \( j (j \geq 0) \). Differentiating the expression of \( t_3 \) in \( D_0 \) we have

\[ \sum \gamma^j \tau^{i+p} + \left( \sum (i+p) \gamma_i \tau^{i+p-1} \right) \left( \sum f_i \tau^{i+d} \right) \]

\[ = \left( \sum a_i^* \tau^{i+p} \right) \left( \sum \gamma_i \tau^{i+p} \right) + \sum b_i^* \tau^{i+d} \quad (0 \leq i < \infty) \]

by \( t_3 = a_3 t_3 + b_3 \). We shall see that \( s \geq 0 \). To the contrary suppose that \( s < 0 \). Then comparing the coefficients of \( \tau^p \) in (2), we have

\[ \gamma_j + \sum (i+p) \gamma_i f_m \quad (i+m+d-1-s=j; \ i \geq 0) \]

\[ = b_{p, p+j-2}^* + \sum a_i^* \gamma_m \quad (i+m=j; \ m \geq 0); \]

here we assume that \( \gamma_{s+j}=0 \) if \( s+j < 0 \). Since \( a_s^* \neq 0 \), we have \( \gamma_j \in A_1 \). This is a contradiction. Hence \( s \geq 0 \). Comparing the coefficients of \( \tau^p \) in (2), we have

\[ \gamma_j' + \sum (i+p) \gamma_i f_m \quad (i+m+d-1=j; \ i \geq 0) \]

\[ = b_{p, p+j-2}^* + \sum a_i^* \gamma_m \quad (i+m+s=j; \ m \geq 0). \]

Hence, \( \gamma_j' = a \gamma_j + b \) with \( a = a^*_{s-j} (j+p) f_{1-d} \) and

\[ b = b_{p, p+j-2}^* + \sum \left( a_{j+1-s}^* (i+p) f_{j-1-d} \right) \gamma_i. \]

Here \( i \) runs through \( 0, \cdots, j-1 \), because \( d > 0 \) and \( s \geq 0 \). We have \( b \in A_1 \). Since \( \gamma_j \) is transcendental over \( A_1 \), we can take \( \gamma_j' \) as \( t \).
§ 3. Proof of Lemma.

Let \( A \) be the set of all pairs \((k^*, L^*)\) satisfying the assumption of Lemma. For each pair \((k^*, L^*)\) of \( A \) there exists such an extending chain \( N_0 \subseteq N_1 \subseteq \cdots \subseteq N_f \) in \( \Omega \) that satisfies the following condition:

(iv) \( N_s \) is an algebraic extension of \( k^* \) of finite degree; \( L^* \) is contained in \( N_f \); \( \text{tr. deg}_{k^*} N_f = f \); the field of constants of \( N_f \) is \( k_0 \); for each \( i \) (\( 1 \leq i \leq f \)) there exist an element \( t_i \) of \( N_i \) and elements \( a_i, b_i \) of \( N_{i-1} \) such that \( t_i = a_i t_{i-1} + b_i \) and \( N_i \) is an algebraic extension of \( N_{i-1}(t_{i-1}) \) of finite degree. For example we can make it from a Liouville chain over \( k^* \) which ends with \( K^* \). For a pair \((k^*, L^*)\) of \( A \) let us define \( f(k^*, L^*) \) as the minimum of those \( f \) such that the condition (iv) is satisfied. Then

\[
f(k^*, L^*) \geq \text{tr. deg}_{k^*} L^*.
\]

Our Lemma asserts that the equality holds. To the contrary suppose that the subset \( \Gamma \) of all pairs \((k^*, L^*)\) of \( A \) for which the equality does not hold is not empty. Let \( \Gamma_e \) be the set of all pairs \((k^*, L^*)\) of \( \Gamma \) such that \( \text{tr. deg}_{k^*} L^* = e \). Let \( n \) be the minimum of those \( e \) such that \( \Gamma_e \) is not empty, and \( m \) be the minimum of \( f(k^*, L^*) \) where \((k^*, L^*)\) runs over all elements of \( \Gamma_e \). Then, \( m > n \geq 1 \). We assume that \((k^*, L^*) \in \Gamma_n, f(k^*, L^*) = m \) and \( N_0 \subseteq \cdots \subseteq N_m \) satisfies (iv) for \((k^*, L^*)\) with \( f = m \). Consider \((N_i, N_i(L^*))\). It belongs to \( A \) (cf. § 1). We have \( \text{tr. deg}_{L^*} N_i(L^*) = n \) and \( N_1 \subseteq \cdots \subseteq N_m \) satisfies (iv) for \((N_i, N_i(L^*))\). Hence, \((N_1, N_1(L^*)) \in \Gamma \) because of the minimality of \( m \) and \( n \). Let \( e \) be the transcendence degree of \( N_1(L^*) \) over \( N_1 \). Then, there exists an extending chain \( H_0 \subseteq \cdots \subseteq H_e \) satisfying the condition (iv) for \((N_1, N_1(L^*))\). The chain \( N_0 \subseteq H_0 \subseteq \cdots \subseteq H_e \) satisfies the condition (iv) for \((k^*, L^*)\). Hence, \( e = n \) and \( m = n + 1 \). Thus, \( t_1 \) is transcendental over \( L^* \) and \( t_2 \) is algebraic over \( L^*(t_1) \). Two elements \( t_1 \) and \( t_2 \) are algebraically independent over \( k^* \) by the assumption (iv). By Proposition there exist an element \( t \) of the algebraic closure of \( L^* \) and elements \( a, b \) of the algebraic closure of \( k^* \) such that \( t' = at + b \) and \( t \) is transcendental over \( k^* \). The transcendence degree of \( L^*(a, b, t) \) over \( k^*(a, b, t) \) is \( n - 1 \). Hence, \((k^*(a, b, t), L^*(a, b, t)) \in \Gamma \). There exists a chain \( H_0^* \subseteq H_1^* \subseteq \cdots \subseteq H_{n-1}^* \) satisfying the condition (iv) for \((k^*(a, b, t), L^*(a, b, t))\). The chain \( k^*(a, b) \supseteq H_0^* \subseteq H_1^* \subseteq \cdots \subseteq H_{n-1}^* \) satisfies (iv) for \((k^*, L^*)\). Thus \( f(k^*, L^*) \) is \( n \). This is a contradiction.

§ 4. An example.

We assume that \( k = k_0(x) \) with \( x' = 1 \). In the differential polynomial algebra \( k[u_1, \ldots, u_n] \) over \( k \) we define \( F_i \) (\( 1 \leq i \leq n \)) by
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\[ F_i = u_i - u_i/(\alpha x) - 1/(\alpha x + 1), \quad \alpha \in k_0, \alpha \neq 0, \]
\[ F_i = (u_{i-1} + 1)(u_{i-1} - u_i) - u_{i-1}, \quad (2 \leq i \leq n). \]

There exists a solution \((y_1, \ldots, y_n)\) of \(F_1 = F_2 = \cdots = F_n = 0\) in \(\Omega\). Suppose that \(\alpha\) is not a rational number. Then, the element \(y_n\) is proved to be a liouvillian one over \(k\). It satisfies an algebraic differential equation over \(k\) of order \(n\). It can be shown that the order of \(y_n\) over \(k\) is \(3n\).

\textbf{Bibliography}


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