§ 1. Introduction.

Let $Q$ be a domain exterior to a finite number of bodies in $E^3$ with the smooth boundary $\partial Q$. The motion of the incompressible viscous fluid in $Q$ is described by the following system of the Navier-Stokes equations for the velocity $u=(u_1(x, t), u_2(x, t), u_3(x, t))$ of the fluid and the pressure $p=p(x, t)$:

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial t} u - \nu \Delta u + (u \cdot \nabla) u + \text{grad } p &= 0 \\
\text{div } u &= 0
\end{aligned}
\end{equation}

(1.1) at $(x, t) \in Q_T$, where $\nu$ is a positive constant — "viscosity constant", $(u \cdot \nabla) u = u_i u_j / x_i$, $0 < T \leq \infty$ and $Q_T$ is the space time region $Q \times (0, T)$. Here and in what follows we use the conventional rule of tensor that repeated suffix means the summation with respect to the suffix.

We consider a flow $u$ satisfying initial-boundary conditions;

\begin{align}
(1.2) & \quad u(x, 0) = a(x), \quad x \in Q, \\
(1.3) & \quad u(x, t) = b(x, t), \quad x \in \partial Q, \quad 0 \leq t < T, \\
(1.4) & \quad u(x, t) \to u_\infty \quad \text{as } |x| \to \infty, \quad 0 \leq t < T,
\end{align}

where $a$ and $b$ are given smooth and bounded functions such that $\text{div } a = 0$ and $a(x) = b(x, 0)$ for $x \in \partial Q$, and $u_\infty$ is a prescribed constant vector. We are mainly concerned with the decay rate of $|u(x, t) - u_\infty|$ as $|x| \to \infty$; for the existence of solutions, see [7], [11], [14], [15] and especially [9], [10], [17] and [18].

In the case that $b(x, t)$ is independent of $t$, R. Finn [4, 5, 6] showed that if a stationary solution $u_s$ of (1.1), (1.3), and (1.4) has finite Dirichlet norm; $\| \nabla u_s \|_{L^2(\Omega)} < \infty$, and satisfies

\begin{equation}
\begin{aligned}
u u_s &= u_\infty + O(|x|^{-\alpha})
\end{aligned}
\end{equation}

(1.5) where $\alpha$ is a constant, $\alpha > 1/2$, then
\( u_s(x) = u_\infty + O(\| x \|^{-1}(1 + s_x)^{-1}) \)

where \( s_x = \| x \| - u_\infty \cdot x / \| u_\infty \| \). Later, K. I. Babenko [1] proved that (1.5) with \( \alpha = 1 \) holds if \( u_s \) has finite Dirichlet norm. Further, D. Clark [3] and K. I. Babenko-M. M. Vasil'ev [2] independently proved that (1.5) implies, for arbitrarily fixed \( \varepsilon > 0 \),

\[
\text{rot } u_s(x) = (R/4\pi)\mathbf{7} s_x \times f_o | x |^{-1} \exp(-R s_x) + O(| x |^{-1} \exp(-(R-\varepsilon)s_x))
\]

where \( R = | u_\infty | / 2\nu \) (the Reynolds number), \( f_o \) is the vector of force exerted by the flow on the bodies, and the notation \( \times \) is the vector product. The equalities (1.6) and (1.7) explain the existence of a paraboloidal wake region behind the bodies.

In the case of non-stationary solutions, G. Knightly [12, 13] obtained a space-time estimate of \( | u(x, t) - u_\infty | \). From his result, one can deduce \( | u(x, t) - u_\infty | \leq M | x |^{-1/2}(1 + s_x)^{-3/2} \), where \( M \) is a constant and \( 0 < \varepsilon \leq 1 \). But it is impossible to deduce \( | u(x, t) - u_\infty | \leq M | x |^{-1}(1 + s_x)^{-1} \). Moreover, his assumptions seem too strong. It remains to investigate whether every classical non-stationary solution of (1.1)-(1.4) satisfies \( | u(x, t) - u_\infty | \leq M | x |^{-1}(1 + s_x)^{-1} \).

Before stating our results, we define a notation. By the equality

\[
f(x, t) = g(x, t) + O(\phi(x)) \quad \text{in } Q_T
\]

we mean that there is a constant \( M \) such that

\[
| f(x, t) - g(x, t) | \leq M | \phi(x) | , \quad (x, t) \in Q_T.
\]

We assume \( 0 \notin \bar{Q} \) without loss of generality.

Let us state our results.

**Theorem 1.** Let \( T < \infty \) and \( u \) be a classical solution of (1.1)-(1.4) in \( Q_T \) with \( \nabla u \in L^\infty(0, T ; L^2(\Omega)) \). Suppose

\[
a(x) = u_\infty + O(| x |^{-1})
\]

where \( \lambda > 0 \). Then

\[
u(x, t) = u_\infty + O(| x |^{-\mu}) \quad \text{in } Q_T
\]

where \( \mu = \min(2, \lambda) \). If the total flux through \( \partial \Omega \) is 0;

\[
\int_{\partial \Omega} u(x, t) \cdot n \, dx = 0 \quad 0 \leq t < T
\]

where \( n \) is the outer normal vector on \( \partial \Omega \), then we can take \( \mu = \min(3, \lambda) \).

**Theorem 2.** Let \( T = \infty \). Let \( u \) be a classical solution of (1.1)-(1.4) in \( Q_\infty \) with \( \nabla u \in L^\infty(0, \infty ; L^2(\Omega)) \). Suppose there exists a stationary solution \( u_s \) of (1.1) and (1.4) with \( \nabla u_s \in L^3(\Omega) \), such that
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(1.11) \[ \alpha(x) = u_s(x) + O(|x|^{-2}). \]

If \( u \) satisfies the following conditions: (i) there is \( r \), \( 1 \leq r < 3 \), such that

(1.12) \[ u - u_\infty \in L^\infty(0, \infty; L^r(\Omega)) \]
and (ii)

(1.13) \[ \lim_{t \to \infty} |u(x, t) - u_\infty| = 0, \]

then

(1.14) \[ u(x, t) = u_\infty + O(|x|^{-1}(1 + sx)^{-1}) \quad \text{in} \quad Q_\infty. \]

**Remark 1.** The assumptions (i) and (ii) seem reasonable. Indeed, the solutions constructed by Heywood [10] and Masuda [17] satisfy (1.13) and (1.12) with \( r = 2 + \varepsilon, \varepsilon > 0 \).

**Remark 2.** We shall give the decay rate of mean value of a weak solution (see § 3.3).

If \( b(x, t) \) is time-independent and \( u \) converges (as \( t \to \infty \)) to a stationary solution \( u_s \) of (1.1), (1.3) and (1.4), it is possible to ask whether \( u - u_s \) decays like \( |x|^{-\alpha}(1 + s_x)^{-1/\alpha}, \alpha > 0 \), as \( t, |x| \to \infty \). Let us introduce a class \( S(\alpha) : \alpha > 0 \).

**Definition 1.** Let \( u, p \) be a classical solution of (1.1)-(1.4) in \( Q_\infty \). Suppose \( b(x, t) \) is independent of \( t \). Then \( u \in S(\alpha) \) if and only if there is a stationary solution \( u_s, p_s \) of (1.1), (1.3) and (1.4) such that

(1.15) \[ \sup_{x \in \Omega} |u(x, t) - u_s(x)| + \|\nabla u(x, t) - \nabla u_s\|_{L^2(\Omega)} \]
\[ \leq M(1 + t)^{-\alpha}, \quad t \geq 0 \]

(1.16) \[ \int_\Omega \{|p(x, t) - p_s(x)| + |\nabla u(x, t) - \nabla u_s(x)|\} \, dx \]
\[ \leq M(1 + t)^{-\alpha}, \quad t \geq 0 \]

where \( M \) is a constant independent of \( t \).

**Remark 3.** J. Heywood [9, 10] and K. Masuda [17, 18] showed that if \( a \) and \( b \) satisfy some additional conditions, then there is a solution of (1.1)-(1.4) contained in \( S(1/4) \).

**Corollary 1.** In addition to the assumptions of Theorem 2, assume \( u \in S(\alpha) \) and

(1.17) \[ \alpha(x) = u_s(x) + O(|x|^{-2-\beta}) \]
where \( \beta > 0 \). Then

(1.18) \[ |u(x, t) - u_s(x)| \leq M|x|^{-2}(1 + s_x)^{-1}(1 + t)^{-1}, \]
for $x \in \Omega$, $0 \leq t$, where $\gamma$ is an arbitrary number satisfying the following conditions: if $\beta \neq 1$, $\gamma \leq \min(\beta/2, 1/2)$ and $\gamma < \alpha/3$ and if $\beta = 1$, $\gamma < 1/2$ and $\gamma < \alpha/3$. $M$ is a constant depending on $u$ and $\gamma$.

The decay of vorticity can be obtained from Theorem 2 and a proposition of Babenko-Vasil'ev [2].

**COROLLARY 2.** In addition to the assumptions of Theorem 2, assume

$$\text{rot } a(x) = \text{rot } u_s(x) + O(|x|^{-\lambda} \exp(-\mu_1 s_x))$$

(1.19)

where $0 < \mu_1 \leq 1/2$, $2 < \lambda$. Then,

$$\text{rot } u(x, t) = O(|x|^{-\lambda/2} \exp(-\mu_2 s_x)) \text{ in } Q_\infty$$

(1.20)

where $\mu_2$ is an arbitrary number $0 < \mu_2 < \mu_1$.

To prove the theorems, we shall apply the method of Babenko. § 2 contains some preliminaries. The proof of the theorems is done in § 3. The corollaries are proved in § 4. In these sections we shall use the same $c$ for various constants independent of variables $x$, $t$, given data, or parameters. We shall use the same $M$ for various constants depending on some data. We shall use $c_p$, $c_\alpha$, $c_p, \alpha$, etc. for various constants depending only on the parameters $p$, $\alpha$, $p$ and $\alpha$, etc. Some notations, e.g. $\xi$, $\alpha$, will be used to represent various objects when no confusion occurs.

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§ 2. Preliminaries.

The main contents of this section are estimates for the fundamental solution tensor of the linearized system of equations for (1.1), reduction of equations (1.1)-(1.4) into integral equations, and proof of a fundamental lemma of Babenko [1].

We first normalize variables. Through a suitable change of variables, we can assume $v = 1$, $u_s = (1, 0, 0)$ without loss of generality. We set $v = u - u_\infty$. Then the equations (1.1)-(1.4) are transformed into

$$\frac{\partial}{\partial t} v - \Delta v + (v \cdot \nabla) v + \frac{\partial}{\partial x_1} v + \text{grad } p = 0$$

(2.1)

$$\text{div } v = 0.$$  

(2.2)

$$v(x, 0) = a(x) - u_\infty, \quad x \in \Omega$$

(2.3)

$$v(x, t) = b(x, t) - u_\infty, \quad x \in \partial \Omega, \quad 0 \leq t < T$$

(2.4)

$$v(x, t) \to 0 \quad \text{as } |x| \to \infty, \quad 0 \leq t < T.$$
2.1. Fundamental solution tensor.

The fundamental solution tensor $E=(E_{ij}, Q_i)$ of the linearized system of equations for (2.1):

\[
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} v - \Delta v + \frac{\partial}{\partial x_1} v + \nabla p &= f \\
\text{div } v &= 0
\end{cases}
\end{align*}
\]

is given by

\[
\begin{align*}
E_{ij}(x, t) &= \delta_{ij} \Gamma(x, t) + \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int \int \int_{E_3} \Gamma(x-y, t) |y|^{-1} dy \\
Q_i(x, t) &= -\frac{1}{4\pi} \frac{\partial}{\partial x_i}(\delta(t) \frac{\partial}{\partial x_i} (|x|^{-1}))
\end{align*}
\]

where $\delta_{ij}$ is the Kronecker's delta, $\delta(\cdot)$ is the delta function, and $\Gamma$ is given by

\[
\Gamma(x, t) = (4\pi t)^{-3/2} \exp(-|x-tu_0|^2/4t).
\]

$E$ is the fundamental solution tensor in the sense

\[
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} E_{ij}(x, t) + \frac{\partial}{\partial x_i} Q_i(x, t) &= \delta_{ij} \delta(x) \delta(t) \\
\frac{\partial}{\partial x_j} E_{ij}(x, t) &= 0
\end{cases}
\end{align*}
\]

This follows from the obvious equality

\[
\left( \frac{\partial}{\partial t} - \Delta + \frac{\partial}{\partial x_1} \right) \Gamma(x, t) = \delta(x) \delta(t).
\]

We shall give some estimates of $E$ for later use. We first state the estimates essentially due to V. A. Solonnikov [20].

**Lemma 1.** (Solonnikov) The inequalities

\[
\begin{align*}
\operatorname{div} \left[ \int \int \int_{E_3} \Gamma(x-y, t) (1+|y|)^{-1} dy \right] & \leq c_1 \\
\left( \frac{\partial}{\partial t} - \Delta + \frac{\partial}{\partial x_1} \right) \Gamma(x, t) & = \delta(x) \delta(t)
\end{align*}
\]

\[
\begin{align*}
\left| \nabla \left[ \int \int \int_{E_3} \Gamma(x-y, t) |y|^{-1} dy \right] \right| & \leq c_1 k(x, t; -(1+l)/2) \\
|E_{ij}(x, t)| & \leq c k(x, t; -3/2) \\
|\nabla E_{ij}(x, t)| & \leq c k(x, t; -2)
\end{align*}
\]

hold for $x \neq 0$, $t \geq 0$, where $\nabla^l$ stands for an arbitrary $l$-th order $x$-derivative and
$k(x, t ; \lambda)$ for real $\lambda$ is given by

\begin{equation}
(2.11) \quad k(x, t ; \lambda) = (t + |x - tu|)^2.
\end{equation}

We evaluate $k(x, t ; \lambda)$ and its certain integrals. To this end, we set $\rho_x = (x^2 + t)^{1/2}$ and

\begin{equation}
(2.12) \quad \Phi_{p,q}(x) = \begin{cases} 
|x|^{-p/2}(1+s)^{-p/2} & \text{for } |x| > 1 \\
|x|^{-q} & \text{for } |x| \leq 1
\end{cases}
\end{equation}

where $p, q > 0$.

**Lemma 2.** Let $\alpha > 0$, $p > 1$ and $q > 0$. The following inequalities (2.13)-(2.17) hold for $x \neq 0$, $t \geq 0$;

\begin{align}
(2.13) & \quad k(x, t ; -q) \leq c^q(x^2/(1+t)^{3/2})^{-p/q} \\
(2.14) & \quad k(x, t ; -q) \leq c^q \Phi_{q,q}(x) \\
(2.15) & \quad \int_0^t k(x, \tau ; -3/2) d\tau \leq c t(1+t)^{3/2} |x|^3 \\
(2.16) & \quad \int_0^t k(x, \tau ; -2) d\tau \leq c t(1+t)^2 |x|^2 \\
(2.17) & \quad \int_0^t k(x, \tau ; -p)(1+t-\tau)^{-\alpha} d\tau \\
& \quad \leq c_{\alpha, p} \Phi_{a_p-1, 2p-2}(x) \\
& \quad \times \begin{cases} 
J_\alpha(t-x_1)(1+t+\rho_x)^{-\min(1, \alpha)/2} & \text{if } |x| > 1, \ x_1 > \rho_x/2 \\
J_\alpha(t)(1+t+|x|)^{-\min(1, \alpha)} & \text{otherwise},
\end{cases}
\end{align}

where

\begin{equation}
J_\alpha(t) = \begin{cases} 
1, & \alpha \neq 1 \\
1 + \log(1+t), & \alpha = 1.
\end{cases}
\end{equation}

**Proof.** Simple calculation can show (2.13) and (2.14). (2.15) and (2.16) follow from (2.13). To show (2.17) in the case $|x| \leq 1$ or $2x_1 \leq \rho_x$, we note

\begin{equation}
k(x, t ; -p) \leq c^p(\tau + |x|)^{-p}, \quad |x| \leq 1 \text{ or } 2x_1 \leq \rho_x, \ 0 \leq \tau.
\end{equation}

Integration of the both sides multiplied by $(1+t-\tau)^{-\alpha}$ gives (2.17) in this case.

We show (2.17) in the excluded case $|x| > 1$ and $2x_1 > \rho_x$. We set $\eta = \rho_x^2 + x_1 - 1/4$. Note $\eta > 3/4$. We put $\xi(t) = (\tau - x_1 + 1/2)/\sqrt{\eta}$ and set $\xi_0 = \xi(0)$, $\xi_1 = \xi(t)$. Changing variables from $\tau$ to $\xi$, we get

\begin{equation}
(2.18) \quad \int_0^t k(x, \tau ; -p)(1+t-\tau)^{-\alpha} d\tau \\
= \eta^{-\alpha/2+1/2} \int_{\xi_0}^{\xi_1} (1+\xi^2)^{-p/2}(1/\sqrt{\eta} + \xi_1 - \xi)^{-\alpha} d\xi.
\end{equation}
Evaluation of the right hand side of (2.18) is carried out separately in cases $\xi_1 < -1$, $|\xi_1| \leq 1$ and $\xi_1 > 1$. In the case $\xi_1 < -1$, put $\zeta = \xi_1/\xi_i$. Evaluating $(1 + \xi^a)^{-1}$ by $|\xi|^{-2}$ and changing variables from $\xi$ to $\zeta$, we have

\[
(2.19) \quad \int_{-\infty}^{\xi_1} (1 + \xi^a)^{-p}(1/\sqrt{\eta} + \xi_1 - \xi)^{-\alpha} d\xi
\]

\[
\leq |\xi_1|^{-2p+1-\alpha} \int_{\xi_1}^{\infty} |\xi|^{-2p}(1/\sqrt{\eta} |\xi_1| - 1 + \xi)^{-\alpha} d\xi
\]

\[
\leq c_{a, p} \left\{ \begin{array}{ll}
|\xi_1|^{-2p+1-\alpha}, & 0 \leq \alpha < 1, \\
|\xi_1|^{-2p} \log (\sqrt{\eta} |\xi_1| + 1), & \alpha = 1, \\
|\xi_1|^{-2p} (\alpha-1)/\alpha, & \alpha > 1.
\end{array} \right.
\]

A brief observation for $|\xi_1| \leq 1$ shows

\[
(2.20) \quad \int_{-\infty}^{\xi_1} (1 + \xi^a)^{-p}(1/\sqrt{\eta} + \xi_1 - \xi)^{-\alpha} d\xi
\]

\[
\leq c_{a, p} \left\{ \begin{array}{ll}
1, & 0 \leq \alpha < 1, \\
\log \sqrt{\eta}, & \alpha = 1, \\
\eta^{(\alpha-1)/2}, & \alpha > 1.
\end{array} \right.
\]

In the remaining case $\xi_1 > 1$, separate the interval of integration into two parts: $(-\infty, \xi_1/2]$ and $(\xi_1/2, \xi_i)$. We obtain

\[
(2.21) \quad \int_{-\infty}^{\xi_1} (1 + \xi^a)^{-p}(1/\sqrt{\eta} + \xi_1 - \xi)^{-\alpha} d\xi
\]

\[
\leq \sup_{\xi_1/2} (1/\sqrt{\eta} + \xi_1 - \xi)^{-\alpha} \int_{-\infty}^{\xi_1} (1 + \xi^a)^{-p} d\xi
\]

\[
+ \sup_{\xi_1/2 < \xi_1 < \xi_i} (1 + \xi^a)^{-p} \int_{\xi_1/2}^{\xi_1} (1/\sqrt{\eta} - \xi_1 - \xi)^{-\alpha} d\xi
\]

\[
\leq c_{a, p} (1/\sqrt{\eta} + \xi_1)^{-\alpha} + c_{a, p} \xi_1^{-2p} \log (\sqrt{\eta} \xi_1 + 1), \quad \alpha = 1,
\]

\[
\leq c_{a, p} (1/\sqrt{\eta} + \xi_1)^{-\alpha} + c_{a, p} \xi_1^{-2p} \eta^{(\alpha-1)/2}, \quad 1 < \alpha.
\]

On the other hand we get

\[
(2.22) \quad \eta(1 + \xi^a) = t + (t-x_1)^2 + \rho^2 \geq c(1 + t + \rho^2)
\]

since $x_1 > c > 0$. The estimate (2.17) follows from (2.18)-(2.22) and the following inequality:

\[
\eta^{-p+1/2} \leq c \Phi_{2p-1,2p-2}(x), \quad x \neq 0.
\]

Lemma 3. Let $p > 3/2$, $1 \leq q < 3/2$, $1 \leq r < 5/4$. Then
\begin{align}
\int_0^t \left\{ \int_{E_3} k(x, \tau; -2r) \, dx \right\}^{1/\nu} \, d\tau \leq c_t \nu^{-2+5/2\nu} \\
\int_0^t \left\{ \int_{E_3} k(x, \tau; -2q) \, dx \right\}^{1/q} \, d\tau \leq c_q \nu^{-1+3/q} \\
\int_0^t \left\{ \int_{\rho \leq R} k(x, \tau; -2r) \, dx \right\}^{1/p} \, d\tau \leq c_p R^{-2+3/p} \\
\int_0^t \left\{ \int_{|z| > R} k(x, \tau; -2r) \, dx \right\}^{1/q} \, d\tau \leq c_q \\
\int_0^t \left\{ \int_{\rho \leq 1} k(x, \tau; -2z) \, dx \right\}^{1/q} \, d\tau \leq c_q
\end{align}

hold for \( t \geq 0 \), where \( R > 1 \).

**Proof.** Putting \( z = x - \tau u \), the integrations with respect to \( x \) in (2.23) and (2.24) can be reduced to integrations with respect to \( |z| \). Then, changing variables from \( |z| \) to \( |z|/\sqrt{\tau} \), straightforward calculation gives (2.23) and (2.24). (2.25) and (2.27) can be shown analogously. Let us show (2.26). Integrating \( k(x, \tau; -2p) \) with respect to \( \rho \), we get

\begin{align}
\int_{|x| > R} k(x, \tau; -2p) \, dx \\
= \frac{\pi}{2p-1} \left\{ \int_0^R \left( \tau + (x_1-\tau)^2 \right)^{-2p+1} \, dx_1 \right\} \\
\leq \frac{2\pi}{2p-1} \tau^{-2p+3/2} \int_{(R-\tau)/\sqrt{\tau}}^\infty (1+\xi^2)^{-2p+1} \, d\xi
\end{align}

where \( \xi = (x_1-\tau)/\sqrt{\tau} \). For \( \tau \leq R/2 \) we have

\begin{align}
\int_{(R-\tau)/\sqrt{\tau}}^\infty (1+\xi^2)^{-2p+1} \, d\xi \leq \frac{1}{4p-3} \left( (R-\tau)/\sqrt{\tau} \right)^{-4p+3},
\end{align}

and for \( \tau > R/2 \),

\begin{align}
\int_{(R-\tau)/\sqrt{\tau}}^\infty (1+\xi^2)^{-2p+1} \, d\xi \leq c_p.
\end{align}

Thus we get

\begin{align}
\int_0^t \left\{ \int_{|x| > R} k(x, \tau; -2p) \, dx \right\}^{1/p} \, d\tau \\
= \int_0^{R/2} \{ \} \, d\tau + \int_{R/2}^t \{ \} \, d\tau \\
\leq c_p R^{-3+3/p} + c_p R^{-1+3/p}.
\end{align}

Thus (2.26) has been proved.
We give some estimates for $I'(x, t)$, which will be used in the proof of Corollary 2.

**Lemma 4.** Let $l$ be a non-negative integer and $0 < \mu < 1/2$. Then

\begin{equation}
\int_0^\infty |\nabla I'(x, t)| \, dt \leq c_{l, \mu} |x|^{-l-1/2}(1 + |x|^{-l/2}) \exp(-\mu s_x)
\end{equation}

holds for $x \neq 0$.

**Proof.** Let $l = l_1 + l_2 + l_3$, where $l_1$, $l_2$, and $l_3$ are non-negative integers. The derivative $\frac{\partial}{\partial x_1}^i \frac{\partial}{\partial x_2}^j_1 \frac{\partial}{\partial x_3}^j_3 \Gamma'(x, t)$ of $I'(x, t)$ is a linear combination of

\begin{equation}
t^{-\frac{l_2}{2} + j_1 + j_2 + j_3} (x_1 - t)^{l_1 - 3j_1} x_2^{l_2 - 2j_2} x_3^{l_3 - 2j_3} \exp\left\{ -\frac{(\sqrt{x_1 - t} + j_1 x_2 + j_3 s_x)}{4t} \right\},
\end{equation}

where $0 \leq j_i \leq l_i/2$, $i = 1, 2, 3$. We can show

\begin{equation}
\int_0^\infty t^{-p} |x_1 - t|^q \exp\left( -\frac{|x_1 - t|^2 + \rho_x^2}{4t} \right) \, dt \leq c_{p, q} |x|^{-p+1/2}(1 + |x|^{-p+q/2+3/2})(s_x + \sqrt{|x|})^q \times \exp(-s_x/2)
\end{equation}

for $x \neq 0$, where $0 \leq q < p$. To show this, put $\zeta = (|x_1 - t|^2 + \rho_x^2)/4t - s_x/2$ and change the variable of integration of the left hand side of (2.31), obtaining

\begin{equation}
\int_0^\infty t^{-p} |x_1 - t|^q \exp\left( -\frac{|x_1 - t|^2 + \rho_x^2}{4t} \right) \, dt = I^+ + I^-
\end{equation}

where

\begin{align*}
I^+ = & \int_0^\infty (|x| + 2\zeta \pm \sqrt{\zeta^2 + \zeta |x|})^{-p} |s_x + 2\zeta \pm \sqrt{\zeta^2 + \zeta |x|}|^q \\
& \times \left( \pm 2 + \frac{2\zeta + |x|}{\sqrt{\zeta^2 + \zeta |x|}} \right) \exp(-\zeta - s_x/2) \, d\zeta.
\end{align*}

Calculation yields

\begin{align*}
I^+ \leq & c_{p, q} |x|^{-p+1/2}(s_x + \sqrt{|x|})^q \exp(-s_x/2) \\
I^- \leq & c_{p, q} |x|^{-p+1/2}(s_x + \sqrt{|x|})^q(1 + |x|^{-p+q/2+3/2}) \\
& \times \exp(-s_x/2)
\end{align*}

for $x \neq 0$. Then the inequality (2.31) immediately follows from these inequalities. By (2.31) we can prove this lemma. Indeed, setting $p = l - (j_1 + j_1 + j_3)/2$, $q = l_1 - 2j_1$, we obtain (2.30) since $|x_i| \leq \sqrt{2} |x| s_x$, $i = 2, 3$. 

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LEMMA 5. Let \( \lambda > 2 \) and \( 0 < \mu' \leq 1/2 \). Then,

\[
\iint_{E_0} \Gamma(x-y, t)(1+|y|)^{-\frac{3}{2}} \exp(-\mu's_y) dy \\
\leq c_{\lambda, \mu'}(1+|x|)^{-\frac{3}{2}} \exp(-\mu's_x)
\]

holds for \( x \in E_0, 0 \leq t \).

PROOF. We put \( \xi = x/\sqrt{t}, \eta = y/\sqrt{t}, r = |\eta|, \) and \( \cos \theta = \cos(\xi, \eta) \). In the left hand side of (2.33), we evaluate \( \exp(-|x-y-tu_\omega|^2/4t) \) by \( \exp(-\mu' \times |x-y-tu_\omega|^2/2t) \) and change the variables of integration from \( y \) to polar coordinates \( r, \theta, \phi \). Since the integrand does not depend on \( \phi \), we have

\[
\iint_{E_0} \Gamma(x-y, t)(1+|y|)^{-\frac{3}{2}} \exp(-\mu's_y) dy \\
\leq (4\pi)^{-3/2} \exp(-\mu't/2 + \mu'x_t) \\
\times \int_0^\infty (1+\sqrt{t} r)^{-\frac{3}{2}} r^3 \exp \left\{ -\frac{\mu'}{2} (|\xi|^2 + r^2 + 2\sqrt{t} r) \right\} dr \\
\times 2\pi \int_0^\pi \exp(\mu'|\xi|r \cos \theta) \sin \theta \ d\theta
\]

\[
\leq c_{\mu'} \exp(-\mu's_x)(I_i + I_2)
\]

where

\[
I_i = \int_0^{1/|\xi|} (1+\sqrt{t} r)^{-\frac{3}{2}} r^3 \exp \left\{ -\frac{\mu'}{2} (r + \sqrt{t} - |\xi|)^2 \right\} dr,
\]

\[
I_2 = \frac{1}{|\xi|} \int_{1/|\xi|}^\infty (1+\sqrt{t} r)^{-\frac{3}{2}} r \exp \left\{ -\frac{\mu'}{2} (r + \sqrt{t} - |\xi|)^2 \right\} dr.
\]

We give a list of estimates of \( I_i, j=1, 2 \). For \( t \geq 2|x| \),

\[
I_i \leq \int_0^{1/|\xi|} r^3 \exp \left\{ -\frac{\mu'}{2} (r + \sqrt{t}/2)^2 \right\} dr \\
\leq \frac{1}{3} \left( \frac{\sqrt{t}}{|x|} \right)^3 \exp(-\mu't/8)
\]

and

\[
I_2 \leq \frac{1}{|\xi|} \int_{1/|\xi|}^\infty r \exp \left\{ -\frac{\mu'}{2} (r + \sqrt{t}/2)^2 \right\} dr
\]

\[
\leq c_{\mu'} \frac{1}{|x|} \exp \left( -\frac{\mu'}{8} |x| \right).
\]

For \( 0 \leq t < 2|x| \),

\[
I_i \leq \int_0^{1/|\xi|} r^3 dr \leq c |x|^{-3/2}.
\]

For \( |x|/2 \leq t < 2|x| \),

\[
I_i \leq \int_0^{1/|\xi|} r^3 dr \leq c |x|^{-3/2}.
\]
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\[ I_2 \leq \frac{1}{|\xi|} \int_0^{\infty} (1+\sqrt{r})^{-1} r \, dr \]
\[ \leq c_1 |x|^{-3/2}. \]

For \( 0 \leq t < |x|/2 \),

\[ I_2 \leq \frac{1}{|\xi|} \left\{ \int_0^{3|x|/16} r \exp \left\{ -\frac{\mu^r}{2} \left( r/2 + |\xi|/4 \right)^2 \right\} \, dr + \int_{3|x|/16}^{\infty} r \exp \left\{ -\frac{\mu^r}{2} \left( r + \sqrt{r} - |\xi|/2 \right)^2 \right\} \, dr \right\} \]
\[ \leq \frac{1}{|\xi|} \left\{ c_{\mu} \exp \left( -\frac{\mu^r}{32} |\xi|^4 \right) + c_{\mu} |\xi|^2 (1+\sqrt{r}-|\xi|/6)^{-1} \right\} \]
\[ \leq c_{\mu} \sqrt{|x|} \exp \left( -\frac{\mu^r}{16} |x| \right) + c_{\mu, \mu'} (1+|x|)^{-1}. \]

Since the left hand side of (2.33) is clearly bounded for \( x \in E_3, t \geq 0 \), the above estimates together with (2.34) assert (2.33).

2.2. Reduction to integral equations.

Let \( v, p \) be a classical solution of (2.1)-(2.4) in \( Q_T \). Let us compute

\[ \begin{align*}
\int_0^1 \int_{\partial D} \left\{ \left( \frac{\partial}{\partial \tau} + \Delta_x + \frac{\partial}{\partial y_1} \right) E_{ij}(x-y, t-\tau) \\
+ \frac{\partial}{\partial y_j} Q_i(x-y, t-\tau) \right\} v_j(y, \tau) \\
+ \left\{ \left( \frac{\partial}{\partial \tau} - \Delta_x + \frac{\partial}{\partial y_1} \right) v_j(y, \tau) + \frac{\partial}{\partial y_j} p(y, \tau) \right\} \\
\times E_{ij}(x-y, t-\tau) \right\} dy \, d\tau
\end{align*} \]

where \( D \) is an arbitrary bounded subdomain of \( \Omega \). Using (2.6) and the Green formula in the hydrodynamics, we obtain

\[ \begin{align*}
v(x, t) &= \sum_{i=1}^k L_i^D [v](x, t) \\
\end{align*} \]

for \( x \in D \) and \( 0 \leq t < T \). Here, the \( i \)-th component \( L_i^D [v] \) of \( L_i^D [v] \) is given by

\[ \begin{align*}
L_i^D [v]_j(x, t) &= \int_D \left( \frac{\partial}{\partial \tau} + \Delta_x + \frac{\partial}{\partial y_1} \right) v_j(y, 0) dy, \\
L_i^D [v]_j(x, t) &= -\frac{1}{4\pi} \int_{\partial D} \left( \frac{\partial}{\partial y_1} \right) v_j(y, 0) \cdot n dy
\end{align*} \]
\[(2.38) \quad L^B[v](x, t) = -\frac{1}{4\pi} \int_{\partial D} \frac{\partial}{\partial y_i} (|x-y|^{-1})v(y, t) \cdot n \, dy, \]

\[(2.39) \quad L^B[v](x, t) = \int_{\partial D} \left[ \left( -\delta_{ij} p + \frac{\partial v_j}{\partial y_k} + \frac{\partial v_k}{\partial y_j} \right) - v_j \left( \frac{\partial E_{ij}}{\partial y_k} + \frac{\partial E_{ik}}{\partial y_j} \right) \right] n_k \, dy \, \tau, \]

\[(2.40) \quad L^B[v](x, t) = \int_{\partial D} \left( \sum_{i \in J} \frac{\partial}{\partial y_k} E_{ij} v_k \right) dy \, \tau, \]

where \( E_{ij} = E_{ij}(x-y, t-\tau), \ v = v(y, \tau), \ p = p(y, \tau), \) and \( n \) is the outer normal vector on \( \partial D. \) We also define \( L^B[f] \) by (2.36)-(2.40) for any vector valued function \( f \) and for any domain \( D. \) Especially if \( D = \Omega, \) we use the notation \( L^{(k)}[v] \) for \( L^B[v]. \)

The formula (2.35) is shown in [12]. To establish this formula for a weak solution and for \( D = \Omega, \) we need

**DEFINITION 2.** A vector valued function \( u \) associated with a scalar function \( p \) is a weak solution in \( Q_T \) of (1.1)-(1.4) if and only if \( u, p \) satisfy (1.1) a.e. in \( Q_T, \) \( \nabla u \in L^4(Q_T), \) \( u(x, t) \to a(x) \) as \( t \to 0 \) in the sense of distribution, and \( u(x, t) = b(x, t) \) a.e. in \( \Omega \times (0, T). \)

We say \( v \) is a weak solution of (2.1)-(2.4) if and only if \( v+u_w \) is a weak solution of (1.1)-(1.4). By the properties of our weak solutions studied by Golovkin and Ladyzenskaya [8] and Solonnikov [20],

\[(2.41) \quad \int_0^T \int_{|x-y|<1, y \in \Omega} \left\{ \left| \frac{\partial u}{\partial t} \right| + |\nabla p| + |\Delta u| \right\}^{5/4} dy \, \tau \leq M, \quad x \in \Omega, \]

where \( p \) can be chosen so that

\[(2.42) \quad \int_{\partial D} \left( |p(y, \tau)| + |\nabla u(y, \tau)| \right) dy \leq M \sup_{\partial \Omega \times (0, T)} \|\nabla u(\cdot, t)\|_{L^4(\Omega)}, \]

and \( M \) is a constant depending on \( a, b \) and \( \|\nabla u\|_{L^4(Q_T)}. \)

**PROPOSITION 1.** Let \( v \) be a weak solution of (2.1)-(2.4) in \( Q_T. \) Then,

\[(2.43) \quad v(x, t) = \sum_{k=1}^s L^{(k)}[v](x, t), \quad a.e. \ (x, t) \in Q_T. \]

**PROOF.** By (2.41) and the theorem of trace, the right-hand side of (2.35) has meaning, and (2.35) holds a.e. in \( D \times (0, T). \) Let \( D = D_R = \Omega \cap \{ x : |x| < R \} \) in (2.35) and integrate both sides with respect to \( R \) in \( R_0 \leq R < R_0+1. \) Then let \( R_0 \) go to infinity. We obtain
The equation for incompressible viscous fluid motions is given by

\[ v_i(x, t) = \sum_{k=1}^{s} L^{(i)}[v]_k(x, t) \]

and

\[ \lim_{R_0 \to \infty} \int_{0}^{T} \int_{|x-y| \leq R_0 |y|} E_{ij}(x-y, t-\tau) p(y, \tau) \frac{y_j}{|y|} dyd\tau, \]

for \( a.e. \ (x, t) \in Q_T \).

By Lemma 1 and by the fact \( v \in L^2(0, T; L^2(\Omega)) \), which is assured by the assumption \( \nabla v \in L^2(Q_T) \) and the imbedding theorem of Sobolev. To prove the proposition, it remains to show the last term of (2.44) vanishes. We set

\[ P_i^\infty(x, t) = \lim_{R_0 \to \infty} \int_{0}^{T} \int_{|x-y| \leq R_0 |y|} E_{ij}(x-y, t-\tau) p(y, \tau) \frac{y_j}{|y|} dyd\tau. \]

We can choose \( p \) such that

\[ \int_{Q_T} p(y, t) dy = 0, \quad 0 \leq t < T. \]

Then by (2.41) we obtain

\[ \int_{Q_T} p(y, t) dy dt \leq M(1 + |x|) \]

for \( x \in \Omega \), where \( M \) is a constant depending on \( a, b \) and \( \| \nabla v \|_{L^2(Q_T)} \). Then, by Lemma 1, we have \( |P_i^\infty(x, t)| \leq M, \ P_i^\infty(x, t) - P_i^\infty(x, t) = 0, \) and \( |P_i^\infty(x, t) - P_i^\infty(x, \tau)| \leq M(t-\tau) \) for \( x, z \in \Omega, \ 0 \leq t, \tau < T \). Since \( P_i^\infty(x, t) \to 0 \) as \( |x| \to \infty \) a.e. in \( 0 \leq t < T \), we obtain \( P_i^\infty(x, t) = 0 \) in \( Q_T \). This completes the proof.

2.3. On \( L^{(s)} \).

Here we prove

PROPOSITION 2. \( L^{(s)} \) is a bounded operator from \( L^s(0, T; L^r(\Omega)) \) to \( L^s(0, T; L^r(\Omega)) \), where \( s, s', r, r' \) are arbitrary numbers satisfying either \( 1 < s, s', r, r' < \infty, 2/s' + 3/r' \geq 2(2/s + 3/r) - 1 \) and \( 0 \leq 2/r - 1/r' < 1/3 \), or \( s = s' = \infty \) and \( 0 \leq 2/r - 1/r' < 1/3 \).

To prove this proposition we use the following theorem of Mihlin-Lizorkin.

THEOREM [16, theorem 8]. Let \( \Phi : R^n \to C \). Suppose \( \Phi \), the derivative \( \partial^n \Phi/\partial \xi_1 \cdots \partial \xi_n \) and all lower derivatives are continuous for \( \xi_j \neq 0, j=1, \cdots, n \). Let \( \mathcal{F}[f] \) be the Fourier transform of \( f \). Then the following transformation

\[ f \longrightarrow (2\pi)^{-n/2} \int_{R^n} \Phi(\xi) e^{i\xi \cdot \cdot x} d\xi \]

is a bounded operator \( L^p(R^n) \to L^q(R^n) \) if

\[ \left| \xi_1^{a_1+} \cdots \xi_n^{a_n+} \frac{\partial^{a_1} \Phi}{\partial \xi_1^{a_1}} \right| \leq M, \quad \xi \neq 0 \]

where \( s = 1/p - 1/q \), \( \alpha_j \) take the values 0 or 1 and \( M \) is a...
constant.

Proof of Proposition 2. The Fourier transform \( \mathcal{F}(\nabla E(\cdot, t)) \) of \( \nabla E(\cdot, t) \) is given by
\[
\mathcal{F}\left(\frac{\partial}{\partial x_k} E_{ij}(\cdot, t)\right)(\xi) = (2\pi)^{-3/2} \sqrt{-1} \xi_k \left( \frac{\partial_{ij} - \xi_i \xi_j}{|\xi|^2} \right) \exp(-t|\xi|^2 - \sqrt{-1} t \xi_i).
\]
The \( \alpha \)-th derivative for any multi-index \( \alpha \) is estimated by
\[
|\xi^\alpha \partial_\xi^\alpha \mathcal{F}(\nabla E(\cdot, t))| \leq |\xi| \left( p_a(t|\xi|^2) + q_a(t|\xi|^2) \right) \exp(-t|\xi|^2)
\]
where \( p_a(\zeta) \) and \( q_a(\zeta) \) are polynomials in \( \zeta \) of order \( |\alpha| \). Hence,
\[
|\xi_i \xi_j |^{1/2} |\xi^\alpha \partial_\xi^\alpha \mathcal{F}(\nabla E(\cdot, t))| \leq c_{\sigma, a} t^{-1/2-3/2} Q(t_{1/2} a/2)
\]
for \( t > 0 \), where \( \sigma > 0 \). By virtue of the theorem of Mihlin-Lizorkin, we obtain
\[
\|L^{(a)}[f](\cdot, t)\|_{L^p(\Omega)} \leq \int_0^t \left\| \nabla E(x-y, t-\tau) \right\|_{L^p(\Omega)} \left\| f(y, \tau) \right\|_{L^q(\Omega)} d\tau 
\]
where \( 1 < p \leq p' < \infty \) and \( 1/\sigma = 1/p - 1/p' \). Hence by virtue of the same theorem, we get
\[
\|L^{(a)}[f]\|_{L^q(0, T; L^p(\Omega))} \leq c_{p, p', q, \sigma} \left(1 + T^{3/2}\right) \|f\|_{L^q(0, T; L^p(\Omega))}^2 
\]
where \( 1 < q \leq q' < \infty, 1/q' \geq 1/4 - (1/2 - 3/2\sigma) \), \( \sigma > 3 \). Setting \( 2p = r, 2q = s, p' = r' \) and \( q' = s' \), we have proposition 2 in the case \( 1 < s, s' < r' < \infty \), \( 2/s + 3/r' \leq 2(2/s + 3/r) - 1 \), and \( 0 \leq 2/r - 1/r' < 1/3 \). The conclusion of Proposition 2 also follows from (2.46) in the other case.

2.4. A fundamental lemma.

Here we prove a fundamental lemma due to Babenko [1].

Lemma 6. (K.I. Babenko) Let \( R_0 \) be a fixed positive number and \( \psi(R) \) be a non-negative function defined for \( R \geq R_0 \). Suppose
\[
\lim_{R \to \infty} \sup \psi(R) < \left(2^{\alpha+1}c_\beta\right)^{-1/(\beta-1)}
\]
(2.48) \[ \phi(R) \leq c_1 R^{-\alpha} + c_2 (\psi(R/2))^{\beta}, \quad R > 2R_0, \]

where \( c_1, c_2, \alpha, \beta \) are constants, \( \alpha > 0, \beta > 1 \). Then for any \( \varepsilon > 0 \) there exists a number \( R_1 \) such that

(2.49) \[ \phi(R) \leq (c_1 + \varepsilon) R^{-\alpha}, \quad R > R_1. \]

**PROOF.** We set \( c_3 = (2a+1c_2)^{-1/(\beta-1)} \) and \( R_1 = \max \{(2c_1/c_3)^{1/\alpha}, 1 + \sup \{R : \phi(R) \geq c_3\} \} \). By (2.47), \( R_1 \) is finite. For \( R \geq R_1 \), there is a non-negative integer \( m \) and a number \( R_2, R_1 \leq R_2 < 2R_1 \) such that \( R = 2^m R_2 \). Using (2.48), we can show \( \phi(2^m R_2) \leq 2^{-m} c_3 \) for all \( m \) by induction on \( m \). Hence

\[ \phi(R) \leq c_3 (2R_1)^{-\alpha}, \quad R \geq R_1. \]

We set \( R_1 = \max \{2R_1, (\varepsilon^{-1} c_3 c_2 (4R_1)^{-\beta})^{1/(\alpha \beta - \alpha)} \} \). Then, we obtain by (2.48)

\[ \phi(R) \leq c_1 R^{-\alpha} + c_2 (c_3 (2R_1)^{\alpha} (R/2)^{-\alpha})^{\beta} \leq (c_1 + \varepsilon) R^{-\alpha} \]

for \( R \geq R_1 \). Thus we have proved the lemma.

### §3. Proofs of theorems.

We shall show Theorem 1 follows from Lemma 6 by setting \( \phi(R) = \sup_{|x| \geq 2 \text{diam}(Q^t)} |v(x, t)| \). To prove Theorem 2, we shall use the theorem of Finn as well as Lemma 6. In this section, every equality and inequality shall hold for \( |x| \geq 2 \text{diam}(Q^t), 0 \leq t < T \).

#### 3.1. Proof of Theorem 1.

We first show under the assumptions of Theorem 1

(3.1) \[ |L^{(k)}[v](x, t)| \leq M |x|^{-\nu}, \quad 1 \leq k \leq 4, \]

where \( \mu \) is given in the statement of Theorem 1. Indeed, one sees by (2.36), (1.8), (2.7) and (2.13),

(3.2) \[ |L^{(1)}[v](x, t)| \leq c_3 M |x|^{-\min(\alpha, 2)} \]

where \( M \) is a constant depending on \( a \) and \( T \). To evaluate \( L^{(2)}[v] \), we set

\[ K_i(x, t) = \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int_{R^3} \Gamma(x-y, t) |y|^{-1} dy, \quad i=1, 2, 3. \]

It follows from the mean-value theorem that

\[ K_i(x-y, t) = K_i(x, t) - y \cdot \nabla_i K_i(x, t - \theta_i y, t) \]

where \( 0 < \theta_i < 1 \). Hence the identity (2.37) and the estimates (2.8) and (2.13) yield
We notice that by the divergence theorem
\[ a_2 v(y, t) \cdot n dy = a_2 (v(y, t) + u_2) \cdot n dy = 0_2 u(y, t) \cdot n dy. \]
Thus we have
\[ (3.3) L(2)[v](x, t) \leq M(1+t)^{3/2} |x|^{-p}, \]
where \( M \) is a constant depending on \( a \) and \( Q \). Similarly we get
\[ (3.4) L(3)[v](x, t) \leq M|x|^{-p}, \]
by (2.38), where \( M \) is a constant depending on \( b \) and \( Q \). Let us estimate \( L(4)[v] \).
To do so, we can assume (2.42). Then the identity (2.39) and the estimates (2.9), (2.10), (2.15) and (2.16) assure
\[ (3.5) L(4)[v](x, t) \leq M(1+t)^2 |x|^{-3} \]
where \( M \) is a constant depending on \( a, b, Q \) and \( \sup_{t = t'} |\nabla v(y, t)| L^2(Q) \). The inequalities (3.2) to (3.5) prove (3.1).

Let us evaluate \( L(5)[v] \). Let us fix \( x \) and set \( D(R) = \{ y \in Q : |x-y| > R \} \),
\[ D(2(R)) = Q - D(1(R)) \text{ for } R > 0. \]
It is clear that
\[ \sum_{k=1}^{\infty} L(6)[v]_k = L(5)[v]. \]
Applying Hölder's inequality and using the estimates (2.10) and (2.13), we get
\[ (3.6) L(6)[v](x, t) \leq c_\varphi t(1+t)^3 R^{-p+1/p'} \sup_{n \in Q} |v(y, \tau)|^{2p'}, \]
where \( p \geq 1, 1/p + 1/p' = 1. \) The estimate (2.24) with \( q = 1 \) yields
\[ (3.7) L(6)[v](x, t) \leq c \sqrt{t} \sup_{x-y \in Q, \tau} |v(y, \tau)|^2. \]
To choose a suitable value of \( p \) in (3.6), we need to prove

**Proposition 3.** Under the assumptions of Theorem 1, \( v \in L^\infty(0, T; L^r(Q)) \),
where \( r \) satisfies either \( r \geq 6 \) or \( r > 3/p \).

**Proof.** The case \( r = 6 \) follows from the Sobolev imbedding theorem and the case \( r = \infty \) from the assumption. Thus for \( r \geq 6 \) Proposition 3 holds. For \( r < 6 \),
we note
\[ |\mathbf{v}(x, t)| \leq M |x|^{-\sigma} + L^{(s)}[\mathbf{v}](x, t) , \]
which follows from Proposition 1 and (3.1). Proposition 2 with \( s=s'=\infty, r=6, \]
r' \( \geq 3 \) assures \( L^{(s)}[\mathbf{v}] \in L^\infty(0, T; L^{r'}(\Omega)), r' \geq 3 \). Hence \( \mathbf{v} \in L^\infty(0, T; L^{r'}(\Omega)) \) for \( r>3/\mu \) and \( r \geq 3 \). But then we have \( L^{(s)}[\mathbf{v}] \in L^\infty(0, T; L^{r'/2}(\Omega)) \) for \( r>3/\mu \) and \( r \geq 3 \), owing to Proposition 2. Repeating the similar arguments proves Proposition 3.

We choose \( p=\max(1, \mu) \) in (3.6) so that \( 2p'>3/\mu \). Then Proposition 3, combined with the estimates (3.6) and (3.7), gives
\[
(3.8) \quad \left| L^{(s)}[\mathbf{v}](x, t) \right| \leq M t(1+t)^2 R^{-\max(1, p)} + c \sqrt{t} \sup_{|x-y| \leq R, 0 \leq \tau \leq t} |\mathbf{v}(y, \tau)|^2.
\]
We choose \( R=|x|/2 \) in (3.8). Then
\[
(3.9) \quad |\mathbf{v}(x, t)| \leq M(1+t)^2 |x|^{-\sigma} + c \sqrt{t} \sup_{|x-y| \leq |x|/2, 0 \leq \tau \leq t} |\mathbf{v}(y, \tau)|^2.
\]
Hence if we set \( \psi(R) = \sup_{|x| \leq R, 0 \leq t \leq T} |\mathbf{v}(x, t)| \), then
\[
(3.10) \quad \psi(R) \leq M(1+T)^2 R^{-\sigma} + c T^{1/2} (\psi(R/2))^{1/2}.
\]
Thus Theorem 1 immediately follows from Lemma 6 if (2.47) holds. Let us verify (2.47). We have by (2.10), (2.23), (2.40) and Hölder’s inequality
\[
(3.11) \quad \left| L^{(s)}[\mathbf{v}](x, t) \right| \leq c t^{1/2+12/r} \|\mathbf{v}\|_{L^{2r'}(0, T; L^{2r'}(D(2)))}
\]
where \( 1<r<5/4 \) and \( 1/r + 1/r' = 1 \). We choose \( R=|x|/2 \) in (3.11), obtaining by Proposition 3
\[
\|\mathbf{v}\|_{L^{2r'}(0, T; L^{2r'}(D(2)))} \to 0 \quad \text{as} \quad |x| \to \infty,
\]
since \( r'>5 \). Hence we have (2.47) by (3.1), (3.6) and (3.11). This completes the proof of Theorem 1.

3.2. Proof of Theorem 2.

Here we shall make use of a trivial modification of a theorem of Finn [5, Theorem 5.1, Corollary 5.1]:

**Theorem.** (R. Finn) Let \( f \) and \( g \) be functions in \( Q_\infty \) satisfying
\[
(3.12) \quad \sup_{0 \leq t \leq T} |f(x, t) - g(x, t)| \leq c \int \int Q \Phi_{x,y}(x-y) \sup_{0 \leq t \leq T} |f(y, t)|^2 dy, \quad x \in \Omega
\]

where \( \Phi_{x,y} \) is an exponential function of \( |x-y| \).
(3.13) $\sup_{0 \leq t} |g(x, t)| \leq c|x|^{-1}(1+s_x)^{-1}, \quad x \in \Omega$

(3.14) $\sup_{0 \leq t} |f(x, t)| \leq c|x|^{-\alpha}, \quad x \in \Omega$

where $\Phi_{3, 2}$ is given by (2.12) and $\alpha > 1/2$. Then, there holds

(3.15) $f(x, t) = g(x, t) + O(|x|^{-3/2} \log (1+|x|)(1+s_x)^{-5/2})$ in $Q_m$.

We set $f = u, g = \sum_{k=1}^{4} L^{(k)}[v]$. Then Theorem 2 is an immediate consequence of this theorem if the conditions (3.12) to (3.14) hold.

First we show (3.12). Since $f - g = L^{(5)}[v]$, (3.12) follows from (2.40), (2.10) and (2.17) with $\alpha = 0, \rho = 2$.

Next we verify (3.13). The terms $L^{(2)}[v]$ and $L^{(3)}[v]$ are estimated by (2.37), (2.38), (2.8) and (2.14), leading us to

(3.16) $\sum_{k=2}^{4} |L^{(k)}[v](x, t)| \leq M|x|^{-1}(1+s_x)^{-1}$

where $M$ is a constant depending on $a$ and $b$. We evaluate $L^{(4)}[v]$ by (2.39), (2.9), (2.10), (2.42) and (2.17) with $\alpha = 0, \rho = 3/2$ or 2, concluding

(3.17) $|L^{(4)}[v](x, t)| \leq M|x|^{-1}(1+s_x)^{-1}$

where $M$ is a constant depending on $a, b$ and $\sup_{0 \leq t} \|\nabla v(\cdot, t)\|_{L^2(\Omega)}$. We evaluate $L^{(1)}[v]$ by decomposing it into $L^{(1)}[v - \nu_\alpha]$ and $L^{(1)}[\nu_\alpha]$. First we evaluate $L^{(1)}[v - \nu_\alpha]$. Since $v - \nu_\alpha = u - \nu_\alpha$, we have

(3.18) $|v(x, 0) - \nu_\alpha(x)| \leq M|x|^{-\alpha}$

by the assumption (1.11), where $M$ is a constant depending on $a$. Hence we get by (2.36), (2.7) and (2.14)

(3.19) $|L^{(1)}[v - \nu_\alpha](x, t)| \leq M|x|^{-1}(1+s_x)^{-1}$.

We evaluate $L^{(1)}[\nu_\alpha]$ by applying Proposition 1 to $v_\alpha$, i.e. by

$L^{(1)}[v_\alpha] = v_\alpha - \sum_{k=2}^{5} L^{(k)}[v_\alpha].$

By (1.6), (3.16) and (3.17) for $v = v_\alpha$, we obtain

$|v_\alpha(x)| + \sum_{k=2}^{4} |L^{(k)}[v_\alpha](x, t)| \leq M|x|^{-1}(1+s_x)^{-1}$,

where $M$ depends on $v_\alpha$. By the theorem of Finn [5, theorem 5.1] and by (1.6) we get

$|L^{(5)}[v_\alpha](x, t)| \leq M|x|^{-1}(1+s_x)^{-1},$
where \( M \) depends on \( \nu \). Thus we have
\[
(3.19) \quad |L^1[\nu](x, t)| \leq M|x|^{-1}(1 + s_x)^{-1}.
\]
Combining (3.16), (3.17), (3.18) and (3.19), we get (3.13).

It remains to show (3.14). Let us set \( \phi(R) = \sup_{I(x, t) \in \Omega} |\nu(x, t)| \). Then the condition (3.14) follows from Lemma 6 if the assumptions (2.47) and (2.48) of the lemma are valid. The inequality (2.47) follows from Theorem 1 and the assumption (1.13) of Theorem 2. Let us prove (2.48) for some \( \alpha > 1/2 \). Since \( |g(x, t)| \leq M|x|^{-1} \), we need to estimate \( L^6[\nu] (= f - g) \). To this end, we fix \( x \in Q \) and \( R \geq 1 \) and define
\[
D(3) = \{ y \in Q : |x - y| > R \},
\]
\[
D(4) = \{ y \in Q : |x_1 - y_1| > R, |x - y| \leq R \},
\]
\[
D(5) = \{ y \in Q : |x_1 - y_1| \leq R, 1 < |x - y| \leq R \},
\]
\[
D(6) = \{ y \in Q : |x_1 - y_1| < R, |x - y| < 1 \}.
\]
It is clear that
\[
L^6[\nu] = \sum_{k=3}^6 L^6(y_3)[\nu] .
\]
Using Hölder’s inequality and the estimates (2.25) and (2.26) we get
\[
(3.20) \quad \sum_{k=3}^6 |L^6(y_3)[\nu](x, t)|
\leq c_{p_1} R^{-1/2} \sup_{\Theta \in \mathfrak{T}} \|\nu(\cdot, \tau)\|_{L^{p_1}(Q)}^p
\leq c_{p_1} R^{-1/2} \sup_{\Theta \in \mathfrak{T}} \|\nu(\cdot, \tau)\|_{L^{p_1}(Q)}^p
\leq c_{p_1} R^{-1/2} \sup_{\Theta \in \mathfrak{T}} \|\nu(\cdot, \tau)\|_{L^{p_1}(Q)}^p \sup_{y \in D(3), \Theta \in \mathfrak{T}} |\nu(y, \tau)|^{1+\epsilon}
\]
where \( 1/p_1 + 1/p_1' = 1, 2p_1' = r \). Note that we can use (2.25) since \( p_1 > 3 \) by the assumption (1.12) of Theorem 2. We assume \( 2 < r < 3 \) without loss of generality. Then by Hölder’s inequality and by (2.25) with \( R = 1 \), we get
\[
(3.21) \quad |L^6(y_3)[\nu](x, t)|
\leq c_{p_1} \sup_{\Theta \in \mathfrak{T}} \|\nu(\cdot, \tau)\|_{L^{p_1}(Q)}^p
\leq c_{p_1} \sup_{\Theta \in \mathfrak{T}} \|\nu(\cdot, \tau)\|_{L^{p_1}(Q)}^p \sup_{y \in D(3), \Theta \in \mathfrak{T}} |\nu(y, \tau)|^{1+\epsilon}
\]
where \( 0 < \epsilon < 1 - r/3, 1/p_2 + 1/p_2' = 1 \) and \( p_2' = r(1 - \epsilon)^{-1} \). We note we can use (2.25) since \( p_2 > 3/2 \). By (2.27) we get
\[
(3.22) \quad |L^6(y_3)[\nu](x, t)| \leq c \sup_{y \in D(3), \Theta \in \mathfrak{T}} |\nu(x, t)| ^\epsilon .
\]
Let \( R = |x|/2\sqrt{2} \). Then the estimates (3.20) to (3.22) gives
\[
|L^6[\nu](x, t)|
\leq M|x|^{-1+3/2} + M \sup_{\Theta \in \mathfrak{T}} |\nu(y, \tau)|^{1+\epsilon}
\]
where $M$ is a constant depending on $v$, $r$, and $\varepsilon$. Hence it follows from (3.13) with $g = \frac{1}{k=1} L^{(k)}[v]$ that

$$|v(x, t)| \leq M|x|^{-1 + \varepsilon}$$

Taking the supremum of the both sides of this inequality over the domain $\{x \in \Omega : |x| > R\} \times (0, \infty)$, we obtain (2.48) for $\alpha = 1 - 3/2p_1 > 1/2$. This completes the proof of Theorem 2.

### 3.3. A remark on the decay of weak solutions.

Here we prove

**Proposition 4.** Let $v$ be a weak solution of (2.1)-(2.4) in $Q_T$. Suppose $v \in L^r(0, T; L^s(\Omega))$ with $r > 3$ and $3/r + 2/s = 1$. Then under the assumption (1.8), there holds

$$\int_0^T \left( \int_{|x-y|<1} |v(y, \tau)|^r \, dy \right)^{s/r} \, d\tau$$

$$\leq M|x|^{-\mu}$$

where $M$ is a constant depending on $v$, $T$, and $\mu$, and $\mu$ is the constant given in the statement of Theorem 1.

**Remark 4.** If $3/r + 2/s < 1$, then $v$ is a classical solution (Serrin [19]) and Proposition 4 follows from Theorem 1.

**Proof.** We define

$$J_{r,s}(x, t) = \left( \int_0^T \left( \int_{|x-y|<1} |f(y, \tau)|^r \, dy \right)^{s/r} \, d\tau \right)^{1/t}$$

where $f$ is a function on $Q_T$. Proposition 1 yields

$$\bar{v}_{r,s}(x, t) \leq \frac{1}{k=1} L^{(k)}[\bar{v}]_{r,s}(x, t), \quad (x, t) \in Q_T.$$

It is clear that

$$\sum_{k=1}^{1} L^{(k)}[\bar{v}]_{r,s}(x, t) \leq M|x|^{-\mu}, \quad (x, t) \in Q_T,$$

where $M$ is a constant depending on $v$, $T$ and $\mu$. To evaluate $L^{(k)}[\bar{v}]_{r,s}$, let $x_o$, $|x_o| > 3$, be arbitrarily fixed in $\Omega$. Let us express a ball centred at $x$ and of a radius $R$ by $B(x, R)$. Let $\{B(y^{(s)}, 1)\}_{s=1}$ be a finite covering of $B(x_0, |x_0|/2)$. We assume $y^{(s)} \in B(x_o, |x_o|/2)$ and the multiplicity of the covering is not more than 8, without loss of generality. Let $(\lambda_j)_{j=1}$ be a partition of unity subordinate to this covering. We define
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\[ I_j(x, t) = \int_0^t \int_\Omega (\lambda_j(y) |\nabla E(x-y, t-\tau)| |v(y, \tau)|^g dy d\tau, \]

\( j=1, 2, \ldots, N. \) Then, by Proposition 2 and the estimates (2.10) and (2.13), the following inequality holds:

\[ \int_{-t}^t \sum_{j=1}^N (1- \sum_{j=1}^N \lambda_j(y)) |\nabla E(x-y, t-\tau)| |v(y, \tau)|^g dy d\tau \]

\[ \leq M t (1+t)^2 |x_o|/2-1 -u \]

\[ \leq M t (1+t)^2 |x_o| -u, \]

where \( M \) is a constant depending on \( v \) and \( \mu \). Hence we have

\[ \mathcal{L}(\bar{v}, \sigma)(x_o, t) \leq M |x_o| -\mu + \sum_{j=1}^N \int_{-t}^t \sum_{j=1}^N \lambda_j(y) \]

\[ \leq M |x_o| -\mu + M \max_{1 \leq j \leq N} \| \bar{v}_r, \sigma(y, \tau) \|^2 \]

where \( M \) is a constant depending on \( v, \mu \) and \( T \). Finally we get

\[ \sup_{x_0 \in R, \sigma \in \Theta} \bar{v}_r, \sigma(x_0, t) \]

\[ \leq M |x_o| -\mu + M \| \bar{v}_r, \sigma(y, \tau) \|^2, \]

which implies, as a consequence of Lemma 6, Proposition 4, if one sets \( \phi(R) = \sup_{x \in \Omega, \tau \in [0, T]} v_r, \sigma(x, \tau) \).

§ 4. Proofs of corollaries.

4.1. Proof of Corollary 1.

We set \( v_v = u_v - u_a \) and \( w = v - v_v \). Let us define

\[ L^{(i)}[f, g](x, t) = \int_0^t \int_\Omega \frac{\partial}{\partial y_k} E_{ij}(x-y, t-\tau) \]

\[ \times \{ f_j(y, \tau) g_k(y, \tau) + f_k(y, \tau) g_j(y, \tau) \} dy d\tau, \]

\( i=1, 2, 3, \) where \( f \) and \( g \) are vector valued functions defined in \( Q_{\infty} \). Then Proposition 1 yields

\[ w = \sum_{k=1}^n L^{(k)}[w] + L^{(\sigma)}[w, v_0]. \]

We evaluate each term of the right hand side of (4.2). Let \( \sigma^* \) be \( \min(\sigma, 1) \) if
\( \sigma \neq 1 \) and represent any number less than 1 if \( \sigma = 1 \). The assumption (1.17), on account of the estimates (2.7) and (2.14), implies

\[
| L^{(1)}[w](x, t) | \leq M(1+t)^{-\beta^*/2}\Phi_{\sigma, \delta}(x), \quad (x, t) \in Q_\infty,
\]

where \( M \) is a constant depending on \( \alpha \) and \( \beta^* \). Since \( w \big|_{\partial Q} = 0 \), \( L^{(k)}[w] \), \( k = 2, 3 \), also vanish. Owing to Lemma 2, the assumption (1.16) implies

\[
| L^{(4)}[w](x, t) | \leq c_M(1+t)^{-\sigma^*/2}\Phi_{\sigma, \delta}(x), \quad (x, t) \in Q_\infty,
\]

where \( M \) is a constant depending on \( \nu \). Let us evaluate \( L^{(5)}[w, v] \). By the assumption (1.15) and Theorem 2, we get

\[
| w(x, t) | \leq \sup_{x \in \Omega} | w(x, t) |^{1/2} \sup_{x \in \Omega} | w(x, t) |^{1/2}
\]

\[
\leq M(1+t)^{-\sigma^*/2}\Phi_{\sigma, \delta}(x), \quad (x, t) \in Q_\infty,
\]

where \( M \) is a constant depending on \( \nu \). The inequalities (1.6) and (4.5) imply

\[
| w(x, t)v(x) | \leq M(1+t)^{-\sigma^*/2}\Phi_{\sigma, \delta}(x), \quad (x, t) \in Q_\infty,
\]

where \( M \) is a constant depending on \( \nu \) and \( v \). We note that Finn's theorem 5.1 ([5]) assures

\[
\int_{\Omega} \int_{\Omega} \Phi_{\sigma, \delta}(x-y)\Phi_{\sigma, \delta}(y)dy \leq cM^2, \quad x \in \Omega.
\]

Hence the following inequality follows from Lemma 2 and (2.10);

\[
| L^{(5)}[w, v](x, t) |
\]

\[
\leq \int_{\Omega} \int_{\Omega} M(1+t)^{-\sigma^*/2}\Phi_{\sigma, \delta}(y)k(x-y, t-\tau, -\xi)dyd\tau
\]

\[
\leq c_M M(1+t)^{-\sigma^*/2} \int_{\Omega} \int_{\Omega} \Phi_{\sigma, \delta}(y)\Phi_{\sigma, \delta}(x-y)dy
\]

\[
\leq c_M M^2(1+t)^{-\sigma^*/2} \Phi_{\sigma, \delta}(x), \quad (x, t) \in Q_\infty.
\]

Similarly we get

\[
| L^{(6)}[w](x, t) | \leq cM(1+t)^{-\gamma_1/2}\Phi_{\sigma, \delta}(x), \quad (x, t) \in Q_\infty,
\]

where \( M \) is a constant depending on \( \nu \). Thus we have

\[
| w(x, t) | \leq M(1+t)^{-\gamma_1/2}\Phi_{\sigma, \delta}(x), \quad (x, t) \in Q_\infty,
\]

where \( \gamma_1 = \min(\beta^*/2, (\alpha/2)^*/2) \) and \( M \) is a constant depending on \( \alpha, \beta, \nu, \Omega \) and \( \gamma_1 \). By (4.6) and (1.15) we get

\[
| w(x, t) | \leq M(1+t)^{-\sigma^*/2} \Phi_{\sigma, \delta}(x), \quad (x, t) \in Q_\infty.
\]

Substituting the estimate (4.5) by this and repeating the argument, we get instead of (4.6).
where \( \gamma = \min(\beta^*/2, (\alpha/2+\gamma_1/2)^*/2) \) and \( M \) is a constant depending on \( \alpha, b, v, v_\gamma, \Omega \) and \( \gamma_\epsilon \). A finite number of iterations complete the proof.

4.2. Proof of Corollary 2.

Corollary 2 is a consequence of a result of Babenko-Vasil’ev [2], which states

**Proposition 5.** (Babenko-Vasil’ev) Let \( \phi(x) \) be a nonnegative function in \( \Omega \) satisfying

\[
\phi(x) \leq M|x|^{-3/2}(1+s_x)^{-1}, \quad x \in \Omega
\]

\[
\phi(x) \leq M \int_{\Omega} \phi(y) |y|^{-1}(1+s_y)^{-1}G_\mu(x-y)dy
\]

\[
+ M|x|^{-1/2}\exp(-\mu s_x), \quad x \in \Omega
\]

where \( 0<\mu<1/2, M \) is a constant and \( G_\mu \) is given by

\[
G_\mu(x) = |x|^{-3/2}(1+|x|^{-1/2})\exp(-\mu s_x).
\]

Then \( \phi(x) \) satisfies

\[
\phi(x) \leq M_\epsilon |x|^{-3/2}\exp\{-\mu(1-s_x)\}, \quad x \in \Omega
\]

where \( 0<\epsilon<\mu \) and \( M_\epsilon \) is a constant depending on \( \epsilon \) and \( M \).

We derive a representation formula for \( \text{rot} \, u \) and prove the inequalities (4.7) and (4.8) for \( \phi(x) = |\text{rot} \, u(x)| \). We start from the obvious formula

\[
\omega = \sum_{k=1}^{k=3} \text{rot} \, L^{(k)}[v]
\]

where \( \omega = \text{rot} \, u \). The terms \( \text{rot} \, L^{(k)}[v], k=2, 3 \) clearly vanish. Note

\[
\text{rot}_z(E* f)(x, y, t, \tau) = \text{rot}_z \Gamma(x-y, t-\tau)f(y, \tau)
\]

for a vector valued function \( f \), where the \( i \)-th component of \( E* f \) is given by

\[
(E* f)_i(x, y, t, \tau) = E_{ij}(x-y, t-\tau)f_j(y, \tau).
\]

Hence the linear part of \( \text{rot} \, L^{(1)}[v] \) with respect to \( (v, p) \) can be rewritten as \( \text{rot} \, L^{(1)}[v] \), where

\[
L^{(1)}[v](x, t)
\]

\[
= \int_{\Omega} \int_{\partial\Omega} \Gamma \left[\left(-\partial_{ij} p + \frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right)n_j - v_i n_j \right] dyd\tau
\]

\[
- \int_{\Omega} \int_{\partial\Omega} \{v_i(n \cdot \nabla)\Gamma + n_i(v \cdot \nabla)\Gamma\} dyd\tau,
\]

where \( \Gamma = \Gamma(x-y, t-\tau), v=v(y, \tau), p=p(y, \tau) \). The sum of the nonlinear part
of rot$L^{(4)}[\nu]$ and rot$L^{(5)}[\nu]$ equals the left hand side of the following equality
\[-rot\int_0^t\int_\Omega \Gamma(\nu \cdot \nabla)\nu \, dy \, d\tau = \sum_{k=7}^{8} L^{(k)}[\nu] \]
where
\[
L^{(\nu)}[\nu](x, t) = \frac{1}{2} \int_0^t \int_\Omega \nu \times \text{grad}(|\nu|^2) \, dy \, d\tau
\]
\[
L^{(8)}[\nu](x, t) = \int_0^t \int_\Omega ((\nu \cdot \nabla)\Gamma)\omega - ((\omega \cdot \nabla)\nu) \, dy \, d\tau .
\]
Thus the representation formula is obtained:
\[
\omega = \text{rot} L^{(1)}(\nu) + \text{rot} L^{(5)}[\nu] + \sum_{k=7}^{8} L^{(k)}[\nu] .
\]
Let us prove (4.7) and (4.8) for $\phi(x) = \sup_{y \in \Omega} |\omega(x, t)|$. The inequality (4.7) is a consequence of Theorem 2 and of a theorem of Finn [5, theorem 5.3]. We show (4.8) by (4.14). Theorem 2 and Lemma 4 implies
\[
|L^{(\nu)}[\nu](x, t)| \leq M \int_\Omega \phi(y) |y|^{-\alpha}(1 + s_y)^{-1} G_m(x - y) \, dy , \quad (x, t) \in Q_m ,
\]
where $0 < \mu < 1/2$ and $M$ is a constant depending on $\nu$ and $\mu$. By Lemma 4 we also get
\[
|\text{rot} L^{(\nu)}[\nu](x, t)| \leq M |x|^{-\alpha/2} \exp(-\mu s_x) , \quad (x, t) \in Q_m ,
\]
where $0 < \mu < 1/2$ and $M$ is a constant depending on $\nu$ and $\mu$. To evaluate $L^{(\nu)}[\nu]$, we note
\[
\int_\Omega \nu \times \text{grad}(|\nu|^2) \, dy = \int_\Omega \text{rot}(\text{grad}(|\nu|^2)) \, dy = \int_\Omega \text{rot}(\text{grad}(\nu)) \, dy = 0 .
\]
Let us expand $\Gamma(x - y, t - \tau)$ into the series of powers in $y$ and apply Lemma 4, concluding
\[
|L^{(\nu)}[\nu](x, t)| \leq M |x|^{-\alpha/2} \exp(-\mu s_x) , \quad (x, t) \in Q_m ,
\]
where $0 < \mu < 1/2$ and $M$ is a constant depending on $\nu$ and $\mu$. The evaluation for $\text{rot} L^{(\nu)}[\nu]$ remains. To do this, let us decompose $\text{rot} L^{(1)}[\nu]$;
\[
\text{rot} L^{(1)}[\nu] = \text{rot} L^{(1)}[\nu] + L^{(1)}[\text{rot}(\nabla \nu)] + \int_\Omega \Gamma(x - y, t) \nu \times (\nabla \nu) \, dy .
\]
By the assumption (1.19) and by Lemma 5 we have
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$$|L^{(1)}[\text{rot}(\mathbf{a} - \mathbf{v}_3)](x, t)| \leq M|x|^{-3/2}\exp(-\mu_1 s_x), \quad (x, t) \in Q_\infty,$$

where $M$ is a constant depending on $\mathbf{a} - \mathbf{v}_3$. We also obtain

$$\left|\sum_{\sigma \in \mathcal{I}} \int_{\partial \Omega} \Gamma(x - y, t) \mathbf{n} \times (\mathbf{a} - \mathbf{v}_3) dy\right| \leq M\sup \Gamma(x, t) \leq M|x|^{-5/2}\exp(-s_x/2),$$

for large $|x|$, where $M$ is a constant depending on $\mathbf{a} - \mathbf{v}_3$. An application of the formula (4.14) to $\mathbf{v} = \mathbf{v}_3$ gives us

$$\text{rot} \ L^{(1)}[\mathbf{v}_3] = \text{rot} \ \mathbf{v}_3 - \text{rot} \ L^{(\sigma)}[\mathbf{v}_3] - \sum_{k=1}^{\infty} L^{(k)}[\mathbf{v}_3].$$

Then, the inequality (1.7) and the inequalities (4.15) to (4.17) for $\mathbf{v} = \mathbf{v}_3$ yield

$$|\text{rot} \ L^{(1)}[\mathbf{v}_3](x, t)| \leq M|x|^{-3/2}\exp(-\mu s_x), \quad (x, t) \in Q_\infty,$$

where $0 < \mu < 1/2$ and $M$ is a constant depending on $\mathbf{v}_3$ and $\mu$. Thus the inequalities (4.18) to (4.20) give

$$|\text{rot} \ L^{(1)}[\mathbf{v}](x, t)| \leq M|x|^{-3/2}\exp(-\mu_1 s_x), \quad (x, t) \in Q_\infty,$$

which, combined with the estimates (4.15) to (4.17), gives (4.8) with $0 < \mu \leq \mu_1$, $\mu \neq 1/2$. This completes the proof.

References


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