Enumerating embeddings of $n$-manifolds in Euclidean $(2n-1)$-space

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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Introduction.

Throughout this paper, “$n$-manifold” and “embedding” will mean closed connected differentiable manifold of dimension $n$ and differentiable embedding, respectively. Let $[M \subset R^m]$ denote the set of isotopy classes of embeddings of a manifold $M$ into Euclidean $m$-space.

It is known that for an $n$-manifold $M$,
1. (Whitney [24]) the set $[M \subset R^{2n+2}]$ consists of only one element if $n \geq 1$,
2. (Wu [25]) the set $[M \subset R^{2n+1}]$ consists of only one element if $n \geq 2$,
3. (Haefliger [6], Bausum [1], Rigdon [13] etc.) if $n \geq 4$, then, as a set,

$$
[M \subset R^{2n}] = \begin{cases} 
H^{n-1}(M; Z) & \text{for } n \equiv 1 \pmod{2}, w_1(M) = 0, \\
H^{n-1}(M; Z_2) & \text{for } n \equiv 1 \pmod{2}, w_1(M) \neq 0, \\
Z \times \rho_2 H^{n-1}(M; Z) & \text{or } n \equiv 0 \pmod{2}, w_1(M) = 0,
\end{cases}
$$

The purpose of this paper is to inquire into the question of whether or not the set $[M \subset R^{2n-1}]$ for an $n$-manifold $M$, if it is not empty, can be described in terms of the cohomology of $M$, its characteristic classes and the cohomology operations. We shall study $[M \subset R^{2n-1}]$ along the lines of Haefliger [5], [6].

Let $X^2$ be the product $X \times X$ of $X$ and let $\Delta X$ be the diagonal in $X^2$. The cyclic group of order 2, $Z_2$, acts on $X^2$ via the map $t: X^2 \to X^2$ defined by $t(x, y) = (y, x)$, where $\Delta X$ is the fixed point set of this action. The quotient space

$$X^* = (X^2 - \Delta X)/Z_2$$

is called the reduced symmetric product of $X$. Let $P^m$ denote the real projective space of dimension $m$ ($m \leq \infty$) and let

$$\xi: X^* \to P^m$$

denote the classifying map of the double covering $X^2 - \Delta X \to X^*$. Then the first Stiefel-Whitney class of this double covering is given by
For a finitely generated abelian group $G$, let $G'$ denote the new abelian group associated with $G$ defined as follows: A given group $G$ is of the form

$$G = \sum_{i=1}^{b} Z_{r(i)} \langle a_i \rangle + G_0$$

(direct sum),

$$r(i) = \begin{cases} \infty & \text{for } 1 \leq i \leq a, \\ 2^{r(i)}(s(i) \geq 1) & \text{for } a < i \leq b, \end{cases}$$

where $Z_r \langle c \rangle$ denotes a cyclic group of order $r$ ($r \leq \infty$) generated by $c$ and $G_0$ denotes the odd torsion subgroup of $G$. Then $G'$ is defined by

$$G' = \sum_{i=1}^{b} Z_{r(i)} \langle (1/2)a_i \rangle + G_0.$$

Under these notations, we shall prove the following theorem.

**Theorem 1.** Let $n \geq 6$ and let $M$ be an $n$-manifold. Assume that $M$ is orientable or $n$ is even and that there exists an embedding of $M$ into $R^{2n-1}$. Then, as a set, it follows that

$$[(M \subset R^{2n-1}) = (1+(-1)^n t^*)^n (H^{n-1}(M; Z) \otimes H^{n-1}(M; Z)) \times \text{Coker } \Theta$$

$$\times \begin{cases} [H^{n-1}(M; Z)]' \times H^{n-2}(M; Z) & \text{if } n \text{ is even and } w_1(M)=0, \\ [H^{n-1}(M; Z)]' \times H^{n-2}(M; Z) & \text{if } n \text{ is even and } w_1(M) \neq 0, \\ H^{n-2}(M; Z_2) & \text{if } n \text{ is odd and } w_1(M)=0, \end{cases}$$

where

$$\Theta = \left( Sq^2 + \left( \frac{2n-1}{2} \right) v^2 \right) \tilde{p}_2 : H^{2n-3}(M^*; Z[v]) \to H^{2n-3}(M^*; Z_2),$$

$v$ is the first Stiefel-Whitney class of the double covering $M^2 - \Delta M \to M^*$, $Z[v]$ is the sheaf of coefficients over $M^*$, locally isomorphic to $Z$, twisted by $v$, and $\tilde{p}_2$ is the reduction mod 2 twisted by $v$. The following information is sufficient to determine $\text{Coker } \Theta$:

(i) the integral cohomology groups $H^i(M; Z)$ for $n-3 \leq i \leq n$,

(ii) the actions of $Sq^i$ on $H^i(M; Z)$ for $i = n-3, n-2$,

(iii) the action of $w_1(M)$ on $H^{n-2}(M; Z_2)$ for $n=2 (4)$. 

**Remark.** (1) For an $n$-manifold $M (n>4)$, there is an embedding of $M$ into $R^{2n-1}$ if $M$ is orientable or if $n$ is not a power of 2 (cf. [10, (1) and (2) in § 1]).

(2) It is known (see (2.5) below) that

$$Sq^2 x = \begin{cases} (w_2(M) + w_1(M))^2 x & \text{if } x \in H^{n-2}(M; Z_2), \\ (w_2(M) + w_1(M))^2 + w_1(M) Sq^1) x & \text{if } x \in H^{n-2}(M; Z_2). \end{cases}$$
The following corollary is an immediate consequence of the theorem, since $H^{2n-1}(M^*; \mathbb{Z}_2)$ is isomorphic to $H^{n-1}(M; \mathbb{Z}_2)$ by Thomas [17, §2].

COROLLARY (Haefliger [6]). If $H_1(M; \mathbb{Z}) = 0$, then

$$[M \subset R^{2n-1}] = \begin{cases} H_3(M; \mathbb{Z}) & \text{for even } n, \\ H_4(M; \mathbb{Z}_2) & \text{for odd } n. \end{cases}$$

Further, Propositions 2–4 will be shown as corollaries to the theorem.

**PROPOSITION 2.** If $M$ is a spin $n$-manifold $(n \geq 6)$, then

$$[M \subset R^{2n-1}] = (1 + (-1)^n \tau_4)(H^{n-1}(M; \mathbb{Z}) \otimes H^{n-1}(M; \mathbb{Z}))$$

$$\times \begin{cases} [H^{n-1}(M; \mathbb{Z})] \times H^{n-1}(M; \mathbb{Z}_2) / Sq^2 H^{n-2}(M; \mathbb{Z}_2) & n \equiv 0 \pmod{4}, \\ H^{n-2}(M; \mathbb{Z}_2) & n \equiv 1 \pmod{2}. \end{cases}$$

**PROPOSITION 3.** If $M$ is an orientable $n$-manifold $(n \geq 6)$ and if either $Sq^2 H^{n-2}(M; \mathbb{Z}) \neq 0$ or $Sq^2 H^{n-3}(M; \mathbb{Z}) = H^{n-1}(M; \mathbb{Z}_2)$, then

$$[M \subset R^{2n-1}] = (1 + (-1)^n \tau_4)(H^{n-1}(M; \mathbb{Z}) \otimes H^{n-1}(M; \mathbb{Z}))$$

$$\times \begin{cases} [H^{n-1}(M; \mathbb{Z})] \times H^{n-1}(M; \mathbb{Z}) for n \equiv 0 \pmod{2}, \\ H^{n-2}(M; \mathbb{Z}_2) & n \equiv 1 \pmod{2}. \end{cases}$$

**REMARK.** It has been shown by Thomas [17, §4] that if $M$ is orientable and if either $w_3(M) \neq 0$, or $w_2(M) \neq 0$ and $H_1(M; \mathbb{Z})$ has no 2-torsion, then $Sq^2 H^{n-2}(M; \mathbb{Z}) \neq 0$.

**PROPOSITION 4.** Let $n$ be even and assume that an unorientable $n$-manifold $M (n \geq 6)$ is embedded in $R^{2n-1}$. Then

$$[M \subset R^{2n-1}] = (1 + (-1)^n \tau_4)(H^{n-1}(M; \mathbb{Z}) \otimes H^{n-1}(M; \mathbb{Z}))$$

$$\times [H^{n-1}(M; \mathbb{Z})] \times H^{n-2}(M; \mathbb{Z}_2),$$

if one of the conditions (i), (ii) and (iii) is satisfied:

(i) $Sq^2 \rho_2 H^{n-3}(M; \mathbb{Z}_2) = H^{n-1}(M; \mathbb{Z}_2)$,

(ii) $Sq^2 H^{n-3}(M; \mathbb{Z}_2) = H^{n-1}(M; \mathbb{Z}_2)$ and either $w_3(M) = w_3(M)$ or $Sq^2 \rho_2 H^{n-3}(M; \mathbb{Z}) = 0$,

(iii) $n \equiv 2 \pmod{4}$ and $w_1(M) \rho_2 H^{n-3}(M; \mathbb{Z}_2) = H^{n-1}(M; \mathbb{Z}_2)$.

As an example, we shall consider $[P(m, n) \subset R^{2m+2n-1}]$ for the Dold manifold $P(m, n)$ of type $(m, n)$ of dimension $m+2n$.

**PROPOSITION 5.** Let $m, n \geq 1$ with $m+2n \geq 6$ and assume that either $m \equiv 0 \pmod{2}$ or $n \equiv 0 \pmod{2}$ holds. Then
\[ \#[P(m, n) \subset R^{2m+4n-1}] = \begin{cases} 16 & \text{if } n \equiv 3 \pmod{4} \text{ and either } m = 2 \text{ or } n = 0 \pmod{4}, \\ 8 & \text{if } m = 0 \pmod{2}, n = 1 \pmod{4} \text{ or } m = 0, n = 0 \pmod{2}, \\ & \text{if } m \geq 4, m \equiv 2 \pmod{4}, n \equiv 3 \pmod{4}, \\ 4 & \text{if } n = 0 \pmod{2}, m \geq 2, \\ & \text{iff } n = 0 \pmod{2}, m = -1, \\ 2 & \text{if } n = 0 \pmod{2}, m = -1, \\ \infty & \text{if } m = 0 \text{ or } n = 1 \pmod{2}, m = 1, \\ 8 & \text{if } n = 1 \pmod{2}, m \geq 3, m \equiv 1 \pmod{2}, \\ 4 & \text{if } n = 0, m \equiv 3 \pmod{4}, \\ 2 & \text{if } n = 0, m \equiv 3 \pmod{4}, m \neq 2^r (r \geq 3). \end{cases} \]

where \( \#S \) denotes the cardinality of the set \( S \).

**Remark.** For all the other Dold manifolds \( P(m, n) \) with \( m+2n \geq 6 \), the cardinality of \([P(m, n) \subset R^{2m+4n-1}]\) are given as follows:

\[ \#[P(m, n) \subset R^{2m+4n-1}] = \begin{cases} \infty & \text{if } m = 0 \text{ or } n = 1 \pmod{2}, m = 1, \\ 8 & \text{if } n = 1 \pmod{2}, m \geq 3, m \equiv 1 \pmod{2}, \\ 4 & \text{if } n = 0, m \equiv 3 \pmod{4}, \\ 2 & \text{if } n = 0, m \equiv 3 \pmod{4}, m \neq 2^r (r \geq 3). \end{cases} \]

In fact, \([P(m, n) \subset R^{2m+4n-1}]\) for \( m = 1 \pmod{2}, n = 1 \pmod{2} \) is calculated in \([20]\) and that for \( n = 0 \) or \( m = 0 \) is calculated by Bausum \([1]\), Larmore and Rigdon \([9]\), Yasui \([18]\), and Haefliger and Hirsch \([6]\), \([7]\), because \( P(m, 0) \) and \( P(0, n) \) are the real and the complex projective spaces, respectively.

**Remark.** The set \([M \subset R^{2n-1}]\) in case \( n = 1 \pmod{2} \) and \( \omega_1(M) \neq 0 \) is not treated in this paper but this set has been studied in \([20]\) under the condition that \( H_0(M; Z) \) is isomorphic to a direct sum of some copies of \( Z_2 \).

The remainder of this paper is organized as follows: In §1, we state the method for computing \([M \subset R^{2n-1}]\). In §2, the action of \( Sq^i \) \( (i = 1, 2) \) on the mod 2 cohomology of \( M^* \) is studied. The morphism \( i^*: H^*(A^8M, \Delta M; Z_2) \to H^*(M^*; Z_2) \) is determined in §3. Here \( (A^8M, \Delta M) \) is the pair of quotient spaces \((M^2/Z_2, \Delta M/Z_2)\). The cohomology groups \( H^{2n-2}(M^*; Z[v]) \) and \( \tilde{p}_2H^{2n-2}(M^*; Z[v]) \) are given in §4, whose proofs for even \( n \) and for odd \( n \) are given in §5 and §6, respectively. §7 is concerned with computing \( \text{Coker } \Theta \). In the last section, §8, we give the proofs of the results stated in the introduction.

**§1. Method for computing \([M \subset R^{2n-1}]\).**

We now recall Haefliger's theorem \([5, \text{Théorème 1}']\). Let \( S^n \to P^m \) be the universal double covering. Then the bundle \( S^n \times_{S^2} S^m \to P^m \) is homotopically equivalent to the natural inclusion \( P^m \to P^m \). Therefore Haefliger's theorem can be restated as follows, where

\[ [X, P^m ; a] = [X, S^n \times_{S^2} S^m ; a] \quad \text{for } a : X \to P^m \]
denotes the homotopy set of liftings of \( a \) to \( S^n \times \mathbb{Z}_2 S^m \):

**Theorem 1.1** (Haefliger). If \( 2m > 3(n + 1) \), then for an \( n \)-manifold \( M \), there is a bijection

\[
[M \subset \mathbb{R}^m] = [M^*, P^{m-1}; \xi].
\]

In particular

\[
[M \subset \mathbb{R}^{2n-1}] = [M^*, P^{2n-2}; \xi] \quad \text{if} \quad n \geq 6.
\]

As for the right-hand side of this equation, we know

**Proposition 1.2** (Bausum [1], Larmore and Rigdon [9], Yasui [18] etc.). Let \( n \geq 6 \) and assume that there is a lifting of \( \xi : M^* \to P^n \) to \( P^{2n-2} \). Then, as a set,

\[
[M^*, P^{2n-2}; \xi] = H^{2n-2}(M^*; \mathbb{Z}[[v]]) \times \text{Coker} \Theta,
\]

where

\[
\Theta = (S^2 + (\frac{2n-1}{2})p^2) \tilde{\partial}_2 : H^{2n-3}(M^*; \mathbb{Z}[[v]]) \to H^{2n-1}(M^*; \mathbb{Z}_2),
\]

\( v \) is the first Stiefel-Whitney class of the double covering \( M^2 - \Delta M \to M^* \) and \( \mathbb{Z}[[v]] \) is the sheaf of coefficients over \( M^* \), locally isomorphic to \( \mathbb{Z} \), twisted by \( v \).

The mod 2 cohomology of \( M^* \) has been studied by Thomas [17], and Bausum [1]. Therefore it is important for our purpose to study the groups \( H^{2n-2}(M^*; \mathbb{Z}[[v]]) \) and \( \tilde{\partial}_2 H^{2n-3}(M^*; \mathbb{Z}[[v]]) \), and the action of \( Sq^i \) (\( i = 1, 2 \)) on \( H^*(M^*; \mathbb{Z}_2) \).

For a manifold \( M \), let \( PM \) denote the projective bundle associated with the tangent bundle of \( M \) and let

\[
\tilde{j} : PM \to M^*
\]

be the inclusion, which is given by the \( \mathbb{Z}_2 \)-equivariant map \( \tilde{j} \) of the tangent sphere bundle \( SM \) to \( M^2 - \Delta M \) defined by \( \tilde{j}(v) = (\exp(v), \exp(-v)) \), the \( \mathbb{Z}_2 \)-action on \( SM \) being induced from the antipodal map on each fibre. The \( \mathbb{Z}_2 \)-action on \( M^2 \), which is defined by interchanging factors, determines the quotient spaces

\[
A^2 M = M^2 / \mathbb{Z}_2 \quad \text{and} \quad \Delta M = (\Delta M) / \mathbb{Z}_2.
\]

Therefore, it follows that

\[
M^* = A^2 M - \Delta M.
\]

Let

\[
\pi : (M^2, \Delta M) \to (A^2 M, \Delta M),
\]

\[
i : M^* \to (A^2 M, \Delta M)
\]

be the natural projection and inclusion, respectively.

**Lemma 1.3** ([19, Lemma 1.4]). For a manifold \( M \) and a cyclic group \( G \), there is an exact sequence
This lemma will play a dominant role for studying $H^{n-2}(M^*; G[v])$ and $\tilde{p}_*H^{n-2}(M^*; Z[v])$, since the twisted integral cohomology groups of $(A^2M, \Delta M)$ is investigated by Larmore [8] (cf. [19, §5]) and so is $PM$ by Rigdon [13, §9] and since the mod 2 cohomology groups both of $(A^2M, \Delta M)$ and $M^*$ are investigated in [8], [12] and [1], [17], respectively, and that of $PM$ is well-known, and moreover the morphism $j^*$ and $\delta$ are given by [17, §2] and [19, Lemma 1.5]. From now on, we shall make much use of notations introduced in [17] and [8] to represent elements of the mod 2 cohomology of $M^*$ and of the cohomology of $(A^2M, \Delta M)$, respectively. For example, let

$$j^*v = v \in H^i(\tilde{P}M; Z_2).$$

Then there are the following relations in the mod 2 cohomology:

\begin{equation}
(1.4) \quad j^*\rho(u^{i} \otimes x^2) = \sum_{r=0}^{i} \nu^{i+r} Sq^r x \quad \text{for} \quad x \in H^i(M; Z_2),
\end{equation}

\begin{equation}
(1.5) \quad \delta(v^i x) = v^{i+1} Ax \quad \text{for} \quad x \in H^*(M; Z_2).
\end{equation}

§ 2. Actions of $Sq^i$ (i=1, 2) on the mod 2 cohomology of $M^*$.

First, we shall study, along the lines of Thomas [17], the mod 2 cohomology of $M^*$ more exactly than [17] and so we assume that the reader is familiar with [17]. Moreover we shall quote the properties referring to $M^*$ and to spaces related to $M^*$ mainly from [17] rather than [4].

**Theorem 2.1 (Haefliger).** For an n-manifold $M$, there is a commutative diagram, in which each row is exact ($i \geq 0$):

\begin{align*}
0 \rightarrow H^{i-n}(M; Z_2) & \xrightarrow{\phi_i} H^i(M^*; Z_2) \rightarrow H^i(M^*, \Delta M; Z_2) \rightarrow 0 \\
0 \rightarrow H^{i-n}(P^n \times M; Z_2) & \xrightarrow{\phi} H^i(\Gamma M; Z_2) \xrightarrow{q^*} H^i(M^*; Z_2) \rightarrow 0 \quad (\Gamma M = S^n \times Z_2 M^*) \\
0 \rightarrow H^{i-n}(P^n \times M; Z_2) & \xrightarrow{\phi^*} H^i(P^n \times M; Z_2) \rightarrow H^i(\tilde{P}M; Z_2) \rightarrow 0.
\end{align*}
Here, the maps $r$, $q$, $k$, $j$ are inclusions and $p$ is a projection. $\phi_1$, $\phi_2$ can be thought of as Gysin maps. $\rho$ is an $H^*(P^\infty; \mathbb{Z}_2)$-module homomorphism defined by

$$\rho = p'^{-1}i'^*$$

where

$$p' : S^\infty \times \mathbb{Z}_2(M^k - \Delta M) \longrightarrow M^*$$
$$r' : S^\infty \times \mathbb{Z}_2(M^k - \Delta M) \longrightarrow S^\infty \times \mathbb{Z}_2M^* = \Gamma M$$

are the natural projection and inclusion, respectively. $r^*$ is the obvious projection. Moreover, the morphisms $\phi_1$, $\phi_2$, $q^*$, $k^*$ and the mod 2 cohomology of $\Gamma M$ are given in [17, §2]. To determine $\operatorname{Ker} \rho = \operatorname{Im} \phi$, consider the certain operations $Q^i$ ($i \geq 0$) introduced by Yo [21, p. 1481].

**Proposition 2.3** (Yo). For an $n$-manifold $M$, there exist operations

$$Q^i : H^q(M; \mathbb{Z}_2) \longrightarrow H^{q+i}(M; \mathbb{Z}_2) \quad \text{for } i \geq 0,$$

with the following properties:

1. if $x \in H^q(M; \mathbb{Z}_2)$, then $Q^i x = 0$ for $i > (n-q)/2$,
2. $Q^0$ is equal to the identity,
3. for any $z \in H^*(M; \mathbb{Z}_2)$ and $x \in H^*(M; \mathbb{Z}_2)$, $\langle Q^i x, z \rangle = \langle x Q^i (P.D), [M] \rangle$, where $\langle , \rangle$ denotes the Kronecker pairing, $P.D.$ is the Poincaré duality and $[M] \in H_n(M; \mathbb{Z}_2)$ is the generator.

These operations $Q^i$ ($i \geq 0$) are related to the squaring operations $Sq^i$ ($j \geq 0$) and the Stiefel-Whitney classes $w_k(M)$ of $M$ ($k \geq 0$) by the equation ([21, Corollary 4, p. 1485])

$$\sum_{i+j=k} Sq^i Q^j(x) = x w_k, \quad (w_k = w_k(M)).$$

(From now on, $w_k$ will stand for $w_k(M)$.) From (2.4), it follows that

$$Q^1 = Sq^1 + w_1, \quad Q^2 = Sq^2 + w_1 + w_2 + w_1 Sq^1,$$

$$Q^1 = 0 \text{ on } H^{n-1}(M; \mathbb{Z}_2), \quad Q^2 = 0 \text{ on } H^k(M; \mathbb{Z}_2) \text{ for } i \geq n-3.$$

**Proposition 2.6.** Let $x \in H^*(M; \mathbb{Z}_2)$ and let $U \in H^n(M^2; \mathbb{Z}_2)$ be the mod 2 Thom class of $M$. Then the following relations hold:

1. $\phi(1 \otimes x) = \sum_{i=0}^{[(n-r)/2]} u^{n-r-i} \otimes (Q^i x)^2 + U(1 \otimes x)$,
2. $U(1 \otimes x) \in I^* \quad (= (1+t^*)H^*(M^2; \mathbb{Z}_2))$ if $n-r = 1(2)$,
3. $U(1 \otimes x) + 1 \otimes (Q^{(n-r)/2} x)^2 \in I^*$ if $n-r = 0 \quad (2)$;
4. $\phi(u^j \otimes x) = \sum_{i=0}^{[(n-r)/2]} u^{j+n-r-i} \otimes (Q^i x)^2 \quad \text{for } j > 0.$

**Proof.** By [17, Theorem 2.4], $\phi(u^j \otimes x)$ is of the form
\[ \phi(u^j \otimes x) = X_1 \cap X_2, \quad X_1 \in H^*(P^\infty; \mathbb{Z}_2) \otimes \mathbb{K}^*, \quad X_2 \in I^*, \]

where \( \mathbb{K}^* = (\text{Ker}(1 \otimes t^*)) / I^* \). From (2.4) and [17, Proposition 2.5 (iii)] it follows that

\[
\Phi^* \left( \sum_{t=0}^{(n-r)/2} u^{j+n-r-2t} \otimes (Q^t x)^2 \right) = \sum_{m \geq 0} u^{j+n-m} \otimes x \cdot w_m,
\]

and from the fact that \( \Phi^* \phi = \phi \) and [17, Proposition 2.5 (v) and (2.3)], it follows that

\[ k^*(X_1) = \sum_{m \geq 0} u^{j+n-m} \otimes x \cdot w_m. \]

Hence we have

\[ X_1 = \sum_{t=0}^{(n-r)/2} u^{j+n-r-2t} \otimes (Q^t x)^2, \]

since \( k^* | (H^*(P^\infty; \mathbb{Z}_2) \otimes \mathbb{K}^*) \) is injective by [17, Proposition 2.5]. If \( j > 0 \), then \( \phi^* \phi(u^j \otimes x) = 0 \) and so \( X_1 = 0 \) follows from [17, Proposition 2.5]. On the other hand, if \( j = 0 \), then

\[ \phi^* \phi(1 \otimes x) = U(1 \otimes x) \text{ in } H^*(M^2; \mathbb{Z}_2) \]

by [17, (2.2)] and therefore

\[ U(1 \otimes x) = \begin{cases} X_2 & \text{for } n-r \equiv 1 \ (2), \\ X_2 + 1 \otimes (Q^{(n-r)/2} x)^2 & \text{for } n-r \equiv 0 \ (2). \end{cases} \]

This completes the proof.

As a corollary to the proposition, the property of [17, (7.2)] holds and hence we have

**PROPOSITION 2.7** (Thomas). Let \( B_k \) be the \( \mathbb{Z}_2 \)-vector subspace of \( H^k(\Omega^r M; \mathbb{Z}_2) \) generated by all the elements of the form \( u^j \otimes x^2 \) with \( 2(\dim x) + j = k \) and \( j < \dim x < n = \dim M \). Then

1. \( \text{Ker } j^* = \rho(I^*) \),
2. \( j^* : B^* \to \text{Im } j^* \) is an isomorphism,
3. \( \rho : B^* + I^* \to H^*(M^*; \mathbb{Z}_2) \) is an isomorphism,
4. \( \nu \rho(I^*) = 0 \).

In consequence of (2.5), Propositions 2.6 and 2.7, we have

**LEMMA 2.8.** Let \( x_{\pm} \in H^r(M; \mathbb{Z}_2) \). Then the following relations hold:

1. \( \rho(u^j \otimes (x_{\pm} x_{\pm})^2) = \rho(U(1 \otimes x_{\pm} x_{\pm})) \),
2. \( \rho(u^j \otimes (x_{\pm} x_{\pm})^2) = \rho(U(1 \otimes (S_{q^1} + w_1) x_{\pm} x_{\pm})) \),
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\[ \rho(u^2 \otimes (x_{n-2})) = \rho(U(1 \otimes x_{n-2})) \]
\[ \rho(u^3 \otimes (x_{n-3})) = \rho(u \otimes ((Sq^1 + w_1)x_{n-3}) + U(1 \otimes x_{n-3})) \]

Lastly, we study the actions of \( Sq^i \) \((i=1, 2)\) on \( H^*(M^*; Z_2) \).

**Lemma 2.9.** Let \( x_r \in H^r(M; Z_2) \). Then the following relations hold:

1. \( Sq^1 \rho(1 \otimes (x_{n-1}^2)) = \rho(Sq^1 x_{n-1} \otimes x_{n-1} + x_{n-1} \otimes Sq^1 x_{n-1}) \) if \( n=1 \) \((2)\),
2. \( Sq^1 \rho(u \otimes (x_{n-2}^2)) = (n-1) \rho(U(1 \otimes x_{n-2})) \),
3. \( Sq^2 \rho(u \otimes (x_{n-2}^2)) = \begin{cases} 
\rho(U(1 \otimes w_1 x_{n-2})) & \text{if } n=0, 3 \text{ (4)}, \\
\rho(U(1 \otimes Sq^1 x_{n-2})) & \text{if } n=1, 2 \text{ (4)},
\end{cases} \)
4. \( Sq^1 \rho(1 \otimes (x_{n-2}^2)) = n \rho(u \otimes (x_{n-2}^2)) + \rho(Sq^1 x_{n-2} \otimes x_{n-2} + x_{n-2} \otimes Sq^1 x_{n-2}) \),
5. \( Sq^2 \rho(u \otimes (x_{n-2}^3)) = (n-1) \rho(U(1 \otimes x_{n-3})) \).

**Proof.** The relations (1)-(5) are seen to be immediate consequences of Lemma 2.8 and the following fact, which is shown by Bausum [1, Lemmas 11 and 24]: if \( x \in H^r(M; Z_2) \), then

\[ Sq^1(u \otimes x^2) = (i+1)u^{i+1} \otimes x^2 \quad \text{for } i>0, \]
\[ ru \otimes x^2 + Sq^1 x \otimes x + x \otimes Sq^1 x \quad \text{for } i=0, \]
\[ Sq^2(u \otimes x^2) = \begin{cases} 
\left( \frac{i+1}{2} \right) u^{i+2} \otimes x^2 + u^i \otimes (Sq^1 x)^2 & \text{for } i>0, \\
\left( \frac{2}{2} \right) u^i \otimes x^2 + 1 \otimes (Sq^1 x)^2 + Sq^2 x \otimes x + x \otimes Sq^1 x & \text{for } i=0.
\end{cases} \]

**Remark.** This lemma, essentially, overlaps with the results of Bausum [1]. But his relations are not in \( \rho H^*(\Gamma M; Z_2) = H^*(M^*; Z_2) \) and we would like to describe the actions of \( Sq^i \) in terms of the elements in \( \rho H^*(\Gamma M; Z_2) \).

§ 3. On the morphism \( i^* : H^*(A^1 M, \partial M; Z_2) \to H^*(M^*; Z_2) \).

By [8, Theorem 11], as an algebra over \( Z_2[v] \), \( H^*(A^1 M, \partial M; Z_2) \) is generated by \( Ax \) for all \( x \in H^k(M; Z_2) \) \((k>0)\), where \( Ax \) satisfies the condition

\[ \pi^*(Ax) = x \otimes 1 + 1 \otimes x \quad \text{in } H^*(M^*, \partial M; Z_2). \]

The purpose of this section is to determine the morphism \( i^* : H^*(A^1 M, \partial M; Z_2) \to H^*(M^*; Z_2) \). Since

\[ i^*(v^i Ax) = 0 \quad \text{for } i>0 \]

by Lemma 1.3 and (1.5), we prove the following

**Lemma 3.3.** Let \( M \) be an \( n \)-manifold.
\[ i^*(Ax) = \begin{cases} 
\rho(x \otimes 1 + 1 \otimes x) & \text{if } \dim x < n, \\
\rho(x \otimes 1 + 1 \otimes x + \lambda U) & \text{if } \dim x = n.
\end{cases} \]

(2) If \( \dim x + \dim y < 2n \), then
\[ i^*(Ax Ay) = \rho(x \otimes y + y \otimes x + x y \otimes 1 + 1 \otimes x y). \]

**Proof.** Let
\[ q' : \Gamma M = S^m \times Z_2 M^2 \longrightarrow A^s M = M^s / Z_2 \subset (A^s M, \Delta M) \]
be the composition of the natural projection and inclusion. Then the following diagram is commutative:
\[
\begin{array}{ccc}
(M^s, \Delta M) & \xrightarrow{\pi} & M^s \\
\downarrow q' & & \downarrow q \\
(A^s M, \Delta M) & \xleftarrow{i} & M^s - \Delta M
\end{array}
\]
(3.4)

Since \( \rho = p'^{-1} i'^* \) by (2.2), it follows that
\[ i^* = \rho q'^*. \]
Moreover it follows that
\[ \rho q'^*(Ax) \in \rho(I^*), \]
because \( \text{Im} i^* = \text{Ker} f^* = \rho(I^*) \) by Lemma 1.3 and Proposition 2.7. If \( k < n \), then \( \rho : H^k(\Gamma M; Z_2) \to H^k(M^s; Z_2) \) is an isomorphism by Theorem 2.1, and so
\[ q'^*(Ax) \in I^* \quad \text{if } \dim x < n. \]

Now, the relation \( q'^* (Ax) = q^* (x \otimes 1 + 1 \otimes x) \) holds by (3.1), (3.4) and [17, Proposition 2.5] and \( q^* \) is an injection on \( I^* \) by [17]. Therefore we have
\[ q'^*(Ax) = x \otimes 1 + 1 \otimes x \quad \text{if } \dim x < n, \]
and so
\[ i^*(Ax) = \rho q'^*(Ax) = \rho (x \otimes 1 + 1 \otimes x) \quad \text{if } \dim x < n. \]

If \( x \in H^n(M; Z_2) \) is the generator, then there is an element \( z \in I^* \) such that \( \rho q'^*(Ax) = \rho z \) by (3.5). Since \( \text{Ker} \rho = \text{Im} \phi \) and \( \phi H^n(P^s \times M; Z_2) = Z_2 \) generated by \( \phi(1 \otimes 1) \), \( q'^*(Ax) \) is of the form
\[ q'^*(Ax) = z + \lambda \phi(1 \otimes 1) \quad (\lambda = 0 \text{ or } 1). \]

Hence
\[ x \otimes 1 + 1 \otimes x = q^* q'^*(Ax) = z + \lambda U \quad \text{in } H^n(M^s; Z_2) \]
by (3.1), (3.4) and [17], and so
\[ z = x \otimes 1 + 1 \otimes x + \lambda U \quad (\lambda = 0 \text{ or } 1). \]
Thus we have
\[ i^*(Ax) = \rho_Z = \rho(x \otimes 1 + 1 \otimes x + \lambda U) \quad (\lambda = 0 \text{ or } 1) \]
and the proof of (1) is complete. The proof of (2) follows immediately from (1)
and the relation \( U(x \otimes 1) = U(1 \otimes x) \) (cf. [11, Lemma 11.8]).

§ 4. Groups \( H^{2n-2}(M^* ; \mathbb{Z}[v]) \) and \( \tilde{\rho}_s H^{2n-3}(M^* ; \mathbb{Z}[v]) \).

We begin this section by explaining notations.
\( \mathbb{Z}_r\langle a \rangle \) denotes the cyclic group of order \( r \) \( (r < \infty) \) generated by \( a \).
For \( v \in H^i(X ; \mathbb{Z}_r) \), \( \mathbb{Z}_r[v] \) denotes the sheaf of coefficients over \( X \), locally
isomorphic to \( \mathbb{Z}_r \), twisted by \( v \) \( (r < \infty) \), and
\[ \tilde{\rho}_r : H^i(X ; \mathbb{Z}_r[v]) \longrightarrow H^i(X ; \mathbb{Z}_r[v]) \quad (s = 0 \text{ or } s = \infty) \]
and
\[ \tilde{\beta}_r : H^{i-1}(X ; \mathbb{Z}_r[v]) \longrightarrow H^i(X ; \mathbb{Z}_r[v]) \quad (r < \infty) \]
denote the reduction mod \( r \) and the Bockstein operator, respectively, twisted by \( v \). Then \( \tilde{\rho}_r \) and \( \tilde{\beta}_r \) for \( v = 0 \) are the ordinary ones \( \rho_r \) and \( \beta_r \), respectively.
Moreover, the following relations are well-known ([3] and [14]):
(4.1) \( \rho_2 \beta_2 = Sq^1, \quad \tilde{\rho}_2 \tilde{\beta}_2 = Sq^1 + v. \)
Assume that the integral cohomology groups \( H^i(M ; \mathbb{Z}) \) for an \( n \)-manifold \( M \) are of the form
\[ H^n(M ; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} \langle M \rangle & \text{if } M \text{ is orientable,} \\
\mathbb{Z} \langle \beta_2 M' \rangle & \text{if } M \text{ is unorientable,}
\end{cases} \]
(4.2)
\[ H^m(M ; \mathbb{Z}) = \bigoplus_{\ell=1}^{\ell(m)} Z_{r(m, \ell)} \langle x_{m, \ell} \rangle \quad (direct \ sum) \quad \text{for } m \leq n-1, \]
where the order \( r(m, i) \) is infinite for \( 1 \leq i \leq \alpha(m) \), a power of 2 for \( \alpha(m) < i \leq \beta(m) \)
and a power of an odd prime for \( \beta(m) < i \leq \gamma(m) \), and if \( \alpha(m) < i < j \leq \gamma(m) \) then either \( r(m, i), r(m, j) = 1 \) or \( r(m, i) | r(m, j) \) holds.
For brevity,
(4.2)' denote \( \alpha(m), \beta(m), \gamma(m), r(m, i), x_{m, i}, y_{m, i} \) in (4.2), respectively, by
\[ \alpha, \beta, \gamma, r(i), x_i \text{ and } y_i \quad \text{when } m = n-1, \]
\[ \alpha', \beta', \gamma', r'(i), x'_i \text{ and } y'_i \quad \text{when } m = n-2. \]
Using the above notations and the symbols \( Ax \) and \( A(x, y) \) introduced in [8], we have the theorems, postponing the proofs till §§ 5-6.
THEOREM 4.3. Let $M$ be an $n$-manifold ($n \geq 4$) and assume that $M$ is orientable or $n$ is even. Then

$$H^{2n-2}(M^*; \mathbb{Z}[v]) \cong (1+(-1)^{n+1})(H^{n-1}(M; \mathbb{Z}) \otimes H^{n-1}(M; \mathbb{Z}))$$

$$+ \begin{cases} [H^{n-1}(M; \mathbb{Z})] + H^{n-2}(M; \mathbb{Z}) & \text{if } n \equiv 0 \pmod{2}, w_1 = 0, \\ [H^{n-1}(M; \mathbb{Z})] + H^{n-2}(M; \mathbb{Z}) & \text{if } n \equiv 0 \pmod{2}, w_1 \neq 0, \\ H^{n-2}(M; \mathbb{Z}) & \text{if } n \equiv 1 \pmod{2}, w_1 = 0, \end{cases}$$

$$\cong G_1 + G_3 + \begin{cases} G_6' + G_6 & \text{if } n \equiv 0 \pmod{2}, w_1 = 0, \\ G_6' + G_7 & \text{if } n \equiv 0 \pmod{2}, w_1 \neq 0, \\ G_8 & \text{if } n \equiv 1 \pmod{2}, w_1 = 0, \end{cases}$$

where

$$G_6 = \sum_{1 \leq i < a} Z \langle (1/2)i^* \delta(x_i, x_i) \rangle$$

$$+ \sum_{\alpha \leq \beta} Z_{2r(\alpha)} \langle (1/2)i^* \beta_{r(\alpha)} \delta(y_i, \rho_{r(\alpha)} x_i) \rangle$$

$$+ \sum_{\beta \leq r(\gamma)} Z_{r(\gamma)} \langle i^* \beta_{r(\gamma)} \delta(y_i, \rho_{r(\gamma)} x_i) \rangle \quad (\cong [H^{n-1}(M; \mathbb{Z})])$$

$$G_1 = \sum_{1 \leq i < a} Z \langle i^* \delta(x_i, x_j) \rangle,$$

$$G_3 = (\sum_{1 \leq i < a} + \sum_{a \leq j < b}) Z_{r(\gamma)} \langle i^* \beta_{r(\gamma)} \delta(y_j, \rho_{r(\gamma)} x_i) \rangle,$$

$$G_6 = \sum_{a \leq b \leq r(\gamma)} Z_{r(\gamma)} \langle i^* \beta_{r(\gamma)} \delta(y_i, \rho_{r(\gamma)} M) \rangle + \sum_{\gamma \leq k \leq a} Z \langle i^* \delta(x_k, M) \rangle \quad (\cong H^{n-3}(M; \mathbb{Z}))$$

$$G_7 = \{\beta_{r(\gamma)} \rho(x \otimes M' + M' \otimes x) | x \in H^{n-3}(M; Z_2) \} \quad (\cong H^{n-3}(M; Z_2)),$$

$$G_8 = \{\beta_{r(\gamma)} \rho(u \otimes x^2) | x \in H^{n-3}(M; Z_2) \} \quad (\cong H^{n-3}(M; Z_2)),$$

and $t : M^* \to M^*$ is the map defined by interchanging factors. (The definition of $G'$ for a group $G$ is given in the introduction.)

THEOREM 4.4. Under the same condition as in Theorem 4.3, there holds the following equation:

$$\bar{p}_z H^{n-2}(M^*; \mathbb{Z}[v]) = \begin{cases} H_4 + H_5 + H_6 + H_7 & \text{if } n \equiv 0 \pmod{2}, w_1 = 0, \\ H_4 + H_5 + H_6 + H_7 & \text{if } n \equiv 0 \pmod{2}, w_1 \neq 0, \\ H_5 + H_6 + H_7 & \text{if } n \equiv 1 \pmod{2}, w_1 = 0, \end{cases}$$

where
Enumerating embeddings

\[ H_1 = \{ \rho \sigma (\rho_2 x \otimes M) \mid x \in H^{n-3}(M; Z) \}, \]
\[ H_2 = \{ \rho \sigma (\rho_2 x \otimes \rho_2 y) \mid x \in H^{n-2}(M; Z), y \in H^{n-1}(M; Z) \}, \]
\[ H_3 = \sum_{a \leq b} Z^2 \langle \rho \sigma (\rho_2 y_i \otimes M + \rho_2 y_j \otimes M') \rangle, \]
\[ H_4 = Z^2 \langle \rho \sigma (\rho_2 y_i \otimes \rho_2 x_i) \rangle, \]
\[ H_5 = \sum_{a \leq b} Z^2 \langle \rho \sigma (\rho_2 y_i \otimes \rho_2 x_i) \rangle, \]
\[ H_6 = \{ \rho (u \otimes x^2 + \sigma (Sq^1 x \otimes x)) \mid x \in H^{n-2}(M; Z) \}, \]
\[ H_7 = \{ \rho (u \otimes (Sq^1 x)^2 + U(1 \otimes x)) \mid x \in H^{n-3}(M; Z) \}, \]
\[ \text{and } \sigma = 1 + t^*. \]

Remark. In Theorems 4.3 and 4.4, G's and H's, except G_6, G_8, H_5 and H_7, are isomorphic, by i*, to those in [19, § 5].

§ 5. Proofs of Theorems 4.3 and 4.4 for even n.

Assume that n is even in this section and consider the exact sequence of Lemma 1.3 for \( G[\nu] = Z[\nu] \), in which the twisted integral cohomology of \((A^2 M, D M)\) is given by [19, § 5] (cf. [8]) and that of \( PM \) is given by Rigdon [13, § 9] as follows:

\[ \theta : H^{n-2}(M; Z_2) \rightarrow H^{2n-2}(PM; Z[\nu]), \quad \theta(x) = \delta_2(x), \]
\[ \theta' : H^{n-2}(M; Z_2) + H^n(M; Z_2) \rightarrow H^{2n-2}(PM; Z[\nu]), \quad \theta'(x, y) = \delta_2(x^2 + v^{n-1} y). \]

The morphism \( \delta \) on \( H^{2n-2}(PM; Z[\nu]) \) has been studied in [19, Lemma 3.2] while \( \delta \) on \( H^{2n-2}(PM; Z[\nu]) \) is given in the same way as before, i.e., by using (1.5), (5.1) and [8], as follows:

\[ \delta_2(x) = \delta_2(v^{n-2} x), \]
\[ \delta_2(x) = \delta_2(v^{n-2} x), \quad \text{for } x \in H^{n-2}(M; Z_2). \]

Therefore, the exact sequence in Lemma 1.3 leads to the lemma.

**Lemma 5.2.** There are two exact sequences

(1) \[ 0 \rightarrow H^{2n-2}(A^2 M, D M; Z[\nu]) / \text{Im} \delta \rightarrow H^{2n-2}(M^*; Z[\nu]) \rightarrow (Z_2)^{\delta} \rightarrow 0, \]
(2) \[ \cdots \rightarrow H^{2n-2}(A^2 M, D M; Z[\nu]) \rightarrow H^{2n-2}(M^*; Z[\nu]) \rightarrow (Z_2)^{\delta + a} \rightarrow 0, \]
such that

\((3) \text{ Im } \delta = \mathbb{Z}_2 \langle \tilde{\beta}_2 (v^{n-2}\Delta M) \rangle, \)

\((4) \ (Z_2)^{\delta} = \{ \tilde{\gamma}_2 (v^{n-2}\rho_2 x) \mid x \in H^{n-2}(M; \mathbb{Z}) \}, \)

\((5) \ (Z_2)^{\delta', \delta-a} = \{ \tilde{\gamma}_2 (v^{n-2}x + v^{n-4}Sq^2 x) \mid x \in H^{n-2}(M; Z_2) \} \).

**Lemma 5.3.** In the exact sequences in Lemma 5.2, the following properties hold:

1. For any subgroup \( Z_2 \) of \((Z_2)^{\delta} \) in Lemma 5.2 (1), there does not exist a subgroup \( Z_2 \) of \( H^{n-2}(M^*; \mathbb{Z}[v]) \) such that \( j*Z_2 = Z_2. \)

2. \( j^* \tilde{\beta}_2 \rho (1 \otimes x^2) = \tilde{\beta}_2 (v^{n-2}x + v^{n-4}Sq^2 x) \) for \( x \in H^{n-2}(M; Z_2) \) and so \( j^* \) in Lemma 5.2 (2) is a split epimorphism.

**Proof.** To prove (1), it is sufficient to show that \( j^* \tilde{\beta}_2 H^{n-2}(M^*; Z_2) = 0. \) Now, it is shown, by Proposition 2.7, that

\( H^{n-2}(M^*; Z_2) = \rho (B^{n-2}) + \rho (I^{n-2}), \)

\( B^{n-2} = \{ u \otimes x^2 \mid x \in H^{n-2}(M; Z_2) \}, \quad j^* \rho (I^{n-2}) = 0. \)

If \( x \in H^{n-2}(M; Z_2), \) then

\( j^* \tilde{\beta}_2 \rho (u \otimes x^2) = \tilde{\beta}_2 \rho_2 \tilde{\beta}_2 (v^{n-2}x + v^{n-4}Sq^2 x) = 0 \)

by (1.4) and (4.1), and hence (1) is established. On the other hand, (2) is obtained in the same way as above.

**Proof of Theorem 4.3 for even \( n \).** Consider the group extension of the short exact sequence in Lemma 5.2 (1), in which the group \( H^{n-2}(A^4 M, \Delta M; Z[v])/\text{Im } \delta \) is determined by Lemma 5.2 (3) and \([19, \text{Proposition 5.4}].\) By using the Gysin exact sequence of the double covering \( M^2 - \Delta M \rightarrow M^* \) (cf. \([19, (5.6)]\)), it is shown that

\[ \tilde{\rho}_2 \tilde{\beta}_{r, ij} \Delta(y_j, \rho_{r, ij} x_i) - \tilde{\rho}_2 \Delta(x_i, x_i) \in \text{Im } \delta \quad \text{for } 1 \leq i \leq \alpha < j \leq \beta \quad \text{or } \alpha < j < i \leq \beta. \]

Therefore, the \( \tilde{\rho}_2 \)-images of the generators of \( H^{n-2}(A^4 M, \Delta M; Z[v])/\text{Im } \delta \) are given, by using Lemma 3.3 and \([19, \text{Lemma 1.5}].\) as follows:

\[ \tilde{\rho}_2 ^* \Delta(x_i, x_i) = 0 \quad \text{for } 1 \leq i \leq \alpha, \]

\[ \tilde{\rho}_2 ^* \tilde{\beta}_{r, ij} \Delta(y_j, \rho_{r, ij} x_j) = 0 \quad \text{for } \alpha < j \leq \beta, \]

and the \( \tilde{\rho}_2 ^* \)-images of the other generators of \( H^{n-2}(A^4 M, \Delta M; Z[v])/\text{Im } \delta, \) mod odd torsion, are linearly independent in \( \rho (I^{n-2}). \) Hence, from this result, Lemma 5.2 (1), (3) and Lemma 5.3 (1), Theorem 4.3 is established when \( n \) is even.

**Proof of Theorem 4.4 for even \( n \).** By Lemmas 5.2 and 5.3, it is seen that

\[ \tilde{\rho}_2 H^{n-2}(M^*; Z[v]) = i^* \tilde{\rho}_2 H^{n-2}(A^4 M, \Delta M; Z[v]) \]

\[ + \{ \tilde{\rho}_2 \tilde{\beta}_2 \rho (1 \otimes x^2) \mid x \in H^{n-2}(M; Z_2) \} , \]
in which the relation
\[ \tilde{p}_x \tilde{\rho} x = \rho(u \otimes x^2 + \sigma(Sq^1 x \otimes x)) \quad \text{for} \quad x \in H^{n-2}(M; \mathbb{Z}_2) \]
holds by Lemma 2.9 and (4.1). Therefore, Theorem 4.4 for even \( n \) is verified, since \( i^* \tilde{p}_x H^{2n-3}(\Delta M, \Delta M; \mathbb{Z}_2) \) is determined by Lemma 3.3 and [19, Proposition 5.5].

§ 6. **Proofs of Theorems 4.3 and 4.4 for odd \( n \).**

For an orientable \( n \)-manifold \( M \), there is an exact sequence
\[ 0 \rightarrow H^{i-n}(M; \mathbb{Z}) \rightarrow H^i(M^2; \mathbb{Z}) \rightarrow H^i(M^2 - 4M; \mathbb{Z}) \rightarrow 0, \]
where
\[ \phi_1(x) = U(1 \otimes x) \quad \text{for} \quad x \in H^{i-n}(M; \mathbb{Z}) \]
and \( U = \phi_1(1) \) is called the Thom class or the diagonal class of \( M \), e.g., by Milnor [11]. Notice that
\[ t^* \phi_1(x) = (-1)^n \phi_1(x) \quad \text{for} \quad x \in H^*(M; \mathbb{Z}) \]
(e.g., [16, P. 305]) and that
\[ U = \pm (M \otimes 1 + (-1)^n 1 \otimes M) \mod \left( \sum_{i=1}^{n-1} H^{n-i}(M; \mathbb{Z}) \otimes H^i(M; \mathbb{Z}) + \sum_{i=1}^{n-2} H^{n-i}(M; \mathbb{Z}) \otimes H^{i+1}(M; \mathbb{Z}) \right) \]

In the rest of this section, assume that \( n \) is odd and that \( M \) is an orientable \( n \)-manifold.

**Lemma 6.5.** The odd torsion subgroup of \( H^{2n-3}(M^*; \mathbb{Z}_2) \) is equal to
\[ \left( \sum_{\beta \leq j < \beta' \leq \gamma} + \sum_{\beta' < j \leq \gamma} \right) Z_{r(j)} \langle \tilde{p}_x^{r(j)} \Delta (y_j, \rho_{r(j)} x_i) \rangle. \]

**Proof.** By the spectral sequence argument for \( M^2 - \Delta M \rightarrow M^* \rightarrow P^\infty \) (cf. [15, Theorem 2.9]), the odd torsion subgroup of \( H^{2n-3}(M^*; \mathbb{Z}_2) \) is isomorphic, by \( p^* \), to that of \( \{ x \in H^{2n-3}(M^2 - \Delta M; \mathbb{Z}) \mid t^* x = -x \} \), which is easily shown to be equal to
\[ \left( \sum_{\beta \leq j < \beta' \leq \gamma} + \sum_{\beta' < j \leq \gamma} \right) Z_{r(j)} \langle \tilde{p}_x^{r(j)} \Delta (y_j, \rho_{r(j)} x_i) \rangle \]
by (6.1)-(6.4). Now, there hold the relations
\[ p^* t^* \tilde{p}_x^{r(j)} \Delta (y_j, \rho_{r(j)} x_i) = \tilde{p}_x^{r(j)} \Delta (y_j, \rho_{r(j)} x_i) \]
for \( 1 \leq i \leq \alpha < j \leq \gamma \) or \( \alpha < j < i \leq \gamma \).
by \([19, (5.9)]\) and the property of being \(\beta_{r(j)}(y_j \otimes \rho_{r(j)}x_i) = x_j \otimes x_i\), and so the lemma is estabished.

Consider, next, the Bockstein exact sequence associated with \(0 \rightarrow \mathbb{Z}[\nu] \rightarrow \mathbb{Z}[\nu] \rightarrow 0\). In Proposition 2.7, it is shown that

\[
H^{2n-1}(M^*; \mathbb{Z}2) = \{\rho(\sigma(M \otimes x)x) | x \in H^{n-1}(M; \mathbb{Z}2)\},
\]

\[
H^{2n-2}(M^*; \mathbb{Z}2) = \{\rho(1 \otimes x^2) | x \in H^{n-1}(M; \mathbb{Z}2)\} + A_1,
\]

\[
A_1 = \{\rho(U(1 \otimes x)) | x \in H^{n-1}(M; \mathbb{Z}2)\} + \sum_{1 \leq j < \beta} Z_2 \langle \rho \sigma(\rho_x x_j \otimes \rho_x x_i) \rangle
\]

since for \(x \in H^{n-1}(M; \mathbb{Z}2)\),

\[
U(1 \otimes x) = 1 \otimes (S \alpha \beta x^3 + \sigma(M \otimes x)) \mod \sigma(H^{n-1}(M; \mathbb{Z}2) \otimes H^{n-1}(M; \mathbb{Z}2)),
\]

by (2.5), Proposition 2.6 and (6.4). Moreover, it is easily verified by Proposition 2.7 (4), Lemmas 2.8 and 2.9 and (4.1) that

\[
p_2 \rho(1 \otimes x^2) = \rho(\sigma(M \otimes x)) \quad \text{for} \quad x \in H^{n-1}(M; \mathbb{Z}2),
\]

\[
p_2 \rho(1 \otimes x^2) = \rho(U(1 \otimes x)) \quad \text{for} \quad x \in H^{n-1}(M; \mathbb{Z}2),
\]

\[
p_2 \rho(\rho_x x_j \otimes \rho_x x_i) = 0 \quad \text{for} \quad 1 \leq j < i \leq \beta.
\]

Therefore, it follows that

\[
p_2 H^{2n-2}(M^*; \mathbb{Z}[\nu]) = A_1.
\]

On the other hand, it is shown in the same way as in proving Lemma 6.5 that

\[
p^*_* \Delta(x_j, x_i) = i^*(x_j \otimes x_i - x_i \otimes x_j) \quad \text{for} \quad 1 \leq j < i \leq \alpha,
\]

\[
p^*_* \beta_{r(j)} \Delta(y_j, \rho_{r(j)}x_i) = i^*(x_j \otimes x_i - x_i \otimes x_j)
\]

for \(1 \leq i \leq \alpha < j \leq \beta \) or \(\alpha < j < i \leq \beta\),

and by the argument similar to that used in proving Theorem 4.3 for even \(n\), it is shown that

\[
p^*_* \beta \Delta(x_j, x_i) = \rho \sigma(\rho_x x_j \otimes \rho_x x_i) \quad \text{for} \quad 1 \leq j \leq \alpha,
\]

\[
p^*_* \beta_{r(j)} \Delta(y_j, \rho_{r(j)}x_i) = \rho \sigma(\rho_x x_j \otimes \rho_x x_i)
\]

for \(1 \leq i \leq \alpha < j \leq \beta \) or \(\alpha < j < i \leq \beta\).

Therefore, \(i^* \Delta(x_j, x_i)\) is of infinite order for \(1 \leq j < i \leq \alpha\) and \(i^* \beta_{r(j)} \Delta(y_j, \rho_{r(j)}x_i)\) is of order \(r(j)\) for \(1 \leq i \leq \alpha < j \leq \beta \) or \(\alpha < j < i \leq \beta\), since the same is true of \(i^*(x_j \otimes x_i - x_i \otimes x_j)\) for \(1 \leq i < j \leq \beta\) by (6.1)-(6.4). From the argument made above, we have the following lemma:

**Lemma 6.6.** There holds the following congruence mod odd torsion:
Enumerating embeddings

\[ H^{2n-2}(M^*; \mathbb{Z}[v]) = \sum_{i < j < k} \mathbb{Z}\langle i^* \Delta(x_i, x_j) \rangle + \left( \sum_{i < j < k} \sum_{a < j \leq \beta} \mathbb{Z}\langle i^* \rho_{r(j)} \Delta(y_j, \rho_{r(j)} x_i) \rangle \right) \]

Thus Theorem 4.3 for odd n is given by Lemmas 6.5 and 6.6.

We continue to prove Theorem 4.4 for odd n. Lemma 6.6 at once leads to an isomorphism

\[ \beta_2 : A_3 \xrightarrow{\cong} \tilde{\beta}_2 H^{2n-3}(M^*; \mathbb{Z}_2), \]

\[ A_3 = \{ \rho(\nu \otimes x^2) | x \in H^{n-2}(M; \mathbb{Z}_2) \} + \sum_{i < j < k} \mathbb{Z}\langle \rho \sigma(\rho_{2} y_j \otimes \rho_{2} x_i) \rangle, \]

by Lemma 3.3 and [19, Lemma 1.5]. On the other hand, by Proposition 2.7, \( H^{2n-3}(M^*; \mathbb{Z}_2) \) can be described as

\[ H^{2n-3}(M^*; \mathbb{Z}_2) = A_3 + A_2, \]

\[ A_2 = \{ \rho(\nu(1 \otimes x) + u \otimes (S^q x)^2) | x \in H^{n-2}(M; \mathbb{Z}_2) \} + \{ \rho \sigma(\rho_{2} x_1 \otimes \rho_{2} y_j) | x \in H^{n-2}(M; \mathbb{Z}_2), y \in H^{n-1}(M; \mathbb{Z}_2) \} \]

\[ + \sum_{a < j \leq \beta} \mathbb{Z}\langle \rho \sigma(\rho_{2} x_1 \otimes \rho_{2} x_1) \rangle + \sum_{a < j \leq \beta} \mathbb{Z}\langle \rho \sigma(\rho_{2} y_j \otimes \rho_{2} x_1) \rangle. \]

By Lemmas 2.8, 2.9 and 3.3 and [19, Proposition 5.5], it is easy to see that \( A_3 \subset \tilde{\beta}_2 H^{2n-3}(M^*; \mathbb{Z}_2) \) and hence

\[ A_3 = \tilde{\beta}_2 H^{2n-3}(M^*; \mathbb{Z}_2) \]

by (6.7). This completes the proof of Theorem 4.4 for odd n.

§ 7. Coker \( \Theta \).

The purpose of this section is to study the condition for computing Coker \( \Theta \), where

\[ \Theta = (S^q + \frac{2n-1}{2}) \tilde{\beta}_2 : H^{2n-3}(M^*; \mathbb{Z}_2) \to H^{2n-1}(M^*; \mathbb{Z}_2). \]
Here $\hat{\beta}_i H^{2n-3}(M^*; Z[v])$ is given by Theorem 4.4, and $H^{2n-3}(M^*; Z_2)$ is given by Thomas [17, § 2] as follows:

**Lemma 7.1** (Thomas). There exists an isomorphism

$$ \phi : H^{n-1}(M; Z_2) \cong H^{n-3}(M^*; Z_2) $$
defined by

$$ \phi(x) = \rho \sigma(x \otimes M) = \rho(x \otimes M + M \otimes x). $$

**Lemma 7.2.** Let $M$ be an $n$-manifold ($n \geq 4$) and assume that $n$ is even or $M$ is orientable. Then $\text{Im } \Theta$ is a $Z_2$-vector space generated by the elements listed below:

1. $\rho \sigma(Sq^i \rho_2 x \otimes M)$ for $x \in H^{n-5}(M; Z)$ if $n \equiv 0 (2)$,
2. $\rho \sigma(Sq^i \rho_2 x \otimes \rho_2 y)$ for $x \in H^{n-3}(M; Z)$, $y \in H^{n-1}(M; Z)$,
3. $\rho \sigma(\rho_2 x_i \otimes Sq^i \rho_2 y_j + \langle r(j)/r(i) \rangle Sq^i \rho_2 y_i \otimes \rho_2 x_i)$ for $\alpha < i < j \leq \beta$,
4. $\rho \sigma(Sq^i \rho_2 y_i \otimes M + Sq^i \rho_2 y_i \otimes M')$ for $\alpha' < k \leq \beta'$ if $n \equiv 0 (2)$ and $w_1 \neq 0$,
5. $\rho \sigma(Sq^i \rho_2 y_i \otimes \rho_2 x_i)$ for $\alpha < i \leq \beta$ if $n \equiv 1 (2)$,
6. $\rho \sigma(Sq^i x \otimes Sq^i x) + U(1 \otimes Sq^i x)$ for $x \in H^{n-5}(M; Z_2)$ if $n \equiv 0 (4)$,
7. $\rho \sigma(Sq^i x \otimes Sq^i x) + U(1 \otimes w_1 x)$ for $x \in H^{n-3}(M; Z_2)$ if $n \equiv 2 (4)$.

**Proof.** On calculating $(Sq^i + \left(2n-1 \right)/2^i)H_4$ with the help of Lemmas 2.8 and 2.9, the element in (1) in the lemma appears for $1 \leq i \leq 6$, while $Sq^i H_4 = 0$ is verified by using Proposition 2.6 (1), Lemma 2.9 and the equation (cf. [11])

$$ Sq^i U = U(1 \otimes w_1) \quad \text{in } H^*(M; Z_2). $$

**Remark.** The elements of (1) for odd $n$ (not necessarily $w_1 = 0$) always belong to $\text{Im } \Theta$ and so do those of (4) and (5) for odd $n$ and $w_1 \neq 0$. For these elements belong to $Sq^i * \hat{\beta}_i H^{2n-3}(A^* M, D M; Z[v])$ by Lemma 3.2 and [19, Proposition 5.5].

**Corollary 7.3.** Assume that the integral cohomology groups $H^i(M; Z)$ for $n - 3 \leq i \leq n$ are given as (4.2). Then, with the assumption of Lemma 7.2, the following information suffices to determine $\text{Coker } \Theta$:

1. The action of $Sq^i$ on $H^i(M; Z_2)$ for $i = n-2, n-3$,
2. The action of $w_1$ on $H^{n-3}(M; Z_2)$ for $n \equiv 2 (4)$.

**Proof.** This is an immediate consequence of Lemmas 7.1 and 7.2, because
the action of $Sq^1$ on $H^*(M; Z_2)$ is determined by the structure of the integral cohomology group of $M$.

**Remark.** The action of $Sq^2$ on $H^i(M; Z_2)$ ($i=n-2$, $n-3$) is given by (2.5) above as follows:

$$Sq^2 x = \begin{cases} (w_2 + w_1^2) x & \text{for } x \in H^{n-2}(M; Z_2), \\ (w_2 + w_1^2 + w_1 Sq^1) x & \text{for } x \in H^{n-3}(M; Z_2). \end{cases}$$

**Corollary 7.4.** Let $M$ be an $n$-manifold ($n \geq 4$). If one of the conditions (i), ..., (iv) below is satisfied, then $\text{Coker } \Theta = 0$:

(i) $Sq^2 \rho_2 H^{n-2}(M; Z) = H^{n-1}(M; Z_2)$,

(ii) $w_1 = 0$ and $Sq^2 \rho_2 H^{n-3}(M; Z) \neq 0$,

(iii) $w_1 \neq 0$, $Sq^2 H^{n-2}(M; Z) = H^{n-1}(M; Z_2)$

and either $w_3^2 = w_3$ or $Sq^2 \rho_2 H^{n-3}(M; Z) = 0$,

(iv) $n \equiv 2 \pmod{4}$, $w_1 \neq 0$ and $w_1 \rho_2 H^{n-2}(M; Z) = H^{n-1}(M; Z_2)$.

**Proof.** By Lemma 7.1, the conditions (i), ..., (iv) imply that $\Theta$ is surjective by the elements in (1), (2), (1) and (4), and (6) of Lemma 7.2, respectively.

**Remark.** By Lemma 7.1 and the remark following Lemma 7.2, this corollary is valid for the case when $n \equiv 1 \pmod{2}$ and $w_1 \neq 0$.

### § 8. Proofs of results in the introduction.

Let $n \geq 6$ and let $M$ be an $n$-manifold such that $n \equiv 0 \pmod{2}$ or $w_1 = 0$. Assume, further, that there exists an embedding of $M$ in Euclidean $(2n-1)$-space $R^{2n-1}$. As is stated in § 1, there is a bijection, as a set,

$$[M \subset R^{2n-1}] \cong H^{n-2}(M^*; Z[v]) \times \text{Coker } \Theta .$$

Therefore, Theorem 1 follows from Theorem 4.3 and Corollary 7.3. If $M$ is a spin manifold, then $w_1 = w_2 = 0$ and so $Sq^2 = 0$ on $H^i(M; Z_2)$ for $i = n-2$ and $n-3$ by the remark following Corollary 7.3. Therefore by Lemmas 7.1 and 7.2, it is shown that

$$\text{Im } \Theta = \begin{cases} \rho_\sigma(Sq^1 x \otimes M) & \text{if } n \equiv 0 \pmod{4}, \\ 0 & \text{otherwise}, \end{cases}$$

and hence that

$$\text{Coker } \Theta \cong \begin{cases} H^{n-1}(M; Z_2)/Sq^1 H^{n-2}(M; Z_2) & \text{if } n \equiv 0 \pmod{4}, \\ H^{n-1}(M; Z_2) & \text{otherwise}. \end{cases}$$
This and Theorem 4.3 lead to Proposition 2. Propositions 3 and 4 are obtained from Theorem 4.3 and Corollary 7.4.

We conclude this paper with proving Proposition 5. The Dold manifold $P(m, n)$ of type $(m, n)$ of dimension $m+2n$ is obtained from $S^m \times CP^n$ by identifying $(x, z)$ with $(-x, \bar{z})$, $S^m$ and $CP^n$ being the $m$-sphere and the complex projective space of complex dimension $n$, respectively. In particular, $P(m, 0)$ and $P(0, n)$ are the real and the complex projective spaces. Dold [2] has stated that

(8.1) $P(m, n)$ is given a cell decomposition with $k$-cell $(C_i, D_j)$ for every pair $(i, j)$ $(i, j \geq 0)$, for which $i+2j = k \leq m+2n$, and the boundary operator satisfies

$$\partial(C_i, D_j) = \begin{cases} 
(1+(-1)^{i+j})(C_{i-1}, D_j) & \text{for } i>0, \\
0 & \text{for } i=0.
\end{cases}$$

Let $C^iD^j$ denote the cochain which assigns 1 to $(C_i, D_j)$ and 0 to all the other $(i+2j)$-cells or its integral cohomology class if it is a cocycle, and let $c^i d^j$ denote the mod 2 cohomology class defined by $C^iD^j$. Then, in [2], it is shown that

$$H^*(P(m, n); \mathbb{Z}_2) = \mathbb{Z}_2\langle c \rangle/(c^m \otimes \mathbb{Z}_2[d]/(d^{n+1})$$

(8.2) $Sq^i d = cd$, $w_1(P(m, n)) = (m+n+1)c$,

where $c = c^d$ and $d = d^d$.

The integral cohomology group of $P(m, n)$ is determined, by using (8.1), as follows:

(8.3) If $m, n > 0$, then

$$H^{m+n-1}(P(m, n); \mathbb{Z}) = \begin{cases} 
\mathbb{Z}\langle \beta_1(c^{m-3}d^n) \rangle & \text{for } m+n = 1 (2), m \geq 2, \\
\mathbb{Z}\langle C^{m-1}D^n \rangle & \text{for } m+n = 1 (2), m = 1, \\
0 & \text{for } m+n = 0 (2);
\end{cases}$$

$$H^{m+n-2}(P(m, n); \mathbb{Z}) = \begin{cases} 
\mathbb{Z}\langle \beta_2(c^{m-1}d^{n-1}) \rangle & \text{for } m+n = 1 (2), \\
\mathbb{Z}\langle C^{m-1}D^{n-1} \rangle + \mathbb{Z}\langle \beta_2(c^{m-3}d^n) \rangle & \text{for } m+n = 0 (2), m \geq 3, \\
\mathbb{Z}\langle C^{m}D^{n-1} \rangle + \mathbb{Z}\langle C^{m-1}D^d \rangle & \text{for } m+n = 0 (2), m = 2, \\
\mathbb{Z}\langle C^{m}D^{n-1} \rangle & \text{for } m+n = 0 (2), m = 1;
\end{cases}$$

$$\rho_2 H^{m+n-3}(P(m, n); \mathbb{Z}) = \begin{cases} 
\mathbb{Z}\langle c^{m-3}d^n \rangle & \text{for } m+n = 1 (2), m \geq 3, \\
0 & \text{for } m+n = 1 (2), m = 1, 2
\end{cases}$$

$$\rho_3 H^{m+n-4}(P(m, n); \mathbb{Z}) = \begin{cases} 
\mathbb{Z}\langle c^{m-1}d^{n-1} \rangle & \text{for } m+n = 0 (2).
\end{cases}$$
From Theorem 4.4 and (8.3), we see easily that, under the condition

\[ m \geq 1, n \geq 1 \text{ and either } m + n \equiv 1 \pmod{2} \text{ or } m \equiv 0 \pmod{2}, \]

there hold the relations

\[ p_2H^{m+n-3}(P(m, n)^*; \mathbb{Z}[v]) = \mathbb{Z}^2 \langle p(u \otimes (cm-1dn-1)) \rangle + \mathbb{Z}^2 \langle p(u \otimes (cm-2dn)^2 + (cm-1dn \otimes cm-2dn)) \rangle \]

\[ + \mathbb{Z}^2 \langle p(u \otimes cm-3dn \otimes cmn) \rangle \]

for \( m = 0, 2, n = 0 \),

\[ \mathbb{Z}^2 \langle p(u \otimes cm-1dn-1 \otimes cmn) \rangle \]

for \( m = 0, n = 1 \),

\[ \mathbb{Z}^2 \langle p(u \otimes cmn) \rangle \]

for \( m = 1, n = 0 \).

By Lemmas 2.8, 2.9 and 7.1, (8.2) and (8.4), it is easy to see that if the condition (*) is satisfied then

\[ \text{Coker } \Theta = \begin{cases} 
\mathbb{Z}^2 & \text{for } m \equiv 3 \pmod{4} \text{ and either } m \equiv 0 \pmod{4} \text{ or } m = 2, \\
0 & \text{otherwise.}
\end{cases} \]

Therefore, Proposition 5 follows from Theorems 1 and 4.3, (8.3) and (8.5).

References


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