Enumerating embeddings of homologically
\((k-1)\)-connected \(n\)-manifolds in
Euclidean \((2n-k)\)-space

Dedicated to Professor Nobuo Shimada on his 60th birthday

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§ 1. Introduction.

Throughout this paper, an \(n\)-manifold and an embedding mean a closed connected differentiable manifold of dimension \(n\) and a differentiable embedding, respectively. Let \([M\subset R^n]\) denote the set of isotopy classes of embeddings of \(M\) in Euclidean \(m\)-space \(R^m\). In [5] (cf. [6]), Haefliger has proved the following theorem:

**THEOREM (Haefliger).** If \(k \leq (n-4)/2\) and if \(M\) is an orientable homologically \(k\)-connected \(n\)-manifold, then \([M\subset R^{2n-k}]\) is equivalent to \(H_{k+1}(M; Z)\) or \(H_{k+1}(M; Z_2)\) according as \(n-k\) is odd or even.

Here a space \(X\) is called homologically \(k\)-connected if it satisfies the condition \(H_i(M; Z)=0\) for \(i \leq k\). A \(k\)-connected path connected space is clearly homologically \(k\)-connected.

The purpose of this paper is to prove the following theorem, which is an extension of the above theorem:

**MAIN THEOREM.** If \(2 < k < \frac{n-4}{2}\) and if \(M\) is a homologically \((k-1)\)-connected \(n\)-manifold whose \((n-k)\)-th normal Stiefel-Whitney class vanishes, then the set \([M\subset R^{2n-k}]\) is given as follows:

- (i) if \(k=2\) and \(M\) is not a spin manifold, then

\[
[M\subset R^{2n-k}] = H^{n-3}(M; Z_2) \quad n=0(4),
\]
\[
= H^{n-3}(M; Z_2) \times Z_2 \quad n=2(4),
\]
\[
= H^{n-3}(M; Z) \times H^{n-3}(M; Z_2) \quad n=1(4), \ w_3 \neq 0,
\]
\[
= H^{n-3}(M; Z) \times H^{n-3}(M; Z_2) \times Z_2 \quad n=1(4), \ w_3 = 0, \ \text{or} \ n=3(4);
\]

...
(ii) if \( k \geq 3 \) or \( M \) is a spin manifold, then

\[
[M \subset R^{2n-k}] = H^{n-k-1}(M; \mathbb{Z}_2)
\]

\[
= H^{n-k-1}(M; \mathbb{Z}) \times H^{n-k}(M; \mathbb{Z}_2)
\]

\[
\times H^{n-k}(M; \mathbb{Z})/\text{Sq}^2 H^{n-k-2}(M; \mathbb{Z})
\]

\[\cdots\]

\[
= H^{n-k-1}(M; \mathbb{Z}) \times H^{n-k}(M; \mathbb{Z}_2)
\]

\[
\times H^{n-k}(M; \mathbb{Z})/(\text{Sq}^1 H^{n-k-1}(M; \mathbb{Z}) + \text{Sq}^2 H^{n-k-2}(M; \mathbb{Z}))
\]

In this theorem, the \((n-k)\)-th normal Stiefel-Whitney class \( \overline{w}_{n-k} \) of an orientable \( n \)-manifold \( M \) is defined by \( \overline{w}_{n-k} \) or \( \overline{w}_{n-k-1} \) \( \in H^{n-k}(M; \mathbb{Z}) \) according as \( n-k \) is even or odd, where \( \overline{w}_i \) is the \( i \)-th mod 2 normal Stiefel-Whitney class of \( M \) and \( \beta_i \) is the Bockstein operator, and moreover \( \overline{w}_{n-k} \) is the unique obstruction to embedding a homologically \((k-1)\)-connected \( n \)-manifold in \( \mathbb{R}^{2n-k} \) by the theorem in [5, § 1.3] (cf. [6, Theorem (2.3)]).

The remainder of this paper is organized as follows: In § 2, we shall state a method of computing \([M \subset R^{2n-k}]\) of a homologically \((k-1)\)-connected \( n \)-manifold \( M \) (Theorem 2.5). In § 3, we state the cohomology group of the reduced symmetric product \( M^* := (M \times M - \Delta M)/\mathbb{Z}_2 \) of \( M \) (Theorem 3.3), postponing the proof till § 5, the last section. § 4 is devoted to proving the main theorem.

§ 2. The method of computing \([M \subset R^{2n-k}]\).

We begin this section by explaining notations. Let \( X^2 \) be the product \( X \times X \) of a space \( X \) and let \( \Delta X \) be the diagonal in \( X^2 \). The cyclic group of order 2, \( \mathbb{Z}_2 \), acts on \( X^2 \) via the map \( t: X^2 \to X^2 \) defined by \( t(x, y) = (y, x) \). Then \( \Delta X \) is the fixed point set of this action. The quotient space

\[
X^* = (X^2 - \Delta X)/\mathbb{Z}_2
\]

is called the reduced symmetric product of \( X \). Here the projection \( \pi: X^2 - \Delta X \to X^* \) is a double covering, whose classifying map we denote by

\[
\xi: X^* \to \mathbb{P}^\infty.
\]

For a fibration \( \pi: E \to B \) and a map \( f: Y \to B \), let

\[
Y \times_B E \to Y \quad \text{and} \quad [Y, E; f]
\]

be the pull-back of \( \pi \) along \( f \) and the homotopy set of liftings of \( f \) to \( E \).

Notice that the sphere bundle \( \pi: S^m \times_{\mathbb{Z}_2} S^m \to \mathbb{P}^\infty \) is homotopically equivalent to the natural inclusion \( \mathbb{P}^m \to \mathbb{P}^\infty \) of the real projective \( m \)-space \( \mathbb{P}^m \). Hence we regard them as identical. Using the above notations, we deduce the following
Theorem from Haefliger's theorem [4, Théorème 1'] (cf. Yasui [18, §1]):

THEOREM 2.1 (Haefliger). For an n-manifold $M$, there is a bijection

$$[M \subset R^{n-k}] \cong [M^*, P^{n-k-1}; \xi]$$

if $k \leq (n-4)/2$.

For any abelian group $G$ and a homomorphism $\phi : \pi_1(P^n) = \mathbb{Z}_2 \to \text{Aut}(G)$, let $G_\phi$ be the sheaf over $P^n$, locally isomorphic to $G$, defined by $\phi$, i.e., the local system associated with $\phi$. This homomorphism $\phi$ gives an action of $Z_2$ on $(K(G, m), *)$. Hence we have a fibration

$$q : L_\phi(G, m) = S^\infty \times_{\mathbb{Z}_2} K(G, m) \to P^n$$

with fiber $K(G, m)$ and a canonical cross section $s$. It has been established (see, for example, G. W. Whitehead [17, Chap. VI, (6.13)]) that there exists a unique fundamental class $c \in H^m(L_\phi(G, m), P^n; q^*G_\phi)$, whose restriction to $K(G, m)$ is the ordinary one (it is equal to $\delta(sq, 1)$ up to sign in [17]), and that given $\hat{s} : X \to P^n$, the correspondence $f \mapsto f^*c$ leads to a bijection

$$[X, L_\phi(G, m); \hat{s}] \cong H^m(X; \hat{s}^*G_\phi).$$

Further, if $\hat{s}$ has a lifting $\tilde{s}$ to $P^{n-k-1}$, then there is a bijection

$$[X, P^{n-k-1} \times \mathbb{R} = L_\phi(G, m); \tilde{s}] \cong [X, L_\phi(G, m); \hat{s}]$$

by [8, Theorem 3.1] and hence we have a bijection

$$(2.2) [X, P^{n-k-1} \times \mathbb{R} = L_\phi(G, m); \tilde{s}] \cong H^m(X; \tilde{s}^*G_\phi).$$

Let

$$G_j = \pi_{2n-k-1+j}(S^{2n-k-1}).$$

Since the sphere bundle $\pi : P^{2n-k-1} \to P^n$ is the one associated with $(2n-k)\gamma$, $\gamma$ being the universal real line bundle over $P^n$, the action of $\pi_1(P^n) = \mathbb{Z}_2$ on $G_j$ is given by the homomorphism

$$\phi : \mathbb{Z}_2(=\{1, a\}) \to \text{Aut}(G_j)$$

defined by

$$\phi(a)(x) = (-1)^{2n-k}x \quad \text{for } x \in G_j$$

and moreover the sheaf $(G_j)_\phi$ is given by

$$(G_j)_\phi = \begin{cases} G_j & \text{if } k \text{ is even}, \\ G_j[u] & \text{if } k \text{ is odd}, \end{cases}$$

where $G_j[u]$ is the sheaf over $P^n$, locally isomorphic to $G_j$, twisted by $u(\neq 0) \in H^1(P^n; \mathbb{Z}_2) = \mathbb{Z}_2$. For $\xi : M^* \to P^n$, let

$$(2.3) G_j = \xi^*(G_j)_\phi = \begin{cases} G_j & \text{if } k \text{ is even}, \\ G_j[v] (v = \xi^*u) & \text{if } k \text{ is odd}, \end{cases}$$
and let
\[ \bar{\rho}_2 : H^i(M_*; \mathbb{Z}) \to H^i(M_*; \mathbb{Z}_2), \]
\[ \bar{\beta}_2 : H^{i-1}(M_*; \mathbb{Z}_2) \to H^i(M_*; \mathbb{Z}) \]
be the ordinary reduction mod 2 and Bockstein operator or the ones twisted by
\( v \) according as \( k \) is even or odd. Then

\[ (2.4) \quad \bar{\rho}_2 \bar{\beta}_2 = \begin{cases} 
Sq^1 & \text{if } k \text{ is even,} \\
Sq^{1+v} & \text{if } k \text{ is odd,}
\end{cases} \]

by [2], [14]. With the above notations, we shall prove

**Theorem 2.5.** Let \( 2 \leq k \leq (n-4)/2 \) and let \( M \) be a homologically \((k-1)\)-
connected \( n \)-manifold. If \( M \) can be embedded in \( \mathbb{R}^{2n-k} \), then there exists a bijection

\[ [M \subset \mathbb{R}^{2n-k}] = H^{2n-k-1}(M_*; \mathbb{Z}) \times \text{Coker} \Theta \]

where

\[ \Theta = \left( Sq^2 + \left( \frac{2n-k}{2} \right) \nu \right) \bar{\beta}_2 : H^{2n-k-2}(M_*; \mathbb{Z}) \to H^{2n-k}(M_*; \mathbb{Z}_2). \]

In order to prove this, it is sufficient, by Theorem 2.1, to show that

\[ [M*, P^{2n-k-1}; \xi] = H^{2n-k-1}(M_*; \mathbb{Z}) \times \text{Coker} \Theta. \]

Let \( P = P^{2n-k-1} \) and let \( \pi' : P' \to P \) be the pull-back of \( \pi : P \to P^\infty \) along \( \pi \). If \( M \)
can be embedded in \( \mathbb{R}^{2n-k} \), then \( \xi \) has a lifting \( \xi' : M_* \to P \) by the first half of

[4, Théorème 1'] and so

\[ (2.6) \quad [M*, P'; \xi'] \cong [M*, P; \xi] \]

by [8, Theorem 3.1]. Since \( \pi : P \to P^\infty \) is the sphere bundle associated with
\((2n-k)\gamma\), the Postnikov tower of \( \pi' : P' \to P \) is given as follows:

\[ \cdots \]
\[ E_{j+1} \]
\[ \downarrow p_{j+1} \]
\[ P' \xrightarrow{h_j} E_j \]
\[ \downarrow p_j \]
\[ E_j \xrightarrow{k_j} P \times P^\infty L_{\phi}(G_j, 2n-k+j) \]
\[ \cdots \]
\[ \downarrow k_2 \]
\[ E_2 \xrightarrow{k_2} P \times K(Z_2, 2n-k+2) \]
\[ \downarrow p_1 \]
\[ P \times P^\infty L_\phi(Z, 2n-k-1) \cong E_1 \xrightarrow{k_1} P \times K(Z_2, 2n-k+1) \]
\[ \downarrow \]
\[ P^{2n-k-1} \cong P \]
where \( h_j \) is a \((2n-k-1+j)\)-equivalence, \( p_j: E_{j+1} \to E_j \) is a \( P \)-principal fibration with classifying map \( k_j \) in the category \( TP \) of \( P \)-sectioned spaces and maps. By [10, Part IV, Theorem 1], for \( \xi': M^* \to P \), \( p_j: E_{j+1} \to E_j \) induces an exact sequence

\[
\begin{align*}
&\cdots \to [M^*, P \times_{P^e} L_{p_j}(G_j, 2n-k-1+j); \xi'] \to [M^*, E_{j+1}; \xi'] \\
&\to [M^*, P \times_{P^e} L_{p_j}(G_j, 2n-k+j); \xi'] \\
&\to \cdots
\end{align*}
\]

\( (j \geq 1) \),

where \( \Omega p k_j \) is the map of loops associated with \( k_j \) in \( TP \). With the help of (2.2), (2.3), this is converted into the exact sequence

\[
\begin{align*}
&\cdots \to H^{2n-k-1+j}(M^*; G_j) \to [M^*, E_{j+1}; \xi'] \\
&\to [M^*, E_j; \xi'] \\
&\to \cdots
\end{align*}
\]

\( (j \geq 1) \),

where

\[
[M^*, E_1; \xi'] = H^{2n-k-1}(M^*; Z).
\]

Now it has been shown by Haefliger and Hirsch [7, p. 237] that if \( M \) is a homologically \((k-1)\)-connected \( n \)-manifold \((k \geq 2)\), then

\[
H^{2n-k-1+j}(M^*; G_j) = H^{2n-k-1+j}(M^*; G_{j-1}) = 0 \quad \text{for} \quad j \geq 2.
\]

We know, on the other hand, that \( \Omega p k_1 \) induces an operation \( (Sq^2 + (2n-k)\nu^2)\theta \), i.e.

\[
\begin{align*}
&\cdots \to [M^*, \Omega P E_1; \xi'] \to [M^*, \Omega P (P \times K(Z_2, 2n-k+1)); \xi'] \\
&\to [M^*, \Omega P (P \times K(Z_2, 2n-k)); \xi'] \\
&\to \cdots
\end{align*}
\]

\( \theta = (Sq^2 + (2n-k)\nu^2)\theta \) \( H^{2n-k-2}(M^*; Z) \to H^{2n-k}(M^*; Z) \),

because \( (k_1) \) corresponds to \( (Sq^2 + w_3((2n-k)\tau))\theta \). From the above argument, it is clear that there exists a short exact sequence

\[
0 \to H^{2n-k}(M^*; Z_2)/\text{Im} \theta \to [M^*, P'; \xi'] \to H^{2n-k-1}(M^*; Z) \to 0.
\]

This, together with Theorem 2.1 and (2.6), completes the proof of Theorem 2.5.

\section*{§ 3. The cohomology of \( M^* \).}

The mod 2 cohomology of \( M^* \) has been studied by Bausum [1], Haefliger [3], Thomas [16], Yasui [19], Yo [21] and others. The notations used here are the same as those explained in [19] (most of them are the same as in [16, § 2]). Let \( M \in H^n(M; Z_2) \) be the generator, i.e.
LEMMA 3.1. Assume that $M$ is a homologically $(k-1)$-connected $n$-manifold $(k \geq 2)$. Then

(i) $H^{i}(M^{*}; Z_{2}) = 0$ if $i > 2n - k$,
(ii) $H^{2n-k}(M^{*}; Z_{2}) = \{ \rho \sigma(M \otimes x) \mid x \in H^{n-k}(M; Z_{2}) \}$ ($\equiv H^{n-k}(M; Z_{2})$),
(iii) $H^{2n-k-1}(M^{*}; Z_{2})$

$$= \{ \rho(u^{k-1} \otimes x^{2}) \mid x \in H^{n-k}(M; Z_{2}) \}$ ($\equiv H^{n-k}(M; Z_{2})$)

$$+ \{ \rho(u^{k+1} \otimes x^{2}) \mid x \in H^{n-k-1}(M; Z_{2}) \}$ ($\equiv H^{n-k-1}(M; Z_{2})$),

(iv) $H^{2n-k-2}(M^{*}; Z_{2})$

$$= \{ \rho(u^{k} \otimes x^{2}) \mid x \in H^{n-k-2}(M; Z_{2}) \}$ ($\equiv H^{n-k-2}(M; Z_{2})$)

$$+ \{ \rho(u^{k+2} \otimes x^{2}) \mid x \in H^{n-k-2}(M; Z_{2}) \}$ ($\equiv H^{n-k-2}(M; Z_{2})$)

$$+ \{ \rho \sigma(x \otimes y) \mid x, y \in H^{n-2}(M; Z_{2}), x \neq y \}$$

where the term in the square brackets $[ \ ]$ is present only when $k=2$.

PROOF. (i), (ii) are given by Thomas [16, Proposition 2.9]. By [19, Proposition 2.6], there are two relations:

$$\rho(u^{k+1} \otimes x^{2}) = \rho(U(1 \otimes x) + u^{k-1} \otimes (S_{q}x)^{2}) \quad \text{if } x \in H^{n-k-1}(M; Z_{2}),$$

$$\rho(u^{k+2} \otimes x^{2}) = \rho(U(1 \otimes x) + u^{k} \otimes (S_{q}x)^{2} + u^{k-2} \otimes ((S_{q}^{2} + w_{2})x)^{2}) \quad \text{if } x \in H^{n-k-2}(M; Z_{2}).$$

Moreover $U(1 \otimes x)$ is expressed in the form

$$U(1 \otimes x) = \sigma(M \otimes x) + \sum x' \otimes x'', \quad \text{dim } x', \text{dim } x'' < n.$$

Applying [16, Proposition 2.9], we can prove (iii), (iv) immediately.

The actions of $v \in H^{1}(M^{*}; Z_{2})$ and the square operation $S_{q}^{i} (i=1, 2)$ on $H^{*}(M^{*}; Z_{2})$ are given by Thomas [16, Corollary 2.10] and Bausum [1, Lemmas 11 and 24] as follows:

LEMMA 3.2. There are the following relations in $H^{*}(M^{*}; Z_{2})$:

(i) $v \rho \sigma(x \otimes y) = 0, \quad v \rho(u^{i} \otimes x^{2}) = \rho(u^{i+1} \otimes x^{2})$;

(ii) if $x \in H^{i}(M; Z_{2})$, then

$$S_{q}^{i} \rho(u^{i} \otimes x^{2}) = \begin{cases} (i+r)\rho(u^{i+1} \otimes x^{2}) & i > 0, \\ r \rho(u \otimes x^{2}) + r \rho \sigma(S_{q}^{i}x \otimes x) & i = 0 \end{cases}.$$
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\[ S_q \rho(u^i \otimes x^2) = \begin{cases} \left( \frac{r+i}{2} \right) \rho(u^{i+r} \otimes x^2) + \rho(u^i \otimes (S_q x)^2) & i > 0, \\ \left( \frac{r}{2} \right) \rho(u^r \otimes x^2) + \rho(1 \otimes (S_q x)^2) + \rho(a (S_q x \otimes x)) & i = 0. \end{cases} \]

For a homologically \((k-1)\)-connected \(n\)-manifold \(M\) \((k \geq 2)\), the cohomology groups \(H^i(M*; \mathbb{Z})\) for \(2n-k-2 \leq i \leq 2n-k\) are given in the following theorem, postponing the proof till § 5:

**Theorem 3.3.** Assume that \(M\) is a homologically \((k-1)\)-connected \(n\)-manifold \((k \geq 2)\). Then

(i) \(H^{2n-k}(M*; \mathbb{Z}) \cong \{ H^{n-k}(M; \mathbb{Z}) \text{ if } n-k \text{ is even, } H^{n-k}(M; \mathbb{Z}) \text{ if } n-k \text{ is odd; } \)

(ii) \(H^{2n-k-1}(M*; \mathbb{Z}) \cong \{ H^{n-k-1}(M; \mathbb{Z}) + H^{n-k}(M; \mathbb{Z}) \text{ if } n-k \text{ is even, }\)

\( \{ H^{n-k-1}(M; \mathbb{Z}) \text{ if } n-k \text{ is odd; } \}

(iii) \( \varphi H^{2n-k-2}(M*; \mathbb{Z}) = \{ \rho(u^k \otimes x^2) | x \in H^{n-k}(M; \mathbb{Z}) \} + \{ \rho(u^{k+1} \otimes x^2) | x \in H^{n-k-1}(M; \mathbb{Z}) \}

+ \{ \rho(a (x \otimes y)) | x, y \in H^{n-k}(M; \mathbb{Z}), x \neq y \} \text{ if } n-k \text{ is even, }

= \{ \rho(u^k \otimes x^2) | x \in H^{n-k-1}(M; \mathbb{Z}) \} + \{ \rho(a (x \otimes y)) | x, y \in H^{n-k}(M; \mathbb{Z}), x \neq y \} \text{ if } n-k \text{ is odd,}

where the terms in the square brackets \([-\)] are present only when \(k=2\).

§ 4. Proof of the main theorem.

In this section, let \(M\) be a homologically \((k-1)\)-connected \(n\)-manifold \((k \geq 2)\). If its \((n-k)\)-th normal Stiefel-Whitney class vanishes, then \(M\) can be embedded in Euclidean \((2n-k)\)-space by Haefliger [5, § 1] and there is a bijection

\[ [M \subset R^{2n-k}] = H^{2n-k-1}(M*; \mathbb{Z}) \times \text{Coker } \Theta \]

where

\[ \Theta = (S^2 + \left( \frac{2n-k}{2} \right) u^2) \varphi : H^{2n-k-1}(M*; \mathbb{Z}) \to H^{2n-k}(M*; \mathbb{Z}) \]

by Theorem 2.5. Since \(H^{2n-k-1}(M*; \mathbb{Z})\) is given in Theorem 3.3(ii), we shall concentrate on calculating \(\text{Coker } \Theta\). Notice that there are an isomorphism

\[ \chi : H^{n-k}(M; \mathbb{Z}) \to H^{n-k}(M*; \mathbb{Z}) \]

\((\chi(x) = \rho(a (M \otimes x)))\),

and equalities

\[ \rho(u^k \otimes x^2) = \rho(U(1 \otimes x)) = \rho(a (M \otimes x)) \text{ for } x \in H^{n-k}(M; \mathbb{Z}), \]

\[ \rho(u^{k+1} \otimes x^2) = \rho(U(1 \otimes x)) = \rho(a (M \otimes x)) \text{ for } x \in H^{n-k-1}(M; \mathbb{Z}), \]

\[ \rho(a (x \otimes y)) = \rho(a (x \otimes y)) \text{ for } x, y \in H^{n-k}(M; \mathbb{Z}), x \neq y. \]
which follow from [19, Proposition 2.6] and (*) in § 3.

Case 1: \( n-k \) is even. See Theorem 3.3 (iii) for the group \( \tilde{p}_* H^{n-k-2}(M^*; \mathbb{Z}) \).

If \( x \in H^{n-k-2}(M; \mathbb{Z}_2) \), then

\[
\begin{align*}
\left( Sq^2 + \binom{2n-k}{2} v^2 \right) \rho(u^{k+2} \otimes x^2) \\
= \left( \binom{n}{2} + \binom{2n-k}{2} \right) \rho(u^{k+2} \otimes x^2) + \rho(u^{k+2} \otimes (Sq^1 x)^2) \quad \text{by Lemma 3.2}, \\
= \left( \binom{n}{2} + \binom{2n-k}{2} \right) \rho(u^k \otimes ((Sq^2 + \omega_2) x)^2),
\end{align*}
\]

because there are two relations

\[
\rho(u^{k+2} \otimes x^2 + u^{k+2} \otimes (Sq^1 x)^2 + u^k \otimes ((Sq^2 + \omega_2) x)^2) = 0,
\]

\[
\rho(u^{k+2} \otimes (Sq^1 x)^2) = 0,
\]

which are easily proved by using [19, (2.5) and Proposition 2.6]. Therefore, by (4.2), we have

\[
(4.3) \quad \left( Sq^2 + \binom{2n-k}{2} v^2 \right) \rho(u^{k+2} \otimes x^2) \\
=\lambda \rho \sigma(M \otimes (Sq^2 + \omega_2) x) \quad \text{for} \ x \in H^{n-k-2}(M; \mathbb{Z}_2),
\]

where

\[
\lambda = \begin{cases} 
0 & \text{for} \ n-k=0 \ (4), \\
1 & \text{for} \ n-k=2 \ (4). 
\end{cases}
\]

Similarly, we have a relation

\[
(4.4) \quad \left( Sq^2 + \binom{2n-k}{2} v^2 \right) \rho(u^{k-2} \otimes x^2) \\
=(1-\lambda) \rho \sigma(M \otimes x) + \left[ \rho \sigma(\omega_2 x \otimes x) \right] \quad \text{for} \ x \in H^{n-k}(M; \mathbb{Z}_2).
\]

Moreover the relation

\[
(4.5) \quad \left( Sq^2 + \binom{2n-k}{2} v^2 \right) \rho \sigma(x \otimes y) = \rho \sigma(\omega_2 x \otimes y + \omega_2 y \otimes x) \\
\quad \text{for} \ x, y \in H^{n-k}(M; \mathbb{Z}_2) \text{ with } x \neq y
\]

follows from Lemma 3.2. Therefore, if \( k \geq 3 \) or \( \omega_2 = 0 \), then

\[
\text{Im} \Theta = \{ (1-\lambda) \rho \sigma(M \otimes x) | x \in H^{n-k}(M; \mathbb{Z}_2) \} + \{ \lambda \rho \sigma(M \otimes Sq^2 x) | x \in H^{n-k-2}(M; \mathbb{Z}_2) \}
\]

and so

\[
(4.6) \quad \text{Coker} \Theta = \begin{cases} 
0 & \text{for} \ n-k=0 \ (4), \\
H^{n-k}(M; \mathbb{Z}_2)/Sq^2 H^{n-k-2}(M; \mathbb{Z}_2) & \text{for} \ n-k=2 \ (4),
\end{cases}
\]
by (4.1), (4.2). Next, consider the case $k=2$ and $w_2 \neq 0$. In general, for a simply connected $n$-manifold $M$ with non-trivial second Stiefel-Whitney class $w_2$, the group $H^{n-2}(M; Z_2)$ can be expressed, by using Poincaré duality, in the form

$$H^{n-2}(M; Z_2) = \sum_{i \leq a} Z_2 \langle z_i \rangle, \quad w_2 z_i = \begin{cases} M & \text{if } i=1, \\ 0 & \text{if } 2 \leq i \leq a \end{cases}$$

Then a simple calculation yields that

$$\text{Im} \Theta = \begin{cases} \sum_{i \leq a} Z_2 \langle \rho \sigma(M \otimes z_i) \rangle & \text{if } n-2=0 \quad (4), \\ H^{n-2}(M^*; Z_2) & \text{if } n-2 \neq 0 \quad (4), \end{cases}$$

and hence

$$\text{Coker} \Theta = \begin{cases} Z_2 & \text{if } k=2, w_2 \neq 0 \text{ and } n \equiv 2 \quad (4), \\ 0 & \text{if } k=2, w_2 \neq 0 \text{ and } n \equiv 0 \quad (4), \end{cases}$$

by (4.1). Thus we deduce the main theorem in case $n-k$ is even, from (4.6), (4.8) and Theorems 2.5, 3.3 (ii).

Case II: $n-k$ is odd. See also Theorem 3.3 (iii) for the group $\tilde{\rho}_2 H^{n-k-2}(M^*; Z)$. In the same way as in the case when $n-k$ is even, we have the following relations:

$$\left( S^q + \binom{2n-k}{2} \mu^2 \right) \rho(u^k \otimes x^2) = \mu \rho \sigma(M \otimes S^q x), \quad \mu = \begin{cases} 0 & \text{for } n-k \equiv 1 \quad (4), \\ 1 & \text{for } n-k \equiv 3 \quad (4), \end{cases}$$

if $x \in H^{n-k-1}(M; Z_2)$;

$$\left( S^q + \binom{2n-k}{2} \mu^2 \right) \rho \sigma(M \otimes \rho_2 x) = \rho \sigma(M \otimes S^q \rho_2 x) \quad \text{if } x \in H^{n-k-2}(M; Z);$$

$$\left( S^q + \binom{2n-k}{2} \mu^2 \right) \rho \sigma(x \otimes y) = \rho \sigma(w_2 x \otimes y + w_2 y \otimes x) \quad \text{if } x, y \in H^{n-k}(M; Z_2).$$

If $w_2 = 0$, then (4.1) and the above relations (4.9) lead at once to the relation

$$\text{Im} \Theta = \begin{cases} S^q \rho_2 H^{n-k-2}(M; Z) & \text{for } n-k \equiv 1 \quad (4), \\ S^q \rho_3 H^{n-k-3}(M; Z) + S^q H^{n-k-1}(M; Z_2) & \text{for } n-k \equiv 3 \quad (4). \end{cases}$$

If $w_2 \neq 0$, it is easily verified, in the same way as in the case when $n-k$ is even, that the subgroup of $\text{Im} \Theta$ determined by the last relation of (4.9) is equal to $\sum_{i \leq a} Z_2 \langle \rho \sigma(M \otimes z_i) \rangle$. On the other hand, the following relations hold:

$$w_2 S^q \rho_2 x = S^q S^q \rho_2 x = S^q S^q \rho_2 x = 0 \quad \text{for } x \in H^{n-k}(M; Z),$$

$$w_2 S^q x = S^q (w_2 x) + (S^q w_2) x = w_2 x \quad \text{for } x \in H^{n-k-2}(M; Z).$$

Therefore, it is shown immediately that $\rho \sigma(M \otimes z_i) \in \text{Im} \Theta$ if and only if $n-2 \equiv 3 \quad (4)$ and $w_3 \neq 0$, and hence
Thus (4.10), (4.11), together with Theorems 2.5, 3.3 (ii), deduce the main theorem in case \( n - k \) is odd.

§ 5. Proof of Theorem 3.3.

Throughout this section, we assume that \( M \) is a homologically \((k-1)\)-connected \( n \)-manifold \((k \geq 2)\) and we compute \( H^{2n-k-i}(M*; \mathbb{Z}) \) for \( 0 \leq i \leq 2 \), where \( \mathbb{Z} = \mathbb{Z} \) or \( \mathbb{Z}[v] \) according as \( k \) is even or odd.

Case I: \( n-k \) is even. First we consider the odd torsion subgroup of \( H^{2n-k-i}(M*; \mathbb{Z}) \) for \( i=0, 1 \). Considering the cohomology spectral sequence (cf. [11, Theorem 1.1]) for a fibration \( M^2 - \Delta M \rightarrow S^m \times \mathbb{Z}_2(M^2 - \Delta M) \rightarrow P^\infty \), which is homotopically equivalent to \( M^2 - \Delta M \rightarrow M^2 \rightarrow P^\infty \), we see that the odd torsion subgroup of \( H^{2n-k-i}(M*; \mathbb{Z}) \) is isomorphic, by \( p^* \), to that of

\[ \{ x \in H^{2n-k-i}(M^2 - \Delta M; \mathbb{Z}) \mid t^*x = (-1)^ix \} = H^{2n-k-i}(M^2 - \Delta M; \mathbb{Z})^{(-1)it'}. \]

Since \( M \) is orientable, there is a short exact sequence

\[ 0 \rightarrow H^i(M; \mathbb{Z}) \rightarrow H^{n+i}(M^2; \mathbb{Z}) \rightarrow H^{n+i}(M^2 - \Delta M; \mathbb{Z}) \rightarrow 0, \]

where

\[ \phi_i(x) = U(1 \otimes x) \quad \text{for} \quad x \in H^i(M; \mathbb{Z}), \]

\( U \in H^n(M^2; \mathbb{Z}) \) is called the Thom class or the diagonal cohomology class of \( M \), e.g. by [12], and \( i \) is the natural inclusion. Therefore, \( i^* \) induces an isomorphism

\[ (H^{2n-k-i}(M^2; \mathbb{Z})/\phi_iH^{n-k-i}(M^2; \mathbb{Z}))^{(-1)it'} \cong H^{2n-k-i}(M^2 - \Delta M; \mathbb{Z})^{(-1)it'}. \]

Here \( \phi_iH^{n-k-i}(M^2; \mathbb{Z}) \subset H^{2n-k-i}(M^2; \mathbb{Z})^{(-1)it'} \) by [15, p. 305]. On the other hand, it is easily verified that \( H^{2n-k-i}(M^2; \mathbb{Z})^{(-1)it'} \) is isomorphic to \( H^{n-k-i}(M; \mathbb{Z}) \) for \( i=0, 1 \). Therefore, \( H^{2n-k-i}(M^2 - \Delta M; \mathbb{Z})^{(-1)it'} \) has no odd torsion subgroup and hence

\[ H^{2n-k-i}(M*; \mathbb{Z}) \] has no odd torsion for \( i=0, 1 \).

In order to study \( H^{2n-k-i}(M*; \mathbb{Z}) \), consider the Bockstein exact sequence associated with \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \),

\[ \ldots \rightarrow H^{i-1}(M*; \mathbb{Z}) \rightarrow H^i(M*; \mathbb{Z}) \rightarrow 2H^i(M*; \mathbb{Z}) \rightarrow H^i(M*; \mathbb{Z}) \rightarrow \ldots \]

By using the relations in (2.4) and Lemma 3.2, we have the following relations:
Enumerating embeddings

\[ p_j \tilde{p}_j \rho(x \otimes y) = 0 \quad \text{if} \quad k = 2 \quad \text{and} \quad x, y \in H^{n-2}(M; \mathbb{Z}_2), \]

\[ p_j \tilde{p}_j \rho(u^i \otimes x^j) = \rho(u^{i+1} \otimes x^j) \]

for \((i, \dim x) = (k-1, n-k), (k, n-k-1), (k-3, n-k), (k+1, n-k-2)\).

These relations, (4.2), (5.1) and the exact sequence (5.2), together with Lemma 3.1, lead to Theorem 3.3 in case \(n - k\) is even.

Case II: \(n - k\) is odd. The group \(\mathbb{Z}_2\) acts on \(SM\), the tangent sphere bundle over \(M\), via the antipodal map on each fibre \(S^{n-1}\). Let

\[ PM = SM/\mathbb{Z}_2, \quad (A^sM, \Delta M) = (M^{s}/\mathbb{Z}_2, \Delta M/\mathbb{Z}_2), \]

\[ i: M^* = A^sM - \Delta M \subset (A^sM, \Delta M), \]

and let

\[ j: PM \rightarrow M^* \]

be the embedding such that \(j^*\nu\) is the first Stiefel-Whitney class of the double covering \(SM \rightarrow PM\). We write \(j^*\nu\) as \(\nu \in H^i(PM; \mathbb{Z}_2)\) if no confusion can arise. Then there exists a long exact sequence, cf. [19, Lemma 1.3],

\[ \delta \]

\[ \cdots \rightarrow H^{i-1}(PM; \mathbb{Z}) \rightarrow H^i(A^sM, \Delta M; \mathbb{Z}) \rightarrow H^i(M^*; \mathbb{Z}) \rightarrow H^i(PM; \mathbb{Z}) \rightarrow \cdots, \]

The cohomology of \(PM\) has been given by Rigdon [13, § 9] as follows:

**LEMMA 5.4 (Rigdon).** Assume that \(M\) is a homologically \((k-1)\)-connected \(n\)-manifold \((k \geq 2)\) and that \(n - k\) is odd. Then

(i) \(H^{n-k}(PM; \mathbb{Z}) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \mathbb{Z}_2v & \text{if } k \text{ is odd;} \end{cases} \)

(ii) \(H^{n-k-1}(PM; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2v^2 & \text{if } k \text{ is even,} \\ \mathbb{Z}_2v^2 & \text{if } k \text{ is odd;} \end{cases} \)

(iii) \(H^{n-k-2}(PM; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2v^2 & \text{if } k \text{ is even,} \\ \mathbb{Z}_2v^2 & \text{if } k \text{ is odd.} \end{cases} \)

In the above lemma, and also from now on, \(\tilde{\beta}_s\) denotes the Bockstein operator twisted by \(\nu\).

The cohomology of \((A^sM, \Delta M)\) has been investigated by Larmore [9].

**LEMMA 5.5 (Larmore).** Assume that \(M\) is a homologically \((k-1)\)-connected \(n\)-manifold \((k \geq 2)\) and that \(n - k\) is odd. Then
(i) \( H^{2n-k}(A^k M, \Delta M; Z) \cong \begin{cases} H^{n-k}(M; Z) + Zq\langle \beta_2(u^{n-k+1} AM) \rangle & \text{if } k \text{ is even,} \\ H^{n-k}(M; Z) & \text{if } k \text{ is odd;} \end{cases} \)

(ii) \( H^{2n-k-1}(A^k M, \Delta M; Z) \cong \begin{cases} H^{n-k-1}(M; Z) & \text{if } k \text{ is even.} \\ H^{n-k-1}(M; Z) + Zq\langle \beta_2(u^{n-k+2} AM) \rangle & \text{if } k \text{ is odd;} \end{cases} \)

(iii) \( i^* \tilde{p}_2 H^{2n-k-2}(A^k M, \Delta M; Z) = \{ \rho_2(\rho_2 x \otimes M) \mid x \in H^{n-k-2}(M; Z) \} \)

\[ + \left[ \{ \rho_2(x \otimes y) \mid x, y \in H^{n-k}(M; Z_2), x \neq y \} \right], \]

where the term in the square brackets is present only when \( k = 2 \).

**Proof.** The cohomology groups \( H^{2n-k-1+i}(A^k M, \Delta M; Z) \) for \( i = 0, 1, 2 \) are given directly by [9, Theorem 20]. Their \( i^* \tilde{p}_2 \)-images are easily obtained by using the relations

\[ (5.6) \quad \delta(v^i x) = v^{i+1} Ax, \quad i^*(Ax Ay) = \rho_2(x \otimes y) + \rho_2(x y \otimes 1) \]

in [18, Lemma 1.5], [19, Lemma 3.3] and the two congruences mod Im\( \delta \)

\[ \tilde{p}_2 \tilde{p}_r(A x) \equiv A(\rho_2 \rho_r x) \quad \text{if } x \in H^*(M; Z_2), \]

\[ \tilde{p}_2 \tilde{p}_r d(x, \rho_r y) \equiv \tilde{p}_2 \tilde{p}_d(\beta_r x, y) \quad \text{if } x \in H^*(M; Z_r), y \in H^*(M; Z), \]

which are easily proved.

**Remark.** The author has proved this lemma in the same way as he proved the propositions in [18, § 5], i.e., by using the results on pp. 908–915 in [9]. He thinks that the expression "r is a power of 2 or" in I (iv), II (v) of [9, Theorem 20] should be omitted.

Using the first relation of (5.6) and the relation

\[ j^* \rho(u^r \otimes x^r) = \sum_{q \geq 0} v^{r+q}(S^q x) \quad \text{if } x \in H^q(M; Z_2), \]

in [16, § 2], we have the following relations:

\[ \tilde{p}_2 \tilde{p}_d(u^{n-k-1} M) = \delta(u^{n-k} M) = u^{n-k+1} AM \neq 0 \quad \text{if } k \text{ is odd,} \]

\[ \delta \tilde{p}_d(u^{n-k-2} M) = \beta_d(u^{n-k-1} AM) \quad \text{if } k \text{ is even,} \]

\[ \delta \tilde{p}_d(u^{n-k-3} M) = \beta_d(u^{n-k-2} AM) \quad \text{if } k \text{ is odd,} \]

\[ j^* \tilde{p}_2 \rho(u^{k-2} \otimes x^r) = \begin{cases} \beta_d(u^{n-k} x + u^{n-k+1} S^q x) & \text{if } k \text{ is even and } \dim x = n-k, \\ \beta_2(u^{n-k} x) & \text{if } k \text{ is odd and } \dim x = n-k, \end{cases} \]

\[ j^* \tilde{p}_2 \rho(u^{k-1} \otimes x^r) = \begin{cases} \beta_d(u^{n-k} x) & \text{if } k \text{ is even and } \dim x = n-k-1, \\ \beta_2(u^{n-k} x + u^{n-k-3} S^q x) & \text{if } k \text{ is odd and } \dim x = n-k-1. \end{cases} \]
On considering the exact sequence (5.3), it follows, from Lemmas 5.4, 5.5 and
the above relations, that \( j^*: H^{n-k-i}(M*; Z) \rightarrow \text{Im} j^* \) is a split epimorphism for
\( i=1, 2 \). Further, the relation
\[
\bar{p}_{2} \bar{p}(u^{k-1} \otimes x^{2}) = \rho(u^{k} \otimes x^{2}) \quad \text{for} \quad x \in H^{n-k-1}(M; Z)
\]
follows from Lemma 3.2. Hence, the theorem is established in case \( n-k \) is odd.

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