Spectral synthesis on the algebra of Hankel transforms

By Yûichi KANJIN and Enji SATO

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1. Introduction.

Let $\nu \geq -1/2$. The Hankel transform of order $\nu$ is given by

$$ g(y) = \int_0^\infty g(x) J_\nu(xy) x^{\nu+1} y^{-\nu} dx, \quad y \geq 0 $$

for a function $g(x)$ on $[0, \infty)$, where $J_\nu(t)$ is the Bessel function of the first kind of order $\nu$. Let

$$ A(\nu) = \{ \hat{g} ; \int_0^\infty |g(x)| x^{\nu+1} dx < \infty \}, $$

and introduce a norm to $A(\nu)$ by

$$ \| \hat{g} \| = \frac{1}{2\nu^\frac{\nu+1}{2}} \int_0^\infty |g(x)| x^{\nu+1} dx. $$

Then the followings are known (cf. [10], [7]):

(i) $A(\nu)$ consists of continuous functions on $[0, \infty)$ vanishing at infinity.

(ii) $A(\nu)$ is a semisimple regular Banach algebra with the product of pointwise multiplication, and the maximal ideal space of $A(\nu)$ is identified with the interval $[0, \infty)$.

Let $A(\mathbb{R}^n)$ be the Fourier algebra given by $A(\mathbb{R}^n) = \{ \hat{g} ; g \in L^1(\mathbb{R}^n) \}$, $\| \hat{g} \| = \| g \|_{L^1(\mathbb{R}^n)}$, where $\hat{g}$ is the Fourier transform of $g$, that is, $\hat{g}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(x)e^{-ix\cdot \xi} dx$, $\xi \in \mathbb{R}^n$. Denote by $A_r(\mathbb{R}^n)$ the Fourier transforms of the radial functions $g$, $g(v) = g(|v|)$ a.e. $v \in \mathbb{R}^n$, in $L^1(\mathbb{R}^n)$. From the well-known formula $\hat{g}(\xi) = \hat{g}(|\xi|)$ for a radial function $g$, it follows that $A_r(\mathbb{R}^n)$ is isomorphic and isometric to $A(\nu)$ if $\nu = (n-2)/2$, $n=1, 2, 3, \ldots$.

L. Schwartz [11] showed that the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ is not a set of spectral synthesis for $A(\mathbb{R}^n)$, $n \geq 3$. Reiter [7] proved that, if $n \geq 3$, then the singleton $\{ y_0 \}$, $y_0 > 0$ is not a set of spectral synthesis for $A(\mathbb{R}^n)$, that is, for $A(\nu)$, $\nu = (n-2)/2$. This implies L. Schwartz's result. For, if $\xi_j \rightarrow \xi$ in $A(\mathbb{R}^n)$, then the Fourier transforms of the means of $g_j$ on $S^{n-1}$ converge to the Fourier transform of the mean of $g$ on $S^{n-1}$ in $A_r(\mathbb{R}^n)$. A. Schwartz [10] showed that
Reiter's result holds good for all $\nu \geq 1/2$.

On the other hand, Herz [4] proved that $S^1$ is a set of spectral synthesis for $A(\mathbb{R}^2)$, which implies that $\{y_0\}$, $y_0 > 0$ is a set of spectral synthesis for $A^{(\nu)}$. Since $S^0$ is a set of spectral synthesis for $A(\mathbb{R}^1)$, the set $\{y_0\}$, $y_0 > 0$ is a set of spectral synthesis for $A^{(-1/2)}$.

For $y_0=0$, Reiter [7] proved that, for every $n \geq 1$, the set $\{0\}$ is a set of spectral synthesis for $A_r(\mathbb{R}^n)$, that is, for $A^{(\nu)}$, $\nu = (n-2)/2$.

The purpose of this paper is to show the following:

**Theorem.** If $-1/2 < \nu < 1/2$ and $y_0 > 0$, then $\{y_0\}$ is a set of spectral synthesis for $A^{(\nu)}$. The set $\{0\}$ is a set of spectral synthesis for $A^{(\nu)}$ for every $\nu \geq -1/2$.

Related results will be found in Igari and Uno [5], Cazzaniga and Meaney [1], and Wolfenstetter [12]. They are concerned with spectral synthesis for the algebra of absolutely convergent Jacobi polynomial series.

## 2. A lemma.

First we will prove a lemma.

**Lemma.** Let $\nu > -1$ and let $m$ be the least integer exceeding $\nu + 3/2$. Let $f$ be an infinitely differentiable function with compact support contained in $[-K, K]$. Then

$$
\int_0^\infty |\hat{f}(u)| u^{2\nu+1} du \leq C \sum_{j=0}^m K^j \sup_{y \in \mathbb{R}} |f^{(j)}(y)|,
$$

where $C$ is a constant depending only on $\nu$.

**Proof.** Let

$$
\int_0^\infty |\hat{f}(u)| u^{2\nu+1} du = \left( \int_0^{1/K} + \int_{1/K}^\infty \right) |\hat{f}(u)| u^{2\nu+1} du
$$

$$
= I_1 + I_2.
$$

Since $|f_\nu(t)| \leq Ct^\nu$ for $0 \leq t \leq 1$, we have

$$
I_1 = \int_0^{1/K} \int_0^K |f(y)(yu)^{\nu+1} u^{-\nu} dy | u^{2\nu+1} du
$$

$$
\leq C \int_0^{1/K} \int_0^K |f(y)||yu|^{\nu+1} dy u^{\nu+1} du
$$

$$
\leq C(2\nu+1)^{-2} \sup_{y \in \mathbb{R}} |f(y)|.
$$

Here and below, the letter $C$ means positive constants depending only on $\nu$, and it may be different in each occasion.
Next we will estimate $I_2$. By the formula \((d/dt)^n[t^j J_v(t)]=t^{v-n} J_{v-n}(t)\) (cf. [2, 7.2.8 (52)]), we have
\[
f(u) = u^{-2(v+m)} \int_0^\infty y f(y) \left( \frac{d}{dy} \right)^{v+m} f(y) dy.
\]
Noting the fact \((d/dy)^{v+m} \{y u\} \mid_{y=0} = 0, j=1, 2, \ldots, m,\) and repeating integration by parts, we have
\[
f(u) = (-1)^m u^{-2(v+m)} \int_0^\infty \left\{ \frac{d}{dy} \left( \frac{d}{dy} \right)^{v+m} f(y) \right\} (yu)^{v+m} J_{v+m}(yu) dy,
\]
and thus we have
\[
I_2 \leq \frac{C}{u} \int_1^u \int_0^\infty \left( \frac{d}{dy} \right)^{v+m} f(y) (yu)^{v+m} J_{v+m}(yu) dy u^{-(2m+1)} du.
\]
By a simple calculation, we have
\[
\frac{d}{dy} \left( \frac{d}{dy} \right)^{v+m} f(y) = \sum_{j=1}^m c_j y^{m+j} f^{(j)}(y),
\]
where every $c_j$ is a constant depending only on $j$. Thus we have
\[
I_2 \leq \sum_{j=1}^m (\sup_{y \geq 0} |f^{(j)}(y)|) \int_1^u \int_0^\infty y^{m+j} J_{v+m}(yu) dy u^{-(m+j)} du.
\]
Let
\[
\int_0^\infty \int_1^u y^{m+j+1} J_{v+m}(yu) dy u^{m+j+1} du = \int_1^u \int_0^1 y^{m+j+1} J_{v+m}(yu) dy u^{m+j+1} du + \int_1^u \int_1^\infty y^{m+j+1} J_{v+m}(yu) dy u^{m+j+1} du = S_{1j} + S_{2j}.
\]
Then we have
\[
S_{1j} \leq C \int_1^u \int_0^1 y^{m+j+1} (yu)^{m+j+1} dy u^{m+j+1} du = C ((j+2v+2)j)^{-1} K^j.
\]
By the inequality \(|J_{v+m}(t)| \leq Ct^{-1/2}\) for $t \geq 1$, we have
\[
S_{2j} \leq C \int_1^u \int_1^\infty y^{m+j+1} (yu)^{m+j+1/2} dy u^{m+j+1} du
\]
\[
= C \int_1^u \int_1^\infty u^{m+j+(1/2)} du y^{j-m+v+1} dy
\]
\[
= C ((m-v-(3/2))j)^{-1} K^j.
\]
Thus we have
\[
I_2 \leq C \sum_{j=1}^m (S_{1j} + S_{2j}) \sup_{y \geq 0} |f^{(j)}(y)| \leq C \sum_{j=1}^m K^j \sup_{y \geq 0} |f^{(j)}(y)|,
\]
3. Proof of Theorem.

Let $I$ be a closed ideal in $A^{(\nu)}$ and let $Z(I) = \{ y \in [0, \infty) ; f(y) = 0 \text{ for all } f \in I \}$. Theorem is an immediate consequence of the following proposition.

**Proposition.** Let $\nu \geq -1/2$ and let $I$ be a closed ideal in $A^{(\nu)}$ such that $Z(I) = \{ y_0 \}, y_0 \geq 0$. If $y_0 > 0$, then $I = \{ f \in A^{(\nu)} ; f^{(j)}(y_0) = 0, j = 0, 1, 2, \ldots, M \}$ for some $M \leq \nu + 1/2$. If $y_0 = 0$, then $I = \{ f \in A^{(\nu)} ; f(0) = 0 \}$.

**Proof.** Let $I$ be a closed ideal in $A^{(\nu)}$ such that $Z(I) = \{ y_0 \}, y_0 \geq 0$, and let $\phi$ be a continuous linear functional on $A^{(\nu)}$ such that $\phi(f) = 0$ for all $f \in I$. Let $\mathcal{D}(-\infty, \infty)$ be the test function space on $(-\infty, \infty)$ with usual topology. For $f \in \mathcal{D}(-\infty, \infty)$, put $f_\nu(x) = f(y(x)), x \geq 0$ and $f_N(x) = f(-x), x \geq 0$. Then, by the inversion formula of the Hankel transform and Lemma, we have that $f_\nu$ and $f_N$ are in $A^{(\nu)}$. We define $\Phi_+ = \phi(f_\nu) + \phi(f_N)$ and $\Phi_- = \phi(f_\nu) - \phi(f_N)$ for $f \in \mathcal{D}(-\infty, \infty)$. By Lemma we have

$$|\Phi(f)| \leq \|\phi\| (\|f_\nu\| + \|f_N\|) \leq C\|\phi\| \sum_{j=0}^M K^j \sup_{-\infty < y < \infty} |f^{(j)}(y)|,$$

where $K$ is a positive number such that $\text{supp} f \subset [-K, K]$, and $m$ is the least integer exceeding $\nu + 3/2$. Thus $\Phi_\pm$ are continuous linear functionals on $\mathcal{D}(-\infty, \infty)$ with order not exceeding $m$. Since $A^{(\nu)}$ is semisimple and regular, the ideal $I$ contains the ideal of functions in $A^{(\nu)}$ which vanish on a neighborhood of $y_0$ (cf. [6, Chapter VIII, 5.7]). This implies that the supports of $\Phi_\pm$ are the singleton $\{ y_0 \}$. Thus $\Phi_\pm$ have the forms

$$\Phi_+ = \sum_{j=0}^m a_j \delta_{y_0}^j, \quad \Phi_- = \sum_{j=0}^m a_j \delta_{y_0}^j,$$

where $a_j$ are constants, and $\delta_{y_0}^j$ is the Dirac measure with mass at $\{ y_0 \}$ (cf. [8, 6.25]).

Now we will show that $a_j = 0$ for $j$ exceeding $\nu + 1/2$ if $y_0 > 0$. Let $\nu, \mu > -1$ and put

$$g_{x,\mu}(x) = \frac{2^{\nu+1} \Gamma(\nu+\mu+2)}{\Gamma(\nu+1)s^{(\nu+\mu+1)}} (s^2 - x^2)^\mu \chi_{(0,s)}(x),$$

where $\chi_{(0,s)}(x)$ is the characteristic function of $[0, s)$. Then $\|g_{x,\mu}\| = 1$ and

$$g_{x,\mu}(y) = \frac{2^{\nu+\mu+1} \Gamma(\nu+\mu+1)}{(sy)^{\nu+\mu+1}} J_{\nu+\mu+1}(sy) \quad [3, 8.5(33)].$$

Let $q(y)$ be a function in $\mathcal{D}(-\infty, \infty)$ such that $q(y) = 1$ on a neighborhood of $y_0$ and $\text{supp} q \subset (0, \infty)$. Then $q g_{x,\mu}$ is in $\mathcal{D}(-\infty, \infty)$ and $|\Phi_+ (q g_{x,\mu})| \leq |\phi(q g_{x,\mu})| \leq \|\phi\| (\|q g_{x,\mu}\| + \|q g_{x,\mu}\|),$
on the other hand, by the formula
\[ \frac{d}{dt}(t^{-\nu}J_\nu(at)) = -at^{-\nu}J_{\nu+1}(at), \]
and the asymptotic formula
\[
J_\nu(t) = \left(\frac{2}{\pi t}\right)^{1/2} \cos(t-\nu\pi/2-\pi/4) + O(t^{-3/2}) \quad (t \to \infty),
\]
we have
\[
\delta_{v_0}^{(j)}(q_0 \hat{g}_s, v) = O(s^{-(\nu+1/2)-(\mu+1)}) \quad (s \to \infty)
\]
and
\[
\limsup_{s \to \infty} |\delta_{v_0}^{(j)}(q_0 \hat{g}_s, v)| s^{-(\nu+1/2)-(\mu+1)} > 0
\]
for \( j = 0, 1, 2, \ldots \). This implies that \( \limsup_{s \to \infty} |\Phi_+(q_0 \hat{g}_s, v)| = \infty \) if \( a_j \neq 0 \) for some \( j > (\nu+1/2)+(\mu+1) \). Since \( \mu > -1 \) is arbitrary, we have \( a_j = 0 \) for \( j > \nu+1/2 \).

Next we will show that \( a_j = 0 \) for \( j > 0 \) if \( y_0 = 0 \). First we note that \( \Phi_+ \) have the forms
\[
\Phi_+ = \sum_k a_{2k}^+ \delta_{v_0}^{(2k)} \quad \text{and} \quad \Phi_- = \sum_k a_{2k-1}^- \delta_{v_0}^{(2k-1)}.
\]

Let \( q_0(y) \) be an even function in \( \mathcal{D}(-\infty, \infty) \) such that \( q_0(y)=1 \) for \( y \in [-1, 1] \) and \( q_0(y)=0 \) for \( y \notin (-2, 2) \). Put \( q_s(y) = (sy/2)q_0(sy/2) \). Then we have \( \|q_s\|_2 = O(1) \) as \( s \to \infty \) by Lemma. Let \( g_{s,0} \) be the function \( g_{s,\mu} \) with \( \mu = 0 \). Then \( |\Phi_+(q_s \hat{g}_s, v)| = O(1) \) and \( |\Phi_-(q_s \hat{g}_s, v)| = O(1) \) as \( s \to \infty \). On the other hand, it follows from the power series expansion of the Bessel function that
\[
\delta_{v_0}^{(2k)}(q_s \hat{g}_s, v) = \frac{(-1)^k(2k)!}{2^{2k} k!} \Gamma(\nu+1) s^{2k},
\]
\[
\delta_{v_0}^{(2k-1)}(q_s \hat{g}_s, v) = \frac{(-1)^{k-1}(2k-1)!}{2^{2k-1} (k-1)!} \Gamma(\nu+1) s^{2k-1}.
\]
This implies that \( |\Phi_+(q_s \hat{g}_s, v)| \to \infty (s \to \infty) \) if \( a_{2k} \neq 0 \) for some \( k > 0 \), and \( |\Phi_-(q_s \hat{g}_s, v)| \to \infty (s \to \infty) \) if \( a_{2k-1} \neq 0 \) for some \( k > 0 \). Thus we have \( a_j = 0 \) for \( j > 0 \), and therefore we have that \( \phi(f_\mu) \) is of the form
\[
\phi(f_\mu) = \left( \Phi_+(f) - \Phi_-(f) \right) / 2
\]
for \( f \in \mathcal{D}(-\infty, \infty) \). A. Schwartz [9] showed that if \( f \) is in \( A^{(\nu)} \), then \( f \) has \( p \) continuous derivatives and \( |f^{(r)}(y)| \leq C \|f\|_r \), \( r = 0, 1, 2, \ldots, p \), where \( p \) is the greatest integer not exceeding \( \nu+1/2 \). From this and the fact that \( \{f_\mu; f \in \mathcal{D}(-\infty, \infty)\} \) is dense in \( A^{(\nu)} \), it follows that \( \phi \) is of the form
\[
\phi(f) = \left\{ \begin{array}{ll}
\sum_{j=1}^N a_j \delta_{v_0}^{(j)}(f), & N \leq \nu+1/2, \; \; (y_0 > 0), \\
\sum_{j=1}^N a_j \delta_{v_0}^{(j)}(f), & N < \nu+1/2, \; \; (y_0 = 0), \\
\end{array} \right.
\]
for \( f \in A^{(\nu)} \). We define \( N(\phi) = \max \{ j; \alpha_j \neq 0 \} \). Let \( L_I \) be the space of continuous linear functionals \( \phi \) on \( A^{(\nu)} \) such that \( \phi(f) = 0 \) for all \( f \in I \). Put \( M = \max \{ N(\phi); \phi \in L_I \} \). Then we note that \( M = 0 \) for \( \nu = 0 \) and \( 0 \leq M \leq \nu + 1/2 \) for \( \nu > 0 \). Let \( f \) be in \( I \) and let \( \phi \) be a functional in \( L_I \) such that \( M = N(\phi) \).

For \( h \in A^{(\nu)} \), we have

\[
0 = \phi(h) = \sum_{k=0}^{M} \left( \sum_{j=k}^{M} \binom{M}{j} a_j f^{(j-k)}(\nu) \right) h^{(k)}(\nu).
\]

Since there exist functions \( h_m \in A^{(\nu)} \) such that \( h_m^{(k)}(\nu) = \delta_{m,k} \), \( k, m = 0, 1, 2, \ldots, M \), we have \( \sum_{j=0}^{M} \binom{M}{j} a_j f^{(j-k)}(\nu) = 0 \) for \( k = 0, 1, 2, \ldots, M \). Thus \( f^{(k)}(\nu) = 0 \) for \( k = 0, 1, 2, \ldots, M \). This implies that \( I = \{ f \in A^{(\nu)}; f^{(k)}(\nu) = 0, k = 0, 1, 2, \ldots, M \} \) since \( I \) is a space of \( f \in A^{(\nu)} \) such that \( \phi(f) = 0 \) for all \( \phi \in L_I \). Therefore the proof is complete.

References