§ 8. Introduction.

The purpose of this paper is to prove:

8.1. Theorem. Let $X$ be an irreducible, smooth projective surface over $C$, with the geometric genus $p_g(X) = 0$. Let $D$ be a reduced (not necessarily connected) curve on $X$ with at worst ordinary double point singularities. Suppose each connected component of $\text{Supp} \ D$ is simply connected and the irreducible components of $D$ generate the divisor class group $\text{Pic} \ (X)$. Then $X$ is rational.

In [G-S] we showed that such a surface $X$ cannot be a surface of general type. Thus 8.1 follows from:

8.2. Theorem. Let $X$ and $D$ be as in 8.1. Assume further that $X$ is an elliptic surface. Then $X$ is rational.

We will refer to the paper [G-S] by Part I. As already indicated in Part I, one easily deduces from 8.1, that any complex homology 2-cell is rational, thus answering affirmatively a question of Van de Ven. (For other consequences of 8.1 see Part I.) The reader is assumed to be familiar with Part I of this paper, the notations and conventions of which we continue to use here also. We shall now briefly outline the proof of 8.2 here.

8.3. Recall that (in Part I), we begin with the assumption that $X$ is not rational (or, equivalently, that $|nK| \neq 0$ for some $n > 0$) so that $K + D$ has Zariski decomposition: $K + D = P + N$. Without loss of generality we also assume that there are no $(-1)$-curves $E$ on $X$ such that (i) $E \cdot D = 1$ or (ii) $E \cdot D = 2$ and $E$ meets two different connected components of $D$. We then apply Miyaoka’s inequality to $(X, D)$. Then, by studying the blowing-down process $\pi : X \to X^\prime$, where $X^\prime$ is the minimal model for the function field of $X$, we obtain the auxiliary inequality

\[(2.8) \quad 3(b_2 - \beta_2) + b_2 + \lambda + \tau + e_1 + r_2 + 2r_4 \leq \beta_2' - 5\]

where each term on the left hand side is nonnegative.
8.4. We now assume that $X$ is an elliptic surface, $\varphi : X \to \mathbf{P}^1$, and arrive at a sequence of contradictions thereby proving 8.2. Now $\beta''_2$, on the right hand side of (2.8) is the second betti number of $X''$, which is a minimal elliptic surface, with $p_g(X') = 0 = q(X')$. Hence $\beta''_2 = 10$. (Note that if $X$ is of general type then $\beta''_2 \leq 9$. So there is an increase in the right hand side of (2.8), as compared with the general type case and this contributes to the enormous increase in our task here.) Except for 4.2, all the results in §1-§6 (of Part I) hold here also. Our first aim is to prove the analogue of 4.2 here also, viz., to show that $\lambda \geq 2$. This is achieved in §9 by showing that $D$ has at least two horizontal components with respect to $\varphi$. Once we show that $\lambda \geq 2$, by 3.2, we obtain that $b_2 = \beta_2$, $D$ is unimodular, $r_s = 0$, and $\sigma + r_s + e_1 + r_3 \leq 2$.

In §12, we show that $e_1 = 0$ and in §13 we will show that $r_s \geq 1$. In §14, we show that $(r_s, \sigma) \neq (1, 1)$. In §15, we show that $r_s \geq 2$. Since, any way $r_s \leq 2$, in §16 and §17 we show that even the assumption $r_s = 2$ leads to a contradiction.

In each of these cases we first get hold of the geometry of $D \cup \mathcal{E}$ (where $\mathcal{E}$ is the exceptional set for $\pi : X \to X''$). In §12-§14, often, we apply Miyaoka’s inequality to the curve $D \cup \mathcal{E}$, to arrive at a contradiction. Note that at the end of §14, we have $\sigma = 0$, $e_1 = 0$, $r_s \geq 1$, from which it follows that $\mathcal{E} \subset D$. Thus in §15-§17 we apply Miyaoka’s inequality to a suitable subset of $D$. This is rather fruitful only if we have equality in (2.8). Further, we need to estimate the $N^2$-term in the Zariski decomposition. For this purpose we use results of [F] which we have collected in §10 in a manner suitable for our purpose. The unimodularity of $D$ and the fact that $\pm K$ is not effective are heavily used, In §11, we have collected various technical lemmas, similar to those in §5 and §6, which enable us to use these two properties.

§9. Number of horizontal components.

9.0. Let $X$ and $D$ be as in 8.2. We assume $X$ is not rational, so that $|nK| \neq \emptyset$ for some $n > 0$. Since $X$ is simply connected (see [G-S] p. 3), we have an elliptic fibration $\varphi : X \to \mathbf{P}^1$. Let $\pi : X \to X''$ be the composite of contraction of $(-1)$-curves where $X''$ is the minimal model for the function field of $X$. It follows easily that $\varphi$ defines the minimal elliptic fibration $\varphi'' : X'' \to \mathbf{P}^1$, $\varphi = \varphi'' \circ \pi$. An irreducible curve $C$ on $X$ is called vertical if $C$ is contained in a fibre of $\varphi$; otherwise it is called horizontal. Our main aim here is to show that $D$ has at least two horizontal components, equivalently, $D'' = \varphi''(D)$ has at least two horizontal components with respect to $\varphi''$. On the way, we also determine multiplicities of the singular fibres of $\varphi''$.

Note that components of $D$ generate $\text{Pic} X$ and hence $D$ should have at least one horizontal component. We shall first prove:
9.1. Lemma. There are exactly two multiple singular fibres of \( \varphi^* \) (and hence, of \( \varphi \)), say \( m_1P_1, m_2P_2 \) (respectively, \( m_1F_1, m_2F_2 \)) with \( \{m_1, m_2\} = \{2, 3\} \) or \( \{2, 5\} \). Further, if \( K^* \cdot H^* \leq 2 \) for some horizontal component \( H^* \), then \( \{m_1, m_2\} = \{2, 3\} \).

Proof. Let \( \{m_jP_j\}_{1 \leq j \leq r} \) be the multiple fibres of \( \varphi^* \). By the simply connectivity of \( X^* \) it follows that \( r \leq 2 \) and \( m_1 \) and \( m_2 \) are coprime (see Proposition 2 of [K]). Since \( p_g = 0 = q \), we have the canonical bundle formula

\[
K^* = \varphi^{-1}(P_1(-1)) \otimes [P_1]^{m_1-1} \otimes \cdots \otimes [P_r]^{m_r-1}.
\]

Thus if \( r \leq 1 \), it follows that \( nK^* = \mathcal{O} \) for all \( n \) and hence \( nK = \mathcal{O} \) for all \( n \), contradicting our basic assumption. Hence \( r = 2 \).

Now if \( P \) denotes a general fibre of \( \varphi^* \), we have the linear equivalence

\[
P \sim m_1P_1 \sim m_2P_2
\]

and hence

\[
K^* \sim (m_1-1)P_1P_2 \sim (m_2-1)P_1P_2.
\]

Thus \( K^* \) is numerically equivalent to the \( \mathbb{Q} \)-divisor \( mP \) where \( m = (m_1m_2-m_1-m_2)/m_1m_2 \). Now let \( H^* \) be a horizontal component of \( D^* \). Then \( 1 \leq K^* \cdot H^* = m(P \cdot H^*) \). Since \( m_1 \) and \( m_2 \) are coprime it follows that \( m,m_1-m_1-m_2 \) divides \( K^* \cdot H^* \). On the other hand we have, by (2.8), \( \lambda \leq 1 \) and hence \( K^* \cdot H^* \equiv 4 \). Hence \( \{m_1, m_2\} = \{2, 3\} \) or \( \{2, 5\} \), and if \( K^* \cdot H^* \leq 2 \) for some \( H^* \), then \( \{m_1, m_2\} = \{2, 3\} \), as claimed.

9.2. Lemma. Suppose \( \beta_2(X) = b_2(D) \). Then \( H_1(X-D, \mathbb{Z}) = 0 \) and consequently no fibre of \( \varphi \) is completely contained in \( D \).

Proof. Since components of \( D \) generate \( \text{Pic} X \), \( \beta_2(X) = b_2(D) \) implies that components of \( D \) form a free basis for \( \text{Pic} X \) and hence \( H^2(X, \mathbb{Z}) \to H^2(D; \mathbb{Z}) \) is an isomorphism. From the cohomology exact sequence of \( (X, D) \), we get \( H^1(X, D; \mathbb{Z}) = 0 \). Hence, by duality, \( H_1(X-D; \mathbb{Z}) = 0 \).

Now suppose \( F_0 \) is a fibre of \( \varphi \) such that \( F_0 \subset D \). Then, since \( D \) is simply connected, \( F_0 \) is simply connected, and so, in particular, \( F_0 \neq m_1F_i \), where \( m_1F_i \) and \( m_2F_2 \) are the multiple fibres of \( \varphi \). Let \( p_i = \varphi(F_i) \) and let \( f : \mathcal{A} \to \mathbb{P}^1 \) be a cyclic covering, ramified on \( p_i \) and \( p_1 \) with ramification index \( m_1 \). After normalization of \( X \times p_1 \mathcal{A} \), we obtain a ramified cyclic covering \( \tilde{f} : \tilde{X} \to X \) with ramification locus contained in \( F_0 \) and hence an \( m_1 \)-fold cyclic, unramified covering of \( X-D \). This contradicts the fact that \( H_1(X-D; \mathbb{Z}) = 0 \). Hence no fibre of \( \varphi \) is completely contained in \( D \). This completes the proof of the lemma.

9.3. Lemma. \( D^* \) (and hence \( D \)) has at least two horizontal components. In particular, \( \lambda \geq 2 \), \( \{m_1, m_2\} = \{2, 3\} \) and \( 6 \mid (P \cdot H^*) \) for any horizontal component \( H^* \)
and any general fibre $P$ of $\varphi^s$. Further, if there are only two horizontal components, then exactly one component from each fibre of $\varphi$ is not contained in $D$.

**Proof.** Assume that $D^s$ has only one horizontal component, $H^s$. We first claim that $\varphi^s$ has no fibres of type $m_1$, with $m>1$. Suppose $m_1 P_1$ is such a fibre. Then $P_1 \sim \sum \alpha_i D'_i + \alpha H^s$, where $D'_i \neq H^s$ are other components of $D^s$ (which are vertical). Intersecting $P_1$ with a general fibre, we see that $\alpha=0$. Since each component of $D^s$ is rational, $P_1$ is not contained in $D^s$. On the other hand, linear equivalence relations between fibre components of $\varphi^s$ are all generated by relations of the type $P_1' \sim a D'_i$, for any two scheme-theoretic fibres of $\varphi^s$. Hence in any such relation $P_1$ should occur in multiples of $m_1$, which contradicts the relation, $P_1 \sim a D'_i$, above.

Now let $S_1, \ldots, S_k$ (respectively, $T_1, \ldots, T_l$) denote the simply connected (respectively, not simply-connected) singular fibres of $\varphi$. By the above observation, it follows that $x(T_j) = b_2(T_j)$ and hence we have

\begin{equation}
\beta_s(X) + 2 = x(X) = \sum_{i=1}^k (b_2(S_i)) + \sum_{j=1}^l b_2(T_j).
\end{equation}

Since, by simply-connectivity of $D$, none of the $T_i$ is completely contained in $D$, we get

\begin{equation}
b_2(D) \leq \sum_{i=1}^k b_2(S_i) + \sum_{j=1}^l (b_2(T_j) - 1) + 1.
\end{equation}

Together with (9.4), this yields $k + l \leq 3 + \beta_s(X) - b_2(D)$. If $k=0$, then clearly no fibre of $\varphi$ is contained in $D$. However, by 9.1, it follows that $l \geq 2$. Since $\beta_s(X) \leq b_2(D)$, it follows that $k \leq 1$, and if $k=1$ then $\beta_s(X) = b_2(D)$. Thus if $k \neq 0$, we can appeal to 9.2 to conclude that no fibre of $\varphi$ is contained in $D$.

But then we have

\begin{equation}
b_2(D) \leq \sum_{i=1}^k (b_2(S_i) - 1) + \sum_{j=1}^l (b_2(T_j) - 1) + 1 < \text{rank}(\text{Pic } X)
\end{equation}

which is absurd.

Hence $D^s$ has at least two horizontal components, and hence $\lambda = \sum_i D'_i \cdot K^s \geq 2$. Now by (2.8) we have $\lambda \leq 4$, and hence $K^s \cdot H'_i \leq 2$ for some horizontal component. Hence by 9.1, $\{m_1, m_2\} = \{2, 3\}$, $K^s \cdot H'_i = (1/6) \cdot (P \cdot H'_i)$ and hence $6 \cdot P \cdot H'_i$. By (2.8) it now follows that $b_2 = \beta_s$ and hence by 9.2, no fibre of $\varphi$ is completely contained in $D$.

Finally let there be exactly two horizontal components. Then we have

\begin{equation}
b_2(D) \leq \sum_{i=1}^k (b_2(S_i) - 1) + \sum_{j=1}^l (b_2(T_j) - 1) + 2 = \text{rank Pic } (X) \leq b_2(D).
\end{equation}

Hence equality holds everywhere. Thus exactly one component from each fibre of $\varphi$ is missing from $D$. This completes the proof of the lemma.
9.6. REMARK. Note that, unlike $D, D'$ may contain a fibre of $\varphi^n$. The following lemma gives some idea of such a situation and will be very useful later. We use Kodaira’s list of singular fibres of an elliptic fibration. For any fibre $P$ and a component $C$ of $D$ let $\mu(C)$ denote the multiplicity of $C$ in $P$.

9.7. LEMMA. Suppose $P$ is a fibre of $\varphi^n$ contained in $D'$ and is not of type $\Pi^*$. Assume that

(i) there is at most one point $x \in P$ which is worse than an ordinary double point singularity of $D'$ and

(ii) if $x$ exists, then $P$ is of type $m_1$, $\Pi$, $\Pi^*$ or IV with $x \in P$ being the singularity of $P$ and at most one horizontal component of $D'$ passes through $x$.

Then $b_1(D') \geq 2$.

PROOF. First assume $x$ exists. By 9.3, there is a horizontal component $H'$ of $D'$ not passing through $x$. If $P$ is of type $m_1$, then $m=2$ or $3$, $6 \mid (P \cap H')$ and hence $P \cap H'$ consists of at least two points. Since $b_1(m_1) = 1$, we have $b_1(D') \geq b_1(P \cup H') \geq 2$. If $P$ is of type $\Pi$, $\Pi^*$ or IV, then it follows that $P \cap H'$ consists of at least six points and hence $b_1(D') \geq b_1(P \cup H') \geq 5$.

Next assume that $x$ does not exist. (Clearly $P$ is not of type $m_1$). Hence $P$ is of type $m_{1b}$, $(m=2$ or $3$, $b \geq 1$), $I_\ell^*$, $\Pi^*$ or IV*. Hence for every component $C$ of $P$, $\mu(C) \leq 4$, $P \cap H'$ consists of points which are all ordinary double points of $D'$. Hence it follows that $b_1(P \cup H') \geq 1$ for each horizontal component of $D'$. Hence $b_1(D') \geq b_1(P \cup H' \cup H'') \geq 2$. This completes the proof of the lemma.

9.8. LEMMA. Suppose $\beta_2(X) = b_2(D)$ and there are precisely two horizontal components of $D$. Let $F_0$ be a fibre of $\varphi$ different from the multiple fibres $m_1F_1$ and $m_2F_2$, and let $C_0$ be the component of $F_0$ not contained in $D$. Then the multiplicity $\mu(C_0)$ of $C_0$ in $F_0$ is not equal to $m_1$ or $m_2$.

PROOF. If not, say $\mu(C_0) = m_1$. Then as in 9.2, we can construct an $m_1$-fold unramified cyclic cover of $V$ which is absurd.

9.9. LEMMA. Let $Y$ be a normal, projective surface $\mathcal{C}$ having exactly one singular point $y$. Suppose $\tilde{Y} \rightarrow Y$ is a minimal resolution of singularity. Assume further the following conditions:

i) $\mathcal{O}_{\tilde{Y}, y}$ is a rational singularity

ii) $\mathcal{O}_{\tilde{Y}, y}$ is a unique factorization domain and

iii) $p_*(\tilde{Y}) = 0 = q(\tilde{Y})$.

Then locally analytically, the singularity at $y$ is $E_8$-rational double point, i.e., given by $\{z_1^2 + z_2^2 + z_3^2 = 0\}$.

PROOF. There exists an affine neighbourhood $U$ of $y$ such that $\Gamma(U, \mathcal{O}_Y)$ is a U.F.D. Since $p_*(\tilde{Y}) = 0$, any topological 2-cycle is algebraic. Let $\pi^{-1}(y)$
\[= \bigcup_{i=1}^{r} L_i \] where \(L_i\) are the irreducible components. We can then find divisors \(A_1, \ldots, A_r\) supported on \(\tilde{Y} - \pi^{-1}(U)\) such that \(L_1, \ldots, L_r, A_1, \ldots, A_r\) form a free basis of \(H^2(\tilde{Y}, R)\). Therefore the intersection matrix of these divisors can be assumed to be unimodular. Since the sets \(L_1 \cup \cdots \cup L_r\) and \(A_1 \cup \cdots \cup A_r\) are disjoint, it follows that the intersection matrix \((L_i \cdot L_j)\) is unimodular. Since the singularity of \(y\) is rational, this implies that analytically the singularity is the \(E_8\)-rational double point, by results of Brieskorn [B].

\section*{§ 10. Estimating \(N^2\).}

10.0. Here we briefly recall and modify certain results and terminologies from § 3 and § 6 of [F] which will help us to estimate the term \(N^2\). Let \(Y\) be a smooth projective surface and \(C\) be a reduced curve on \(Y\). For our purpose we shall assume that all components of \(C\) are smooth rational curves. Hence, we shall drop the word ‘rational’ from Fujita’s terminology. We shall also assume that \(C\) is minimal with normal crossings (MNC). For any curve \(\Gamma\) on \(Y\), let \(Q(\Gamma)\) denote the subspace of \(\text{Pic } Y \otimes \mathbb{Q}\) generated by components of \(\Gamma\).

10.1. For any component \(C_0\) of \(C\) the branching number is given by \(\beta(C_0) = C_0 \cdot (C - C_0)\). \(C_0\) is called a tip of \(C\) if \(\beta(C_0) = 1\). A sequence \(\Gamma\) of components \(\{C_1, \ldots, C_r\}\), \(r \geq 1\), is called a twig of \(D\) if \(\beta(C_0) = 1, \beta(C_2) = 2\) and \(C_{j-1} \cdot C_j = 1\) for \(2 \leq j \leq r\). We denote \(\Gamma\) by \([a_1, \ldots, a_r]\) where \(a_i = -C_i\). \(\Gamma\) is a maximal twig if there is a (unique) component \(C_0\) of \(C\) such that \(C_r \cdot C_0 = 1\) and \(\beta(C_0) \geq 3\). Then \(C_0\) is called the branching component of \(\Gamma\). For any twig \(\Gamma\), \(\Gamma^\prime\) denotes the curve \(C_2 \cup \cdots \cup C_r\) (\(\Gamma^\prime = \emptyset\) if \(r = 1\)), and \(e(\Gamma) = d(\Gamma) / d(\Gamma')\) where \(d(-)\) denotes the discriminant, \((d(\phi)) = 1\), by convention). A sequence \(\Gamma\) is called a club if \(\beta(C_1) = \beta(C_r) = 1, \beta(C_2) = 2, 2 \leq j \leq r - 1\) and \(C_j \cdot C_{j+1} = 1, 1 \leq j \leq r - 1\).

10.2. A connected component \(A\) of \(C\) is called an abnormal (rational) club if

(i) \(A\) has a unique component \(C_0\) with \(\beta(C_0) = 3\),

(ii) \(A\) has three maximal twigs \(\Gamma_1, \Gamma_2, \Gamma_3\) with \(C_0\) as their branching component,

(iii) \(A\) is negative definite, and

(iv) \(d(\Gamma_1)^{-1} + d(\Gamma_2)^{-1} + d(\Gamma_3)^{-1} > 1\).

By 6.19 of [F] it follows that an abnormal club \(A\) can be contracted to a rational singularity, i.e. \(A\) is ‘rational’ according to our Definition 5.2.

10.3. Assume now that \(K + C\) is pseudo effective, so that, by 6.13, of [F], all (rational) clubs and maximal twigs of \(C\) are negative definite (contractible).

If \(\Gamma\) is a maximal twig of \(C\), then \(Bk(\Gamma)\) (bark of \(\Gamma\)) is the element \(N_1 \in Q(\Gamma)\) such that \(N_1 \cdot C_i = -1, N_1 \cdot C_j = 0\) for \(j \geq 2\) (equivalently, \(N_1 \cdot C_i = (K + C) \cdot C_i\), for \(1 \leq i \leq r\)). If \(\Gamma\) is a club of \(C\) then \(Bk(\Gamma)\) is an element \(N_i \in Q(\Gamma)\) such that
For an isolated club \( \Gamma = \{ C_1 \} \), we have \( \text{Bk}(\Gamma) = 2(-C_1) \cdot \text{C}_1 \). For any connected component \( A \) of \( C \) we define \( \text{Bk}(A) = \sum \text{Bk}(\Gamma_i) \), where \( \Gamma_i \) are all maximal twigs and clubs of \( A \). We define \( \text{Bk}(C) = \sum \text{Bk}(A_j) \) where \( A_j \) are all the connected components of \( C \).

For an abnormal club \( A \) of \( C \) we define \( \text{Bk}^*(A) \) as the element \( N_0(A) \) such that \( N \cdot L = (K + C) \cdot L = (K + A) \cdot L \) for every component \( L \) of \( A \). If \( A \) is a connected component which is not an abnormal club of \( C \) then we take \( \text{Bk}^*(A) = \text{Bk}(A) \). Finally we define \( \text{Bk}^*(C) = \sum \text{Bk}^*(A_j) \) where \( A_j \) are all the connected components of \( C \).

We introduce the notation \( b_k(A) \) (respectively \( b_k^*(A) \)) for the rational number \( (\text{Bk}(A)) \) (respectively, \( (\text{Bk}^*(A)) \)). Then clearly \( b_k(C) = \sum b_k(I') \) where \( I' \) are all clubs and maximal twigs of \( C \). We have

### 10.4. Lemma

Let \( C \) be as in 10.0 and 10.3, and \( \Gamma = [a_1, \ldots, a_r] \), \( r \geq 1 \) be a twig of \( C \).

(i) If \( \Gamma \) is a maximal twig of \( C \), then \( \text{Bk}(\Gamma) = -e(\Gamma) \). In particular, \( b_k(\Gamma) \leq -1/\alpha_i \).

(ii) If \( \Gamma \) is a club of \( C \) then \( \text{Bk}(\Gamma) \leq -1/\alpha_i + 1/\alpha_r \).

(iii) If \( \Gamma \) is a club of \( C \) with \( r = 1, 2 \) or \( 3 \) then \( \text{Bk}(\Gamma) = -4/\alpha_i - \alpha_i + \alpha_r + 2/(\alpha_i \alpha_r - 1) \) or \( -\alpha_i + \alpha_r / \alpha_i \alpha_r(\alpha_i - \alpha_r) \), respectively.

(iv) If \( \{ L_1, \ldots, L_k \} \) is a set of tips of \( C \), then \( \text{Bk}(C) \leq \sum b_k(I') \).

(v) For any abnormal club \( A \) of \( C \) we have \( \text{Bk}^*(A) \leq \text{Bk}(A) \). Hence \( \text{Bk}^*(C) \leq \text{Bk}(C) \).

**Proof.** (i) That \( \text{Bk}(\Gamma) = -e(\Gamma) \) is proved in 6.16 of [F]. Since \( d(\Gamma) = a_i d(\Gamma) - d(\Gamma) \) it follows that

\[
\text{Bk}(\Gamma) = -e(\Gamma) = -d(\Gamma) / (a_i d(\Gamma) - d(\Gamma)) \leq -d(\Gamma) / a_i d(\Gamma) = -1/\alpha_i .
\]

(ii) If \( r_i = 1 \), then \( \text{Bk}(\Gamma) = -4/\alpha_i \) and so we are done. In general, let \( N_i = \sum \lambda_i C_i, N_2 = \sum \mu_i C_i \in \mathcal{Q}(\Gamma) \) be defined as in 10.3. Then \( N = \text{Bk}(\Gamma) \) whereas \( N_1 \) has the numerical property of \( \text{Bk}(\Gamma) \) if \( \Gamma \) were a maximal twig with \( a_i \) as its tip. Hence by 6.16 of [F], \( \lambda_i = -N_i \geq 1/\alpha_i \) by (i) above. On the other hand we have \( (N_1 - N_i) \cdot C_i \leq 0 \) for every component \( C_i \) of \( \Gamma \) and hence \( N_2 \succeq N_1 \) (by 5.3). In particular \( \mu_i \geq \lambda_i \geq 1/\alpha_i \). By symmetry, we get \( \mu_i \geq 1/\alpha_r \). Now \( \text{Bk}(\Gamma) = N_2 = -(\mu_i + \mu_r) \leq -(1/\alpha_i + 1/\alpha_r) .
\]

(iii) Compute directly.

(iv) Follows easily from (i) and (ii) above.

(v) Let \( A \) be an abnormal club of \( C \) as in 10.2, \( N = \text{Bk}^* A \), and let \( N_i = \text{Bk}(\Gamma_i) \), \( i = 1, 2, 3 \). Let \( \theta = (N_1 + N_2 + N_3) \). Then we have \( \theta \cdot L = 0 \) for every component of \( \Gamma \), \( \Gamma \), \( \Gamma \), and \( \theta \cdot C_i = (K + C) \cdot C_i - (d(\Gamma_1) \cdot d(\Gamma_2) \cdot d(\Gamma_3) < 0 ,\)
by 6.16 of [F]. Hence \( \theta > 0 \) (by 5.3). Now \( N^2 = N_1^2 + N_2^2 + \theta \) and \( \theta^2 = \theta \cdot (\lambda C_0) \) where \( \lambda \geq 0 \) is the coefficient of \( C_0 \) in \( N \). Since \( \theta \cdot C_0 < 0 \), it follows that \( \theta^2 < 0 \). Hence \( b_k(A) = N_1^2 < N_1^2 + N_2^2 + \theta \). Hence \( b_k(C) < b_k(C) \). This completes the proof of the lemma.

Finally the following lemma will enable us to use the above estimates in our situation.

10.5. LEMMA. Suppose \( C \) is a union of trees of rational curves on a smooth projective surface \( Y \), \( C \) is MNC and all \((-1)\)-curves on \( Y \) are contained in \( C \). For any set \( E_1, \ldots, E_r \) of \((-1)\)-curves on \( Y \) are contained in \( C \). Assume that \( |nK_Y| \neq 0 \) for some \( n > 0 \), and \( K_Y + C = P + N \) is the Zariski decomposition. Then \( N = b_k(C) \). In particular \( N^2 \geq b_k(C) \).

PROOF. By 6.20 of [F], if \( N \neq b_k(C) \) then there exists an exceptional curve \( E \) on \( Y \), not contained in \( C \), such that one of the following holds:

(i) \( C \cdot E = 0 \).

(ii) \( C \cdot E = 1 \) and \( E \) meets a component of \( b_k(C) \).

(iii) \( C \cdot E > 1 \) and \( E \) meets precisely two components of \( D \), one of which is a tip of a (rational) club of \( D \).

Note that the word “precisely” in (iii) is not there in [F], though from the proof of 6.20, this is obvious. Now, since \( C \) is MNC, every exceptional curve \( E_i \) not in \( C \) meets at least three components of \( C \). Hence \( N = b_k(C) \). That \( N^2 \geq b_k(C) \) follows from 10.4 (v) above. This completes the proof of the lemma.

§ 11. Some technical lemmas.

11.0. Here we collect a number of technical results similar to those in § 4, § 5 and § 6. The first one below, is analogous to 4.3 and will be used in conjunction with 4.1.

11.1. LEMMA. Let \( Y \) be a smooth projective surface, \( C \) be a curve on \( Y \), with \( C = \sum_{i=0}^{k} C_i \), where either (a) \( C_i \) are smooth rational curves pairwise intersecting transversally as shown in (a) Figure 1, or (b) \( C_i \) are smooth rational curves, \( C_1 \cdot C_2 = 2 \), \( C_0 \cdot C_1 = C_0 \cdot C_2 = 1 \) or (c) \( C_0 \) is a smooth rational curve; \( C_1 \) is a rational curve with an ordinary cusp; \( C_0 \cdot C_1 = 2 \). Suppose, in each of these cases, we have a canonical divisor

\[
K_Y = \sum_{i=0}^{k} t_i C_i + A, \quad t_i \in \mathbb{Z}
\]

for some divisor \( A \) with the property \( \text{supp}(A) \cap C_i = \emptyset \), \( i \geq 1 \), and \( C_0 \notin \text{supp}(A) \). Set \( a_i = -C_i \). Then:

(a) In Figure 1(a), let \( (a_1, a_2, a_3) \) be any of the six triples \((2, 2, 3), (3, 2, 3), \ldots\).
(4, 2, 3), (2, 2, 4), (3, 2, 4) or (2, 3, 4). Then $t_6 > 0$ (resp. $t_6 \leq 0$) implies $t_1 > 0$, $t_2 > 0$, $t_3 > 0$ (resp. $t_1 < 0$, $t_2 < 0$, $t_3 < 0$).

(b) In Figure 1(b) if $(a_1, a_2) = (2, 3)$, then $t_6 > 0$ (respectively, $t_6 \leq 0$) implies $t_1 > 0$ and $t_2 > 0$ (respectively $t_1 \leq 0$ and $t_2 \leq 0$).

(c) In Figure 1(c) if $C_1^2 = -1$, then $t_6 > 0$ (respectively $t_6 \leq 0$), implies $t_1 > 0$ (respectively $t_1 < 0$).

PROOF. (a) Compute $K_T \cdot C_1$ and use adjunction formula to obtain the equations:

\[
\begin{align*}
\alpha_1 - 2 &= t_6 + t_2 + t_3 - a_3 t_1 \\
\alpha_2 - 2 &= t_1 + t_4 - a_2 t_2 \\
\alpha_3 - 2 &= t_1 + t_2 - a_3 t_3.
\end{align*}
\]

These, in turn, yield

\[
\begin{align*}
(a_2 a_3 - 1)t_2 &= (a_2 + 1)t_1 - (a_2 a_3 - a_2 - 2), \\
(a_2 a_3 - 1)t_3 &= (a_2 + 1)t_1 - (a_2 a_3 - a_2 - 2), \\
(a_2 a_3 - 1)t_0 &= (a_1 a_2 a_3 - a_1 - a_2 - a_3 - 2)t_1 + (a_1 a_2 a_3 - a_1 - a_2 - a_3 - 2).
\end{align*}
\]

Using the fact that $t_i$ are all integers, we easily check the assertion of (a) for any of the triples in the statement of (a).

(b) In Figure 1(b), note that $C_1 \cdot C_2 = 2$ and so we obtain

\[
\begin{align*}
0 &= -2t_1 + 2t_2 + t_0 \\
1 &= -3t_2 + 2t_1 + t_0
\end{align*}
\]

and hence $t_1 = (5t_0 - 2)/2$; $t_2 = 2t_0 - 1$, and the conclusion follows easily.

(c) Here again, the adjunction formula yields $C_1 \cdot K_T = 1 = 2t_0 - t_1$ so that $2t_0 = 1 + t_1$ and again the conclusion follows easily.

The next lemma helps us to compute the discriminant of certain weighted trees. The idea used here occurs in [R].

11.2. LEMMA. (a) Suppose $\Gamma$ is a weighted tree, $u_0 \in \Gamma$ is a vertex such that $\Gamma - \{u_0\}$ is negative (or positive) definite. Then $\Gamma$ can be diagonalized in such a way that if $p/q$ is the diagonal entry at $u_0$ with $p, q$ coprime, then
(b) Suppose \( A \) is a tree, \( v_0, u_0 \in A \) are any two adjacent vertices, \( \Gamma \) is the branch of \( A \) at \( v_0 \), \( \Gamma' = A - \Gamma \) is the branch at \( u_0 \) containing \( v_0 \). Suppose both \( \Gamma \) and \( \Gamma' \) can be diagonalized as in (a) above with diagonal entries at \( u_0 \) and \( v_0 \) being \( p/q \) and \( p'/q' \) respectively. Suppose that \( p' \neq 0 \) (or \( p \neq 0 \)). Then \( d(\Gamma) = |pp' - qq'| \).

**Proof.** (a) We induct on the number of vertices of \( \Gamma \). Let \( \Gamma - \{u_0\} = \Gamma_1 \sqcup \cdots \sqcup \Gamma_k, k \geq 1 \), with \( u_i \in \Gamma_i \), being joined to \( u_0 \) in \( \Gamma \). Then each \( \Gamma_i \) can be diagonalized in such a way that if \( p_i/q_i \) is the entry at \( u_i \), then \( d(\Gamma_i) = |p_i| \). By the definiteness, \( p_i \neq 0 \). Hence we can further diagonalize the whole of \( \Gamma \) by taking the diagonal entry at \( u_0 \) as \( \Omega_{u_0 - \sum_{i=1}^{k} (q_i/p_i)} = p/q \), say, where \( q = p_1 \cdots p_k \). Thus \( d(\Gamma) = |\det(\Gamma)| = |(p/q)d(\Gamma_1) \cdots d(\Gamma_k)| = |p| \) as claimed.

(b) Since \( p' \neq 0 \), using the given diagonalization of \( \Gamma \) and \( \Gamma' \) we can further diagonalize \( A \), by changing the entry at \( u_0 \) to \( p/q - q'/p' = (pp' - qq')/p'q \). If \( \Gamma_1, \ldots, \Gamma_k \) are the branches of \( \Gamma \) at \( u_0 \), then we have \( |p| = d(\Gamma) = |(p/q)d(\Gamma_1) \cdots d(\Gamma_k)| \) and hence \( |q| = d(\Gamma_1) \cdots d(\Gamma_k) \). Hence \( d(\Lambda) = d(\Gamma_1) \cdots d(\Gamma_k) \cdot (|pp' - qq'|/p'q) \cdot d(\Gamma') = |pp' - qq'| \) as required.

11.3. **Lemma.** Let \( \Gamma \) be a unimodular tree with all weights \( \leq -1 \), \( v \in \Gamma \) be a vertex, \( \Gamma_1 \) be a branch of \( \Gamma \) at \( v \), with \( w \in \Gamma_1 \) joined to \( v \) in \( \Gamma \). Suppose \( \Gamma_1 \) satisfies the following conditions:

(i) \( \#(\Gamma_1) = 6 \), \( \Omega_w \leq -2 \);

(ii) The weight-set \( \Omega(\Gamma_1 - \{w\}) \) of \( \Gamma_1 - \{w\} \), is one of the following four sets: \( \{-2, -2, -2, -2, -2\}, \{-3, -2, -2, -2, -2\}, \{-4, -2, -2, -2, -2\} \) or \( \{-4, -2, -2, -2\} \).

(iii) \( \Gamma_1 \) is not rational (see 5.2).

Then we have:

(a) \( \Gamma_1 \) is one of the trees listed in Figure 2.

(b) \( \Gamma_1 \) is negative definite and we can diagonalize \( \Gamma_1 \), as in 11.2(a) so that if \( p/q \) is the entry at \( w \) then \( p = |\det \Gamma_1| = d(\Gamma_1) \).

(c) For \( \Omega_w = -2 \), \( p/q \) takes the values, \(-16/23, -4/17, -4/9, -27/26, -32/23, -13/17, -13/42 \) and for \( \Omega_w = -3 \), \( p/q \) takes the values \(-39/23, -21/17, -13/9, -53/26, -55/23, -40/17, -55/42 \) respectively. And,

(d) If \( \{A_i\} \) are the maximal twigs of \( \Gamma_1 \), not containing \( w \), then \( \sum_{b \in \Gamma_1} (A_i) \leq -7/6 \).
PROOF. Since $f_1$ is nonrational, by 5.4, it follows that $I_1$ should have one of the configurations shown in Figure 3. By 6.3 (i) configuration (a) is ruled out easily. For the same reason, in (b) it follows that the tip $u$ must be same as $w$, i.e. $u = w$. And then if $v_1 \in I_1$ is the branch point, the twigs of $I'$ at $v_1$ (which are the twigs of $I_1$ at $v_1$ not containing $w$) should have coprime discriminants. Thus it is easily seen that $I_1$ should be either (i) or (ii) of Figure 2.

Finally, in (c) of Figure 3, upto symmetry there are two choices for the location of $w$, viz., $w = u_1$ or $w = u_2$. The first choice gives precisely the four possibilities (iii)-(vi) for $I'$, and the second choice precisely the last one in Figure 2. This proves statement (a). Statements (b), (c) and (d) are directly checked.

11.4. LEMMA. Let $I'$ be a unimodular tree $v \in I'$ be a vertex at which $I_1$ is a branch of $I'$ with $w \in I_1$ being joined to $v$ in $I'$. Suppose $I_1$ satisfies the following conditions:

![Figure 2](image1)

![Figure 3](image2)
(i) \( \#(\Gamma_1) = 7 \).
(ii) \( Q_w \leq -2 \) and \( Q(\Gamma_1 - \{ w \}) = \{-2, -2, -2, -2, -2, -2 \} \) or \( \{-3, -2, -2, -2, -2, -2 \} \).
(iii) \( \Gamma_1 \) is not rational.

Then we have:
(a) \( \Gamma_1 \) is one of the trees listed in Figure 4.A and Figure 4.B.
(b) \( \Gamma_1 - \{ w \} \) is negative definite and we can diagonalize \( \Gamma_1 \), as in 11.2(a), so that if \( p/q \) is the entry at \( w \), then \( d(\Gamma_1) = |p| \).
(c) For \( Q_w = -2 \), \( p/q \) takes the values \( -3/7, -12/13, -4/3, 16/7, 4/13 \) and for \( Q_w = -3 \), \( p/q \) takes the values \( -10/7, -25/13, -3/7, -9/13 \) respectively.
(d) If \( \{ A_i \} \) are maximal twigs of \( \Gamma_1 \) not containing \( w \), then \( \sum \text{bk}(A_i) \leq -16/15 \).

Figure 4.A.
PROOF. We argue exactly as in 11.3. By 5.4, first of all we see that the non-rationality of \( I_1 \) implies that \( I_1 \) has one of the configurations (a)-(h) in Figure 5. We then employ 6.1 repeatedly, taking these configurations, one by one, first to determine the possible locations for \( w \), and then with each such location of \( w \), the possible weights at other vertices.

The configurations (a), (b) and (d) are quickly ruled out. Configuration (c) is ruled out as follows: Upto symmetry, we may assume that \( w = u \). Then the two isolated twigs of \( I_1 \) at \( v_1 \) should have weights \(-2\) and \(-3\) respectively.
This forces the third twig at $v_1$ to be $[2, 2, 2]$ with discriminant even. This contradicts 6.1. Configuration (e) gives the possibilities (15) and (16) of Figure 4. In (f), we first see that $w$ should be one of the four tips, which are symmetrical and hence we may assume $w=u_1$. Then it follows that $\{Q_{u_2}, Q_{u_3}\} = \{-2, -3\}$, and all other weights are $-2$. But now at $v_1$, $\Gamma$ has two branches, viz. $\{u_4\}$ and $\{v_2, v_3, u_4\}$ with even discriminants, contradicting 6.1. Thus configuration (f) is also ruled out.

Consider now, the configuration (g). There are, upto symmetry, three different possible locations for $w$. The possibility $w=v_1$ is ruled out easily. The other two possibilities precisely yield the first three configurations in Figure 4.

Finally, consider the configuration (h). If $\mathcal{Q}(\Gamma_1 - \{w\}) = \{-2, -2, -2, -2, -2, -2\}$, then upto symmetry there is a unique choice for $w$, viz., $w = u$. This then yields configuration (14) of Figure 4. Now let $\mathcal{Q}(\Gamma - \{w\}) = \{-3, -2, -2, -2, -2, -2\}$. Then upto symmetry, there are six different locations for $w$. If $w = u$, then the $(-3)$-curve can be located in six different places. Five of these are listed in (4)-(8), and the sixth one is ruled out easily. For the remaining five locations of $w$, it is easily seen that the location of $(-3)$-curve is uniquely determined, giving the trees (9)-(13). This proves statement (a).

Statements (b), (c) and (d) are directly checked. This completes the proof of Lemma 11.4.

§ 12. The case $e_1 \geq 1$.

12.0. Returning to the proof of 8.2, note that we have $\lambda \geq 2$ (by 9.3), and hence $r_1 = 0$, $D$ is unimodular, $b_2 = \beta_2$ (by 3.2), and $r_3 + e_1 + \sigma + \tau \leq 2$, by (2.8). By (1.3) we now obtain $K \cdot D \leq \beta_2 - 6 = b_2 - 6$. In this section we shall dispose off the case when $e_1 \geq 1$. However, we shall first prove:

12.1. Lemma. Suppose $D$ is unimodular and has no $(-1)$-curves. Then $b_2 = b_2(D) \geq 12$.

Proof. Assume on the contrary that, $b_2 \leq 11$. Since $D$ is unimodular, each connected component of $D$ is also unimodular. Hence, by 6.2, it follows that $D$ is connected. Let $T$ be the dual tree of $D$; $v_0 \in T$ be a vertex such that all branches of $T$ at $v_0$ have less than six vertices. By 4.1, at least one of them, say $T_1$, is not rational. Hence by 5.4, $\#(T_1) = 5$, and by 6.4(a), there exist $u_1, u_2, u_3 \in T_1$ which are tips of $T$ such that $\sum_{u \in T_1} Q_u \leq -10$. It follows that $\sum_{u \in T_1} Q_u \leq -14$. Let $u_4 \in T_1$ be the vertex joined to $v_0$ in $T$. Again, by 4.1, $T - u_4$ should have a branch $\Gamma$ which is non-rational. Since $\#(T_1 - \{u_4\}) = 4$, it follows that $\Gamma$ is disjoint from $T_1 - \{u_4\}$. Clearly $\#(\Gamma) \leq 6$. Since $\sum_{u \in \Gamma} Q_u \leq -14$, and $K \cdot D \leq 5$, it follows that all components in $\Gamma$ are $(-2)$-curves except
perhaps one which may be a $(-3)$-curve. Hence by 6.4, it follows that $#(F)=6$, and $T$ should be the tree shown in Figure 6. But then one easily checks that $T$ is not unimodular. This contradiction completes the proof of the lemma.

![Figure 6. d=149.](image)

12.2. We now claim that $e_1 \leq 1$. Assume the contrary that $e_1=2$. Then we have $r_3=0$ and an equality in (2.8). Hence, by 2.10, $K \cdot D=\beta_2-6$. Let $L_1$, $L_2$ be the two components of $\mathcal{E}_1-D$. Then $L_1 \subseteq R_2$ since $r_3=0=r_4$. We claim that $L_1 \cap L_2 = \emptyset$. For, if $L_1 \cap L_2 \neq \emptyset$, then, we may assume $L_1 = -1$, say, so that $L_1 \cdot L_2 = 1$, $L_2 \leq -2$. Since components of $D$ generate $\text{Pic } X$, $L_2 \cdot D \geq 1$. Clearly $L_1 \cdot D=2$. Hence it follows that $\beta(L_2) \geq 3$ contradicting the observation that $L_2 \subseteq R_2$. So $L_1 \cap L_2 = \emptyset$. Now it follows easily that for the curve $C = D \cup L_1 \cup L_2$, we have $b_1(C) \leq 2$, $K \cdot C \geq K \cdot D -2 = \beta_2 - 8$, and $b_2(C) = b_2(D) + 2 = \beta_2 + 2$. Hence

$$M(X, C) \leq 4\beta_2 + 2 - 1 - 4 - 3(\beta_2 + 2) - (\beta_2 - 8) = -1$$

contradicting (1.6). Hence $e_1 \leq 1$ as claimed.

For the rest of the section we shall assume that $e_1=1$ and hence $b_0+\lambda+\sigma + \tau + r_3 \leq 4$. Let $L_0$ be the unique component of $\mathcal{E}_1-D$.

12.3. Suppose first that $L_0 \subseteq R_2$. Then $r_3 \geq 1$ and hence $r_1=1$, $b_3=1$, $\lambda=2$, $\sigma+\tau=0$ and we have equality in (2.8). Hence by 2.10, $K \cdot D=\beta_2-6$. Let $D_1$, $D_2$, $D_3$ be the three distinct components of $D$ such that $L_0 \cdot D_i = 1$. Suppose first that all the three points $L_0 \cap D_i$ are distinct. Then we claim that $D_i \leq -3$, for $i=1, 2, 3$. If not, say, $D_1 = -2$. Then, it follows that $D_1 \subseteq \mathcal{E}_1$ and hence $D_1 \subseteq R_2$. This means that $D_1$ is an isolated component of $D$ and hence $1=b_2 \geq 2$ which is absurd. It follows that after blowing down $L_0$, we obtain the surface $X'$. Indeed we claim that $X'=X^\sigma$. For, first of all $\sigma=0$ implies, by 3.1, that $\mathcal{E}_2 \subseteq D'$ has at most one component. Clearly this should be one of $D'_1$, $D'_2$ or $D'_3$. But we have just seen that $D'_1 \leq -3$ and hence $D'_2 \leq -2$. Hence $X'=X^\sigma$. In particular, it follows that $\beta_2=11$ and $D$ has no $(-1)$-curves. By the lemma above, this contradicts the unimodularity of $D$.

Hence the three points $L_0 \cap D_i$, $i=1, 2, 3$, cannot be all distinct. Since $D$ is NC all the three points cannot coincide. Hence it must be the case that $L_0 \cap D_1 = L_0 \cap D_2 \neq L_0 \cap D_3$, say. But then for $C = D \cup L_0$, we have $b_1(C) = 1$, $b_2(C) = \beta_2 + 1$, $K \cdot C = K \cdot D - 1 = \beta_2 - 7$ and hence
We now note that $C$ has tips (since $D$ is not linear) and hence by 1.4, and 1.6, we have a contradiction.

Thus, we have shown that $L_0 \supseteq R_2$.

12.4. Now let $D_1$ and $D_2$ be the components of $D$ such that $D_i \cdot L_0 = 1$. We shall first show that $L_0 \cap D_1 \neq L_0 \cap D_2$. For, if $L_0 \cap D_1 = L_0 \cap D_2$, then for $C = D \cup L_0$, $b_2(C) = 0$, $b_2(C) = \beta_2 + 1$, and $K \cdot C = K \cdot D - 1 \geq \beta_2 - 8$ by 2.6 and hence

$$M(X, C) \leq 4\beta_2 - 1 - 4 - 3(\beta_2 + 1) - (\beta_2 - 8) = 0$$

which again leads to a contradiction as above. Hence $L_0 \cap D_1 \neq L_0 \cap D_2$. Here again, if $K \cdot D \geq \beta_2 - 6$, then, as above we will have $M(X, C) \leq 0$ which leads to a contradiction. So we assume that $K \cdot D \leq \beta_2 - 7$. If $b_0 = 1$, from (2.10) we see that there is no equality in (2.8) and hence $\sigma + \tau + r_3 = 0$. If $b_0 > 1$, we get $b_0 + \lambda + \sigma + \tau + r_3 \leq 4$. Thus $b_0 = 0$, $\lambda = 0$, $r_3 = 0$, $\sigma = 0$ and $\tau = 0$. Now by 3.1, it follows that $n_2 \leq 1$, and $e_2 \subseteq D'$.

12.5. Suppose $n_2 = 1$. Let $E_i = \{E_i\} \subseteq D_i$, $E_i \cdot L_0 = -1$. Since $\tau = 0$, there should exist $\{D_i\}_{1 \leq i \leq s}$ in $D'$, such that $E_i \cdot D_i = 2$, $E_i \cdot D_i = 1$, $2 \leq i \leq s$. Clearly, after blowing down $E_i$, we obtain the minimal surface $X'$. So one can easily see that $K' \cdot D' = s + \lambda = s + 2$ and hence

$$M(X', D') \leq 4s + 1 - b_0 - 4 - 3(s + 2) = 4 - s - b_0.$$ 

Further, since $D$ is not linear, it follows that $D'$ has tips. Hence $M(X', D') > 0$, by 1.6, and 1.4. Hence $s = 1$, $b_0 = 1$. Note that $E_i'$ should meet $D_i$ in two distinct points (otherwise $r_3 > 0$).

Thus on $X'$, all components of $D^*$ are smooth except $D_i$ which has a node at $\pi_d(E_i') = x$, and all the singularities of $D^*$ are ordinary double point singularities. By 3.1 (i), it follows that $L_0 \supseteq \pi^{-1}(x)$. So if $F$ is the singular fibre of $\varphi$, containing $L_0$ it follows that $F - L_0 \supseteq D$, by 9.3. Hence, the fibre $P$ of $\varphi^*$, through $x$ is contained in $D^*$. Since no other component of $D^*$ passes through $x$, we conclude that $P_{\text{red}} = D_i$. But now we can appeal to 9.7 to conclude that $b_0(D^*) \geq 2$ which is absurd, since $b_0(D^*) = b_0(D') = 1$. Thus we have shown that $n_2 = 0$, i.e., $X' = X^*$.

12.6. It follows that $D'$ is NC, $b_0(D') = b_0(D \cup L_0) = 1$, $b_0(D') = 11$, $\lambda = 2$. Let now $\pi(L_0) = x \in X'$. Then by 9.3, it follows that the fibre $P$ through $x$ of $\varphi^*(=\varphi')$ is contained in $D^* = D'$. Since $b_0(D^*) = 1$, by 9.7, we conclude that $P$ is of type $\Pi^*$. Further more, if $H_i^*$ and $H_2^*$ are the two horizontal components, then since $\lambda = 2$, we have $H_i^* \cdot K^* = 1$, $i = 1, 2$. Hence $H_i^* \cdot P = 6$. Again, using the fact that $D' = D^*$ is NC and $b_1(D^*) = 1$, we infer that one of the horizontal
Figure 7. The fibre $P=\Pi^*$. 

Figure 8.
components say $H'_1$ meets $D''_1$ transversely, where $D''_1$ is the component of $P$ with $\mu(D''_1)=6$. The other component $H'_2$ will meet two (not necessarily distinct) components $D'_1$ and $D'_2$ of $P$ transversely, so that $\mu(D'_1)+\mu(D'_2)=6$. This yields eight possible configuration for the dual graph of $D'$, as shown in Figure 8. Consider the configuration (a). It follows that if $D_0$ is the proper transform of $D''_0$ on $X$, then $D$ has two branches $[3]$ and $[2,2]$ at $D_0$ both having discriminant 3. This contradicts 6.1. Thus configuration (a) is ruled out. Exactly for the same reason, (b), (c), (d) and (e) are also ruled out. In (f), $D$ will have two branches $[3]$ and $[2,2,2,2,2]$ at $D_0$ with discriminants 3 and 6 respectively. This again contradicts 6.1. In (g) and (h), we consider the curve $C=D''-D''_0$. Then $b_0(C)=3, b_1(C)=0, b_2(C)=10, K \cdot C=2$, and so

$$M(X'', C) = 40 - 3 - 4 - 30 - 2 = 1.$$ 

If $T_1, T_2$ and $T_3$ denote the components of $C$, then by 10.4, $bk(C)=\sum bk(T_i) < -2 - 4/3 - (1/2 + 1/2 + 1/2) < -4$. Hence by 1.6 and 10.5, we have,

$$0 \leq M(X'', C) + \frac{1}{4} N^2 \leq 1 + bk(C) < 1 - 1 = 0$$

which is absurd.

This proves that $e_i=0$.

§ 13. The case $r_3=0$.

13.0. We shall now dispose of the case $r_3=0$. Since $D$ is MNC, $r_3=r_4=e_1=0$ implies that $n_1=0$, i.e., $X=X'$, and $D$ is free from $(-1)$-curves. By 12.1, it follows that $\beta_2=b_2\geq 12$ and hence $n_2\geq 2$. Hence, by 3.1, it follows that $\sigma\geq 2$ and hence $\sigma=2$, and $n_2=2$. Let $E_i=\{E_1, E_2\}$.

13.1. Suppose $E'_1 \cap E'_2 = \emptyset$. Then, it follows that $E'^2_1=E'^2_2=-1$. Since $\tau=0$ (and $n=2$), $m_{1,i}=2$ for all $(t, i)$. Hence both $E'_i$ are outside $D$. Also it follows that there exist $\{D'_{i,j}\}_{1 \leq j \leq s_{t_i}} \subseteq D'$ such that $E'_i \cdot D'_{i,j}=2, E'_i \cdot D'_1=1, 2 \leq j \leq s_t, s_t \geq 1$, for $i=1, 2$. Note that for a fixed $i, D'_{i,j}$ are distinct. Now, there is an equality in (2.8) and hence $K \cdot D=\beta_2-6=6$. On the other hand using the fact that $\lambda=2$, one also computes easily that $K \cdot D=s_1+s_2+4$. Hence $s_1=s_2=1$. Now for the curve $C=D' \cup E'_1 \cup E'_2$, we have $b_0(C)=1, b_1(C)\leq 2, b_2(C)=b_2(D')+2=14$ and $K \cdot C=4$. Hence,

$$M(X', C') \leq 48 + 2 - 1 - 4 - 42 - 4 = -1$$

contradicting (1.6).

13.2. Hence $E'_1 \cap E'_2 \neq \emptyset$ and we may assume that $E'^1_1=-1, E'_1 \cdot E'_2=1, E'^2_2=-2$. Again by 3.1, $E'_1 \cap D'$, and there exist components $\{D'_{i,j}\}$ of $D'$ such that $E'_1 \cdot D'_{i,j}=2, E'_i \cdot D'_1=1, 2 \leq j \leq s_t, (s_t \geq 1)$, and $E'^2_2 \cdot D'_1=1$, for $1 \leq j \leq s_2, s_2 \geq 0$. 


Now for the curve \( C = D' \cup E_1 \cup E_2 \), it follows that, \( b_0(C) = 1, b_1(C) \leq s_1 + s_2, b_2(C) = 14, \) and \( K \cdot C = 2s_1 + s_2 + 2 \). Hence,

\[
M(X', C) \leq 48 + s_1 + s_2 - 1 - 4 - 42 = (2s_1 + s_2 + 3) < 0
\]

contradicting (1.6).

Thus, we have shown that \( r_i \geq 1 \).

§ 14. The case \( r_i = 1 = \sigma \).

14.0. So far we have proved that \( r_i = \sigma_i = 0 \) and \( r_i \geq 1 \). Hence \( \sigma \leq 1 \). In this section we shall show that \( \sigma = 0 \). Assuming that \( \sigma = 1 \), we have \( r_i = 1, b_0 = 1, \lambda = 2, \tau = 0, b_2 = \beta_2, K \cdot D = \beta_2 - 6, \) etc. from (2.8). Let \( L_0 \) be the unique \((-1)\)-curve in \( R_3 \). Note that since \( \epsilon_1 = 0 \), \( \{ L_0 \} = R_3 \setminus D \). Let \( L_1, L_2, L_3 \) be the three components of \( D \) such that \( L_0 \cdot L_i = 1 \).

14.1. Suppose \( L_i \leq -3, i = 1, 2, 3 \). Then, it follows that after blowing-down \( L_0 \), we obtain \( X' \). Now \( \sigma = 1 \) implies, by 3.1, that \( \mathcal{E}_2 = \{ E_i \} \) is not contained in \( D \), and there exist components \( \{ D_i \} \) of \( D' \) such that \( E'. \cdot D_i = 2 \), and \( E'_i \cdot D_i = 1, 2 \leq j \leq s, s \geq 1 \). It follows that \( \beta_i(X') = 11 - \beta_i(D') \). Since \( \lambda = 2 \), one easily computes that \( K' \cdot D' = s + 3, K \cdot D = s + 5 = 8 - 6 = 6 \). Hence, \( s = 1 \). Take \( C' = D' \cup E' \). Then \( K' \cdot C' = 3, b_1(C') \leq 1 \), and hence

\[
M(X', C') \leq 44 + 1 - 1 - 4 - 36 - 3 = 1
\]

Now consider the (nontrivial) contraction \( \varphi_1 : X \to X' \). After blowing up at \( E' \cap D'_i \) if necessary, we obtain a contraction \( \alpha : \tilde{X} \to X \) such that \( \tilde{C} = \alpha^{-1}(C) \) is NC. Then \( M(\tilde{X}, \tilde{C}) \leq M(X, C) \). And as in the proof of 1.7 we see that \( M(X, C) \leq M(X', C') - 1 \leq 0 \). Hence \( M(\tilde{X}, \tilde{C}) \leq 0 \). On the other hand since \( C \) has tips so does \( \tilde{C} \). Hence by (1.4) and (1.6) \( M(\tilde{X}, \tilde{C}) > 0 \), which is absurd.

Thus, it follows that \( L_i = -2 \), for some \( i = 1, 2, 3 \), say \( L_2 = -2 \). Since \( r_2 = 1 \), it follows that \( L_2 \subseteq R_3 \) and hence \( L_3 \) should be a tip of \( D \). Clearly \( L_3 \leq -3 \), and

\[
L_3 \leq -3.
\]

14.2. Now, suppose that \( L_2 \leq -4, L_3 \leq -4 \). Then, it follows that after blowing down \( L_0 \) and then \( L_2 \), we obtain \( X' \). Now arguing exactly as in 14.1, we obtain \( \{ E'_1 \} = \mathcal{E}_2, E'_i \not\subset D' \) etc., and for \( C' = D' \cup E'_i \), \( M(X', C') \leq 1 \) which leads to a contradiction, as above. Hence we may assume that \( L_2 \leq -3 \), so that the image \( L'_2 \) of \( L_2 \) on \( X' \), is a \((-1)\)-curve. Note that \( L'_1 \cdot L'_2 = 2, L'_1 \cap L'_2 = \{ x \} \), say.

By 3.1, it follows that \( \mathcal{E}_2 \) consists of one more component, besides \( L'_2 \), say, \( \mathcal{E}_2 = \{ E'_i, L'_4 \} \) and we must have one of the following two cases:

(a) \( E'_i \) and \( L'_4 \) are disjoint, \( E'_i \leq -1 \), and \( E'_i \not\subset D' \), or

(b) \( E'_i \cdot L'_4 = 1, E'_i \leq -2 \), and \( E'_i \subset D' \).
14.3. Consider the case (a) above. It follows that there exist components \( \{D_{i,j}\}_{1 \leq j \leq s_1} \) of \( D' \) with \( E_i \cdot D_{i,j} = 2 \), \( E_i \cdot D_{i-1,j-1} = 1 \), \( 2 \leq j \leq s_1 \), \( s_1 \geq 1 \). Let \( \{D_{i,j}\}_{1 \leq j \leq s_2} \) be the components of \( D' \), such that \( L_i \cdot D_{i,j} = 1 \). (Clearly \( D_{i,j} \neq L_i \).) Then one easily computes that \( K' \cdot D' = 2 + (s_1 - 1) + 2 + s_2 + 2 - 1 = s_1 + s_2 + 4 \). Again take \( C' = D' \cup E_i \). Then \( K' \cdot C' = K' \cdot D' = s_1 + s_2 + 4 \), \( b_1(C') \leq s_1 \) and hence \( M(X', C') = 48 + s_1 - 1 - 4 - 39 - (s_1 + s_2 + 3) = 1 - s_2 \leq 1 \). This again leads to a contradiction as in 14.1, using the fact that \( C' \) has tips.

14.4. Consider the case (b). Let now, \( \{D_{i,j}\} \) and \( \{D_{i-1,j}\} \) be components of \( D' \) such that \( D_{i,j} \cdot L_i = 1 \), \( D_{i,j} \neq E_i \), \( 1 \leq j \leq s_2 \), \( s_2 \geq 0 \). Again, one easily computes that \( K \cdot D = s_1 + 2s_2 + 8 = 8 \), and hence \( s_1 = 0 = s_2 \). In particular, it follows that \( E_i \), the proper transform of \( E_i \) on \( X \) is a \((-2)\)-curve and is a tip of \( D \), and \( L_i \) intersects only \( E_i \) and \( L_i \). Thus the curve \( C = D - \{L_i\} \) has three connected components \( T_1, T_2, T_3 \) say, where \( T_3 = \{L_3\} = \{2\} \), and \( T_2 = \{L_2 \cup E_i\} = \{2, 3\} \). On the other hand we have:

\[
M(X, C) = 56 - 3 - 4 - 39 - 9 = 1.
\]

Hence, by 10.5, and 1.6, \( b_1(C) \geq -4 \). On the other hand \( b_1(C) = \sum b_1(T_i) = b_1(T_1) - 7/5 = 2. \) We shall presently show that \( b_1(T_1) < -3/5 \) which leads to a contradiction.

As before, by 4.1, it follows that \( T_1 \) is non-rational, and hence has at least three tips. One easily sees that \( L_i \) is horizontal and so \( L_i \cdot K^* > 0 \). Since there is another horizontal component in \( D^* \), say, \( L_i \), it follows that \( L_i \cdot K^* = L_i \cdot K^* = 1 \). It follows that \( L_i = -9 \) and \( L_i = -3 \), and all other components of \( T_1 \) are \((-2)\)-curves. Hence by 10.4, \( b_1(T_1) \leq -1/9 + 1/3 + 1/2 < -3/5 \) as required.

This completes the proof of the claim \( \sigma = 0 \).

14.5. Remark. At this stage, it is not hard to see that there are only finitely many possibilities for the dual graph \( T \) of \( D \).

§ 15. The case \( r_3 = 1 \).

15.0. To sum-up, so far, we have proved that \( r_3 = e_1 = \sigma = 0 \) and \( r_3 \geq 1 \). Of course, \( 3 \leq \lambda \leq 2 \), \( 2 \leq \beta_2 \), \( \beta_2 \leq 1 \), \( b_0 + r \cdot \tau + \lambda \leq 5 \). In this section we shall dispose of the case \( r_3 = 1 \). So, assume now that \( r_3 = 1 \), so that \( b_0 = \tau + \lambda \leq 4 \). Since \( \sigma = 0 \), by 3.1, there is a unique \((-1)\)-curve \( L_0 \) on \( X \), \( \{L_0\} = R_3 \subset D \) and there are precisely three components of \( D \) which meet \( L_0 \), say, \( L_i, L_j, L_k \).

Again, by 3.1, it follows that either \( \epsilon_1 = \emptyset \) or \( \epsilon_1 = \{E_i\} \subset D' \). In 15.1-15.4 we shall show that \( \epsilon_1 \) has at least two components. Equivalently, we will show that one of the \( L_i, \ i = 1, 2, 3 \), is a \((-2)\)-curve. Then in 15.5-15.9 we will show that \( \epsilon_2 \neq \emptyset \) and in the rest of the section we investigate the case when \( \epsilon_2 \neq \emptyset \).
For a tree $T$ of smooth rational curves let us introduce the notation:

$$\lambda(T) := -\sum_{c \in T}(C^2+2) = \sum_{c \in T}(C \cdot K).$$

15.1. So, assume that $L_i \leq -3$ for $i=1, 2, 3$. It follows that after blowing-down $L_i$, we obtain the surface $X' = X^*$, (using 3.1). Since $b_2(D')= b_2(D)=0$, it follows from 9.7 (or otherwise) that not all $L_i$ are $(-3)$-curves, $i=1, 2, 3$. (For, if so, $L_1 \cup L_2 \cup L_3$ will be a full fibre of $\varphi^*$ contained in $D^*$.) Thus $L_i \leq -4$ for some $i=1, 2, 3$.

15.2. Let now $T_1, T_2, T_3$ be the branches of $T$ at $L_0$, with $L_i \subseteq T_i$, $i=1, 2, 3$. As argued in 12.1, it follows from 4.1 that one of the $T_i$, say $T_1$, is non-rational. Since $\lambda \leq 3$, it follows, from 5.4 and 6.4 that $\#(T_i) \geq 6$. Since $\#(T)=b_2(D)=11$, one easily checks that $T_2 \cup \{L_0\} \cup T_3$ is rational. Hence, again by 4.1, it follows that $T_i - \{L_i\}$ should have a branch $T_i'$ which is non-rational, and then again $\#(T_i') \geq 6$. Since $L_i \leq -4$ for some $i$, it follows that $\lambda(T_i') \leq 2$.

15.3. Suppose $\#(T_i')=6$. By Lemma 11.3, it follows that $T_1$ is one of the trees in Figure 2 where $w$ is the vertex joined to $L_1$, in $T_1$. Using 6.1, we list the possibilities for $T_1$. We see that these trees can be diagonalized in such a way that if $p'/q'$ is the entry at $L_1$, then $d(T_i') = \left|\frac{p'}{q'}\right|$, and $p'/q'$ has values $-3/8, -11/8, -11/13, -6/13, -1/7, -1/4, -4/7, -8/7, -5/4, -1/10, -4/15, -11/10 and -5/14. With $p/q$ given by Lemma 11.3, using 11.2, we see that $d(T) = \left|\frac{pp'-qq'}{q'}\right| \neq 1$, for any of these values.

![Figure 9.](image-url)
15.4. Thus we have shown that \(\#(\Gamma_1) \geq 7\). Indeed \(\#(\Gamma'_1) = 7\), \(\#(\Gamma_1) = 8\), so that \(\#(\Gamma_2) = \#(\Gamma_3) = 1\). We can now apply 11.1(a) with \(Y = X', C_i = L_i, i \geq 1\) and \(C_0\) as the image of the curve dual to \(w \in \Gamma_1\). Since \(\lambda \leq 3\), one easily sees that the triple \((a_1, a_2, a_3) = (-L_1^\prime, -L_2^\prime, -L_3^\prime)\) is one of the six triples mentioned in 11.1(a). Since \(K'\) is supported on \(D'\), this leads to a contradiction by 11.4.

Thus as claimed in 15.0, one of the \(L_i\) is a \((-2)\)-curve, say, \(L_3 = -2\). Then clearly \(L_1^\prime \leq -3\); \(L_2^\prime \leq -3\). We shall now show (in 15.5-15.9) either \(L_1\) or \(L_2\) is a \((-3)\)-curve.

15.5. Suppose that \(L_1^\prime \leq -4\), \(L_2^\prime \leq -4\). Clearly, \(R_3 = \{L_0\}\), and so \(T_3 = \{L_3\} = [2]\). After blowing-down \(L_0\) and \(L_3\) we obtain the surface \(X' = X''\). So \(b_2 = 12\). As before, using 9.7, we see that not both \(L_1\) and \(L_2\) are \((-4)\)-curves. By 4.1, either \(T_1\) or \(T_2\), say \(T_1\), is non-rational, and again by 5.4 and 6.4, \(\#(T_1) \geq 6\), so that \(\#(T_2 \cup \{L_0, L_3\}) \leq 6\). So one easily sees that \(T_2 \cup \{L_0, L_3\}\) is rational and hence, again by 4.1, \(T_1 - \{L_1\}\) should have a branch \(\Gamma_1\) which is non-rational and as before \(\#(\Gamma_1) \geq 6\). Let \(u_0 \in \Gamma_1\) be the vertex joined to \(L_1\) in \(T_1\). Since \(L_1^\prime + L_2^\prime \leq -5\), it follows that \(\mathcal{A}(\Gamma_1) \leq \lambda - 1\).

15.6. Suppose further that \(\lambda = 3\). Then we have \(K \cdot D = 12 - 6 = 6\). Hence, for \(C = D - \{L_0\}\), we have \(K \cdot C = 7\), \(b_0(C) = 3\), \(b_1(C) = 0\), \(b_2(C) = 11\). Hence \(M(X, C) = 48 - 3 - 4 - 33 - 7 = 1\). By 1.6 and 10.5, it follows that \(b_k(C) \geq -4\). On the other hand \(b_k(C) = b_k(T_1) + b_k(T_2) - 2\). Below, we shall show in 15.7, that \(b_k(T_1) + b_k(T_2) < -2\) which is absurd, thus proving that \(\lambda = 2\).

15.7. Clearly, \(\{-L_1^\prime, -L_2^\prime\} = \{2, 3\}, \{3, 3\}, \{2, 4\}\). \((\mathcal{A}(\Gamma'_1) > 0, \text{for otherwise } \Gamma'_1 \text{ will be contained in a fibre of } \varphi^*, \text{hence rational. Thus } \{-L_1^\prime, -L_2^\prime\} \neq \{3, 4\}.\) Accordingly, \(\{-L_1^\prime, -L_2^\prime\} = \{4, 5\}, \{5, 5\}, \{4, 6\}\). Thus it follows that the weight set \(\mathcal{Q}(T_1 \cup T_2)\) is one of the following:

\[
\{-5, -4, -3, -2, -2, -2, -2, -2\}, \\
\{-5, -4, -4, -2, -2, -2, -2, -2\}, \\
\{-5, -5, -3, -2, -2, -2, -2, -2\} \text{ or} \\
\{-4, -6, -3, -2, -2, -2, -2, -2\}.
\]

Suppose \(T_1 \cup T_2\) has at least six tips. Then by 10.4 it follows that \(b_k(T_1 \cup T_2) \leq -\sum_{i=1}^{l} (1/a_i)\), for some \(\{a_1, \cdots, a_l\} \subseteq \mathcal{Q}(T_1 \cup T_2)\), which is easily seen to be less than \(-2\). So we may assume that \(T_1 \cup T_2\) has fewer than six tips.

Suppose \(\#(\Gamma_1) = 6\). Then by 11.3, it follows that \(T_1\) has at least 4 tips. Hence \(t_3 = \#(T_2) = 1\). This means \(T_3 = [5]\), \(b_k(T_2) = -4/5\). And it is easily checked that \(b_k(\Gamma_1) < -6/5\) as desired. So we may assume that \(\#(\Gamma_1) \geq 7\).
Suppose \( \lambda(T_1) \leq 1 \). If \( t_2=2 \) then by 11.4, it follows that \( T_1 \cup T_2 \) has at least six tips, except in the case (1) or (2) of 11.4. In these two cases we directly check that \( d(T) \neq \pm 1 \), for various possibilities of \( T \). If \( t_2=1 \), then \( T_2=[5] \) and one easily verifies that \( bk(T_1) < -6/5 \). Hence \( \lambda(T_1)=2 \).

Now let \( t_2 \neq 1 \). Since \( \lambda(T_1)=2 \) it follows that \( \{-L_1^w, -L_2^w\} = \{2, 3\} \) and hence we can apply 4.3(2) with \( L_1^w = C_3, L_2^w = C_2 \), to conclude that \( I_1' - \{u_0\} \) should have branch \( I_1' \) which is not rational, where \( u_0 \in T_1 \) is the vertex joined \( L_1 \) in \( T_1 \); \( \#(I_1') \geq 6 \), and hence \( \#(I_2') = 6 \). Again, by Lemma 11.3, it follows that \( (T_1 \cup T_2) \) has at least six tips. Hence we must have \( t_2=1 \).

It follows that \( T_2 = \{L_2\} = [5] \), so that \( bk(T_2) = -4/5 \). Thus we have to show that \( bk(T_1) < -6/5 \). Looking at \( \mathcal{Q}(T_1) \), this is obvious if \( T_1 \) has at least four tips. Thus, since \( T_1 \) is non-rational, we have to consider the case when \( T_1 \) has precisely three tips. Let \( A_1, A_2, A_3 \) be its maximal twigs. Since \( T_1 - \{L_1\} \) is also non-rational, we may assume that \( L_1 \in A_1 \). Further, if \( A_2 \) and \( A_3 \) consist of only \((-2)\)-curves then by 6.1, it follows that \( \#(A_2) + \#(A_3) \geq 3 \) and hence \( bk(A_2) + bk(A_3) \leq -16/15 \) so that \( bk(T_1) \leq -(1/2 + 2/3 + 1/6) < -6/5 \). So we may assume that \( A_2 \) has a \((-3)\) or \((-4)\)-curve. In particular, it follows that \( L_1^w = -6 \), i.e. \( L_1^w = -2 \) or \(-3 \).

Thus we apply 4.3(1) with \( Y=X^w, C_3 = L_2^w \) and \( C_2 = L_1^w \). Along with 4.1, this implies that \( I_2 = T_1 - \{L_1, u_0\} \) is nonrational where \( u_0 \) is the vertex joined to \( L_1 \), in \( T_1 \). Note that \( \#(I_2') = 7 \) and hence by 5.4 it follows that \( T_1 \) itself has the configuration as shown in Figure 10. Further if \( L_1^w = -5 \), then \( \lambda(I_2') \leq 1 \) and so, by 11.4 \( I_2' \) is (1) or (2) of Figure 4. Hence \( bk(A_2) + bk(A_3) \leq -16/15 \) and hence \( bk(T_1) \leq -(1/2 + 2/3 + 1/6) < -6/5 \). Finally, if \( L_1^w = -4 \), then it follows that \( A_2 = [4, 2], [3, 3] \) or \([3, 2] \), and hence \( A_3 = [2, 2] \) always (using 6.1). But then \( bk(T_1) \leq -(1/4 + 2/7 + 2/3) < -6/5 \) as desired, in 15.6.

15.8. Thus we have \( \lambda = 2 \), and so \( \lambda(T_1) \leq 1 \). Suppose, now that \( \#(I_1') = 6 \). Then by 11.3, \( I_1' \) should be as in (iii) of Figure 2, with \( u_0 = w, Q_w = -2 \). \( d(I_1') = 4 \), and \( \lambda(I_1') = 1 \). Hence we have \( \{L_1^w, L_2^w\} = \{-2, -3\} \), i.e. \( \{L_1^w, L_2^w\} = \{-4, -5\} \), \( L_3^w = -1 \) and all other components of \( I_1' = T - I_1 \) are \((-2)\)-curves. Also, since \( d(I_1') = 4 \), by 6.1, all other branches of \( T \) at \( L_1 \) should have odd discriminants. Using 6.1, at other vertices of \( T \) also, it follows that the only possibility for \( T \) is the one shown in Figure 11. But one easily checks that even this tree is not unimodular. Hence \( \#(I_1') \geq 7 \).
15.9. We now apply 4.3, with $Y=X^e$, $C_1$ as the component dual to $u_0 \in \Gamma_1$, $C_2=L_1^\prime$ and $C_3=L_2^\prime$. It follows that $\Gamma_1-\{u_0\}$ has a branch $\Gamma_2$ which is non-rational, $\#(\Gamma_2) \geq 6$. Let $u_1 \in \Gamma_2$ be the vertex joined to $u_0$. If $\#(\Gamma_2) = 6$, then $\Gamma_2$ is as in (iii) of Figure 2 with $u_1 = w$, $Q_w = -2$, $d(\Gamma_2) = 4$. Arguing as in 15.8 it follows that $\Gamma_2^\prime = T - \Gamma_1$ is one of the configurations shown in Figure 12. We can diagonalize each of these tree, with the diagonal entry $p'/q'$ at $u_0$ taking the values $1/1$, $+1/2$ and $+4/1$ respectively, where as the diagonal entry $p/q$ at $u_1 = w$, for $\Gamma_2$ is $-4/9$. Hence $d(T) = |pp'-qq'| \neq 1$, which proves that $\#(\Gamma_2) \geq 7$.

Thus $\#(\Gamma_2) = 7$; $\#(T_1) = 9$, so that $L_2$, $L_3$ are tips of $T$ and hence by 6.1, $L_2^\prime$ is odd and so $L_2^\prime = -5$. Lemma 11.4 gives various possibilities for $\Gamma_2$ with $w = u_1$. Since $\Gamma_2$ is non-rational, $\lambda(\Gamma_2) > 0$ and hence $\lambda(\Gamma_2) = 1$. Then $\Gamma_2^\prime$ should be the tree in Figure 13. We can diagonalize this with diagonal entry $p'/q'$, at $u_0$ given by $p'/q' = -1/2$. Using the value of $p/q$ from 11.4, we see that $d(T) = |pp'-qq'| \neq 1$, for any of the values of $p/q$. This contradiction, then proves that either $L_1^\prime = -3$ or $L_2^\prime = -3$, as claimed in 15.4.

15.10. Thus in the remaining paragraphs we shall assume that $L_2^\prime = -3$. It follows that after blowing down $L_0$, $L_3$ and $L_2$ successively, we obtain $X^e$. The image $L_1^\prime$ of $L_1$ is a rational curve with a cusp; $L_1^\prime + L_2^\prime + K^e = 0$. We claim that either $\lambda = 3$ and $L_2$ is a tip of $D$ or $\lambda = 2$ and $L_2$ is linear in $D$ (i.e., $L_2$.
meets at most two components of \(D\). For if \(L_2\) meets \(s\) components of \(D\), then one easily computes that \(K \cdot D = \lambda + s + 3\). On the other hand \(K \cdot D \leq K_2 - 6 = 7\) and \(K \cdot D = K_2 - 6\) if \(\lambda = 3\) (by 2.8 and 2.10). Hence \(s \leq 4 - \lambda\) which proves the claim.

In particular, through the cusp of \(L''_2\) there passes at most one other component of \(D^*\). Hence by 9.7, we conclude that \(L''_2\) is horizontal; \(L''_2 \cdot K^* = 1\) or 2. Accordingly, \(L''_2 = -7\) or \(-8\). In 15.11, below we shall dispose of the case \(\lambda = 3\).

15.11. Assume now that \(\lambda = 3\). As seen above, \(L_2\) is a tip of \(D\). We now apply (1.3) to \(C = D - \{L_0\}\). Letting \(T_1, T_2, T_3\) denote the three branches of \(C\), as before, this yields; \(-4 \leq \text{bk}(C) \leq \text{bk}(T_1) - 4/3 - 2\), and hence \(\text{bk}(T_1) \geq -2/3\).

On the other hand, it follows that \(L''_2\) is horizontal. Since \(\lambda = 3\) and there is at least one more horizontal component of \(D^*\), which is of course a smooth rational curve, it follows that the weight set \(Q(T_1)\) is one of the following:

\[
\begin{align*}
\{-8, -3, -2, -2, -2, -2, -2, -2, -2\}, \\
\{-7, -4, -2, -2, -2, -2, -2, -2, -2\} \text{ or} \\
\{-7, -3, -3, -2, -2, -2, -2, -2, -2\}.
\end{align*}
\]

Moreover \(T_1\) is non-rational and hence it has at least three tips. Hence, as seen before, \(\text{bk}(T_1) < -2/3\). This contradiction shows that \(\lambda = 2\).

15.12. So, from now on, we have \(\lambda = 2\). In 15.12-15.14, we shall show that \(L_2\) is a tip of \(D\). Assuming the contrary, as seen in 15.10, it follows that \(L_2\) will meet another component, say \(L_4\), \(L_2 \cdot L_4 = 1\), and \(L_4 = -3\) or \(-4\); \(L_1 = -7\) (because there is one more horizontal curve) \(K \cdot D = \lambda + s + 3 = 7\). Hence for \(C = D - \{L_0\}\), by (1.3), we obtain, \(\text{bk}(C) \geq -4\). We shall show that \(\text{bk}(T_1) + \text{bk}(T_2) \leq -2\) thereby arriving at a contradiction and proving that \(L_2\) is a tip.

Note that the weight set

\[
Q(T_1 \cup T_2) = \{-7, -4, -3, -2, -2, -2, -2, -2, -2\} \text{ or} \\
\{-7, -3, -3, -2, -2, -2, -2, -2, -2\}.
\]

Hence, if \(T_1 \cup T_2\) has more than five tips then clearly \(\text{bk}(T_1) + \text{bk}(T_2) < -2\) (by 10.4). So we shall assume that \(T_1 \cup T_2\) has at most five tips and hence at least one of them is linear. On the other hand, at least one of them is non-rational and hence \(T_1 \cup T_2\) has at least four tips.

15.13. Assume first that \(T_2\) is linear. Then \(T_2\) has two tips and so \(T_1\) has three tips, \(T_1 - \{L_1\}\) has a non-rational branch \(T_1\), and hence \(T_1\) also has three tips. By 11.3, it follows that \(#(T_1) \geq 7\), and if \(#(T_1) = 7\) then by 11.4, it is either (1) or (2) of Figure 4, with \(w \in T_1\) being the vertex joined to \(L_1\) and
\( \chi(T_1) = 1 \). It follows that \( T_1 = [3, 3, 2] \) and hence \( \chi(T_1) = -15/13 \), by 10.4. By 11.4, we have \( \chi(T_1) \leq -16/15 \) and hence \( \chi(T_1) + \chi(T_2) < -2 \) as required. Thus \( \#(T_1) = 8 \). Again \( T_1 \) should have a \((-3)\)-curve and so \( T_1 = [3, 3], d(T_1) = 8 \). This contradicts 6.1.

15.14. It follows that \( T_1 \) is linear. Note that \( \Gamma_1 = \Gamma_2 - L_2 \) is connected. It should be non-rational and as usual we must have \( \#(\Gamma_1) \geq 6 \). Suppose \( \#(\Gamma_1) = 6 \), so that \( \#(T_1) = 7 \). By 11.3 and 10.4, \( \chi(T_1) \leq -7/6 - 1/3 = -3/2 \). On the other hand, \( \#(T_1) = 4 \), and \( T_1 \setminus \{L_1\} \) consists of \((-2)\)-curves only i.e. \( T_1 = [7, 2, 2, 2] \). Hence \( \chi(T_1) \leq -(1/7 + 1/2) = -9/14 \). Also \( \chi(T_2) = 3 \), \( d(T_2) \) is odd and so \( T_1 \in \{[7, 3, 2], [7, 2, 2]\} \). Hence \( \chi(T_1) \leq -9/14 \) so that \( \chi(T_1) + \chi(T_2) < -2 \).

Now suppose \( \#(\Gamma_1) = 7 \). Then it follows that \( T_1 = [7, 2] \) (since if \( T_1 = [7, 3] \) then \( d(T_1) \) is even). Hence \( \chi(T_1) = -11/13 \). Note that \( L_4 \) is not a tip of \( T_2 \). Hence one of the tips of \( T_2 \) is a \((-2)\)-curve. Hence \( \chi(T_2) \leq -(1/3 + 1/3 + 1/2) = -7/6 \) and again we have \( \chi(T_1) + \chi(T_2) < -2 \).

So finally let \( \#(\Gamma_1) = 9 \), so that \( T_1 = [7] \); \( \chi(T_1) = -4/7 \). We have to show that \( \chi(T_2) < -10/7 \). We can now apply 11.1 (c), with \( Y = X^\ast \), \( C_0 = L_4^\ast \), and \( C_1 = L_1^\ast \). Combined with 4.1, this implies \( \Gamma_1 \setminus \{L_4\} \) is non-rational. Also if \( T_2 \) has four tips then clearly \( \chi(T_2) \leq -(1/3 + 1/3 + 1/2 + 1/2) = -5/3 < -10/7 \). So \( T_2 \) has only 3 tips. Consequently \( \Gamma_1 \setminus \{L_4\} \) has three tips. By 5.4, \( \Gamma_1 \setminus \{L_4\} \) has one of the two configurations in Figure 14. Consequently, \( T_2 \) itself has one of the configurations shown in Figure 15. Note that \( L_4 = -3 \), for otherwise \( L_4 = -4 \) and \( \Gamma_1 \setminus \{L_4\} \) will consist of only \((-2)\)-curves. The rest of the weights are determined by using 6.1.

![Figure 14](image_url)

Figure 14.

Let \( A_1, A_2 \) and \( A_3 \) be the three maximal twigs of \( T_2 \), with \( L_4 \in A_1 \). Then \( \chi(T_2) = \sum_1^n \chi(A_i) \). Now consider Figure 15 (i), in which \( A_1 = [3, 3, 2, 2, 2] \), \( A_2 = [2, 2] \) and \( A_3 = [2, 3] \) or \([3, 2] \). Hence \( \chi(T_2) \leq -(2/5 + 2/3 + 9/23) < -10/7 \) as required. In Figure 15 (ii), if \( A_1 \) and \( A_2 \) consist of only \((-2)\)-curves then \( \chi(A_1) + \chi(A_2) \leq -(3/4 + 2/3) \) where as \( \chi(A_3) \leq -1/3 \). Hence \( \chi(T_2) < -10/7 \) as required. So we may assume that, say \( A_1 \) has a \((-3)\)-curve. Then \( A_3 = [3, 3, 2, 2], \chi(A_3) = -7/18 \). One easily checks that \( \chi(A_1) + \chi(A_2) \leq -(2/5 + 3/4) \).
In Figure 15 (iv), \(b_k(113) = -5/13\), \(b_k(112) = -3/4\) and hence \(b_k(T_2) < -10/7\). Finally in Figure 15 (iii), one checks that \(b_k(T_2) < -10/7\) unless \(A_1 = [3, 2, 2]\), \(A_2 = [2]\). In this last case one directly computes \(d(T)\) and sees that \(T\) is not unimodular.

Thus the claim in 15.12 that \(L_2\) is a tip has been proved.

15.15. As before, we have a branch \(\Gamma_1\) of \(T_1 - \{L_1\}\) which is non-rational and \(\#(\Gamma_1) \geq 6\); \(\lambda(\Gamma_1) \leq 1\). If \(\#(\Gamma_1) = 6\), by 11.3, \(\Gamma_1\) is the tree (iii) in Figure 2, with \(d(\Gamma_1) = 4\); \(\lambda(\Gamma_1) = 1\). Another branch of \(T\) at \(L_1\) is \(L_0 \cup L_3 \cup L_2\), and the rest consists of three \((-2)\)-curves, all contained in \(T_1\). Thus it is easily seen that there will be one more branch of \(T\) at \(L_1\) with even discriminant which contradicts 6.1.

Hence \(\#(\Gamma_1) \geq 7\). Suppose \(\#(\Gamma_1) = 7\). Then by 11.4, the possibilities for \(\Gamma_1\) are given in Figure 4, with \(w \in \Gamma_1\) being the vertex joined to \(L_1\). If \(\lambda(\Gamma_1) \leq 1\), it follows that \(\Gamma' := T - \Gamma_1\) is as shown in Figure 16, (using 6.1) which can be diagonalized with diagonal entry at \(L_1 = -1/3 = (p'/q')\). With the values of \(p/q\) given by 11.4, we see that \(|p'q - q'p| \neq 1\) for any of these possibilities. If \(\lambda(\Gamma_1)\)
=0, then \( \Gamma_1 \) is the unique configuration (14) of Figure 4 and it is easily verified that \( d(T) \neq 1 \).

15.16. Hence \( \#(\Gamma_1) \geq 8 \). We can now apply Lemma 4.4, to conclude that \( \Gamma_1 - \{u_0\} \) should have a branch \( \Gamma_2 \) which is non-rational where \( u_0 \) is the vertex of \( \Gamma_1 \) joined to \( L_1 \). Let \( u_1 \in \Gamma_2 \) be the vertex joined to \( u_0 \). As before, \( \#(\Gamma_2) \geq 6 \), and if \( \#(\Gamma_2) = 6 \), then \( \Gamma_2 \) is the configuration (iii) of Figure 2 with \( u_1 = w \), \( Q_w = -2 \), and \( p/q = -4/9 \). Also \( \Gamma'_2 = T - \Gamma_2 \) is one of the configurations shown in Figure 17. Neither of these possibilities yields a unimodular \( T \) and hence \( \#(T) \geq 7 \).

\[
\begin{array}{ccc}
-3 & -1 & -7 \\
\downarrow & \downarrow & \downarrow \\
-2 & -2 & -2 \\
\end{array}
\quad
\begin{array}{ccc}
-3 & -1 & -7 \\
\downarrow & \downarrow & \downarrow \\
-2 & -2 & -2 \\
\end{array}
\]

Figure 17.

15.17. Suppose \( \#(\Gamma_2) = 7 \). Then 11.4 gives the possibilities for \( \Gamma_2 \) with \( w = u_1 \) etc. as before whereas one easily sees that \( \Gamma'_2 = T - \Gamma_2 \) is one of the configurations shown in Figure 18 (again using \( \lambda(\Gamma_2) = 1 \)), with the diagonal entry \( p'/q' \) at \( u_0 \) taking the values \( p'/q' = 0/1, -1/2 \) respectively. One easily checks that none of these yield \( |pp' - qq'| = 1 \) and so \( \#(T) \geq 8 \).

\[
\begin{array}{ccc}
-3 & -1 & -7 \\
\downarrow & \downarrow & \downarrow \\
-2 & -2 & -2 \\
\end{array}
\quad
\begin{array}{ccc}
-3 & -1 & -7 \\
\downarrow & \downarrow & \downarrow \\
-2 & -2 & -2 \\
\end{array}
\]

(i) \quad (ii)

Figure 18.

15.18. Hence \( \#(\Gamma_3) = 8 \). Note that \( L_1 \) is a horizontal component and so we shall also denote it by \( H_1 \). Since \( \Gamma_2 \) is non-rational, it follows that \( \Gamma_3 \) contains the other horizontal component \( H_2 \), \( \lambda(\Gamma_3) = 1 \). In particular, \( Q_v = -2 \).

We first claim that all connected components of \( \Gamma_3 - \{u_1\} \) are rational. This is obvious, if each of them has less than 6 vertices. Now suppose \( \Gamma_3 \) is a branch of \( \Gamma_2 - \{u_1\} \) with \( \#(\Gamma_3) \geq 6 \). If \( \#(\Gamma_3) = 6 \), then \( \Gamma_3 \) is not rational would imply, by 11.3, that \( \Gamma_3 \) is as in (iii) of Figure 2 with \( Q_w = -2 \) and hence \( T \) itself will be as shown in Figure 19 which is clearly not unimodular.
When \(\#(\Gamma_3)=7\), the non-rationality of \(\Gamma_3\) implies, by 11.4, that \(\Gamma_3\) is one of the trees shown in Figure 4. Again not all curves in \(\Gamma_3\) are \((-2)\)-curves and so \(\lambda(\Gamma_3)=1\). Hence it follows that \(\Gamma_3'=T-\Gamma_3\) is as shown in Figure 20 with \(p'/q'\) at \(u_1\) equal to \(-1\). With \(p/q\) at \(w\) for \(\Gamma_3\) being given by 11.4 (c),

\[
\begin{align*}
-2 & -1 -7 -2 -2 -2 -2 \\
\text{L}_1 & \quad \text{u}_0 & \quad \text{u}_1 \\
-3 & -2 -2 -1 -3
\end{align*}
\]

Figure 20.

we see that \(|pp'-qq'|=1\) if and only if \((|p+q|=1\) and hence) \(p/q=-12/13, -20/21\) or \(-3/4\). In the last mentioned case, it turns out that \(\lambda(\Gamma_3)=2\) and so we are left with only the first two cases, when \(T\) itself has configuration as shown in Figure 21 (a) and (b). In each case, a direct computation shows that \(K\) is effective. In Figure 21, the numbers in bracket give the respective coefficients for the canonical divisor. Thus, we have shown that all branches of \(\Gamma_3-\{u_1\}\) are rational.

\[
\begin{align*}
-2 & -1 -7 -2 -2 -2 -2 \\
(50) & (100)(16) (17) (18) (20) (21)(14) (7) \\
(33)*-3 & (8)*-3 \\
(4)*-2
\end{align*}
\]

(a)

\[
\begin{align*}
-2 & -1 -7 -2 -2 -2 -2 \\
(92) & (184)(30) (31) (32) (33) (34) (18)(12) (6) \\
(61)*-3 & (17)*-2 \\
& -2
\end{align*}
\]

(b)

Figure 21.

15.19. Consider the linear equivalence

\[K^\sigma \sim \mu_1 H_1^\sigma + \mu_2 H_2^\sigma + \sum \lambda_i C_i^\sigma\]

on \(X^\sigma\), where \(C_i^\sigma\) and \(H_i^\sigma\) are the components corresponding to \(u_0\) and \(L_1\). All
the \( C_l \) are \((-2)\)-curves and vertical. \( H^*_1 \) is a rational curve with a cusp, \( (H^*_1)^3 = -1 \), so that \( K'' \cdot H^*_1 = 1 \). \( H^*_2 \) is a \((-3)\)-curve and \( K'' \cdot H^*_2 = +1 \). Hence it follows that \( \mu_1 = -\mu_2 \), and hence \( \mu_2 \neq 0 \). Intersecting with \( H^*_1 \) and \( C'' \) we obtain

\[
K' \cdot H^*_1 = 1 = -\mu_1 + \lambda_0, \\
K' \cdot C'' = 0 = \mu_1 - 2\lambda_0 + \lambda_1
\]

and hence

\[
\lambda_0 = \mu_1 + 1, \\
\lambda_1 = \mu_1 + 2.
\]

As in the proof of 4.1, since all branches of \( I'_2 - \{ u_i \} \) are rational, it follows that if \( \lambda_i \leq 0 \) then \( \mu_i \leq 0 \) and \( \lambda_i \leq 0 \) for all \( i \geq 1 \) and \( \lambda_i \geq 0 \) then \( \mu_i \geq 0 \) and \( \lambda_i \geq 0 \) for all \( i \geq 1 \).

Thus, if \( \lambda_i \leq 0 \) it follows that \(-K\) is effective. Suppose \( \lambda_i > 0 \), \( \lambda_0 \geq 0 \) and \( \mu_1 \geq -1 \). Hence either \( K \) is effective or \( \mu_i = -1 \). Thus we have obtained a linear equivalence

\[
K' \sim H^*_1 - H^*_2 + Z
\]

where \( Z \) is supported on the vertical components of \( D^* \). Now blow-down all the vertical components of \( D^* \) to finitely many rational double-points on a normal projective surface \( Y \), and let \( \pi : X^* \to Y \) be the contraction. Since \( D^* \) is a tree, it follows that \( \pi(H^*_1) \cap \pi(H^*_2) = \{ y_o \} \), a singleton set and \( y_o \) is a singularity of \( Y \). We claim \( y_o \) is the only singularity of \( Y \). For if \( y \neq y_o \) is any other singularity of \( Y \), then \( y \in \pi(H^*_1) \) and \( \pi(H^*_2) \) does not pass through \( y \). On the other hand \( K_Y \sim \pi(H^*_1) - \pi(H^*_2) \) and \( K_Y \) is locally principal on \( Y \). Since every divisor on \( Y \) is linearly equivalent to an integral combination of \( \pi(H^*_1) \) and \( \pi(H^*_2) \), it follows that \( \mathcal{O}_{Y, y} \) is a unique factorization domain. By Lemma 9.9, \( y \) is an \( E_8 \)-singularity, which is not possible since \( \pi \) contracts, in all, eight curves only.

Thus we have shown that \( y_o \) is the only singularity of \( Y \), i.e., all the eight vertical components of \( D^* \) form a connected curve, and both \( H^*_1 \) and \( H^*_2 \) are tips of \( D^* \).

15.20. Let \( F_o \) be the fibre of \( \varphi^* \) containing \( \pi^{-1}(y_o) \). From Kodaira’s list of possible singular fibres of \( \varphi^* \), we see that \( F_o \) is of one of the three types: \( m I_9 \), \( I_9^* \) or \( II^* \). In the first two cases, it follows that \( \pi^{-1}(y_o) \) has configuration \( A_8 \) and \( D_8 \) respectively. If \( F_o \) is of type \( II^* \), then by 9.8, it follows that the component of \( F_o \), not contained in \( D^* \) should occur with multiplicity 1 and hence \( \pi^{-1}(y_o) \) is actually the configuration \( E_8 \).

We can now list all possibilities for the configuration of \( D^* \). Using the unimodularity of \( D^* \), only two possibilities, as shown in Figure 22, may occur.
In both the cases one easily computes and checks that $K$ is effective; this completely disposes off the case $r_3=1$.

§ 16. The case $r_3=2$ and the two components meet.

16.0. So far, we have proved that $r_4=e_1=s=0$, $r_3=2$ and hence $b_5=1$, $\lambda=2$, $\tau=0$ and $D$ is unimodular. In this section we shall show that the two components of $R_3$ do not meet. So, we assume that the two components meet, and denote them by $L_0$ and $L_3$. We can further assume that $L_3=-1$. If follows that $L_0 \cdot L_3=1$; $L_3=-2$. Let $L_1$, $L_2$ be the other two components of $D$ such that $L_0 \cdot L_1=L_0 \cdot L_2=1$. It follows that there is precisely one component $L_4(\neq L_0)$ such that $L_3 \cdot L_4=1$, $L_4 \leq -3$; $L_4 \leq -3$. Also since there is equality in 2.8, we have $K \cdot D=\beta_3-6$. Setting $C=D\setminus \{L_0\}$, it follows that $M(X, C)=1$ and hence $b_k(C) \geq -4$. In the sequel, we shall show that $b_k(C)<-4$ thus arriving at a contradiction and thereby showing that the two components of $R_3$ do not meet. In this, often we use the unimodularity of $T$.

16.1. Blow-down $L_0$ and $L_3$ to obtain a surface $X_1$. We shall assume in 16.1-16.3 that $X_1$ is minimal, (i.e., $X_1=X'=X^*$) and show that $b_k(C)<-4$. By 9.7, it follows that $(L_1^\alpha, L_2^\alpha) \neq (-2, -2)$. Since $\lambda=2$, and there are two horizontal components, $(L_1^\alpha, L_2^\alpha)=(-3, -3)$ or $(-2, -3)$, so that $(L_1^\alpha, L_2^\alpha)=(-5, -5)$ or $(-4, -5)$. Also $L_4=-3$ or $-4$. Setting $T_1\setminus \{L_0\}=T_1 \parallel T_2 \parallel T_3$, with $L_i=T_i$, $i=1, 2, 3$, it follows, by 4.1, as usual, that at least one of the $T_i$ is non-rational.

16.2. Suppose $T_3$ is non-rational. Then, as seen before, it follows that $\#(T_3)\geq 6$ and so $\#(T_3 \cup L_0)\leq 6$. Hence $T_1 \cup T_2 \cup \{L_0\}$ is easily seen to be rational, using 5.4 and 6.1. Hence it follows that $T_1 \cup T_2 \cup \{L_0\}$ is itself non-rational; $\#(T_1 \cup T_2 \cup \{L_0\})\leq 1$.

If $\#(T_1)\geq 6$, then by 11.3, $T_1$ is (iii) of Figure 2 with $w=L_4$, $\Omega_w=-3$. It
follows that \( \{L_1^*, L_2^*\} = \{-4, -5\} \). Keeping in mind 6.1 we obtain only two possibilities for \( T \) as shown in Figure 23. But neither of these is unimodular. Hence \( \#(I_1) \geq 7 \).

\[
\begin{array}{cccccccc}
-2 & -2 & -5 & -1 & -2 & -3 & -2 & -2 \\
& L_3 & L_1 & & & & & \\
-4 & -2 & -3 & & & & & \\
\end{array}
\begin{array}{cccccccc}
-2 & -5 & -1 & -2 & -3 & -2 & -2 & -2 \\
& L_3 & L_4 & & & & & \\
-4 & -2 & -3 & & & & & \\
& & & & & & & \\
\end{array}
\]

Figure 23.

If \( \#(I_1) = 7 \), then 11.4 gives all possibilities for \( I_1 \) with \( w = L_4 \). It follows that except perhaps for (14) of Figure 4 (i.e. when \( \lambda(I_1 - \{L_4\}) = 0 \), \( \Omega_w = -3 \) and \( \{L_1^*, L_2^*\} = \{-4, -5\} \), and in (14), we may even have \( \Omega_w = -4 \) and \( \{L_1^*, L_2^*\} = \{-4, -5\} \) or \( \Omega_w = -3 \) and \( \{L_1^*, L_2^*\} = \{-5, -5\} \). Thus, we list up the configurations for \( I_1' = T - I_1 \) in Figure 24. We can diagonalize each of these with diagonal entry \( \frac{p'}{q'} \) at \( L_3 \) taking values \(-2/9\), \(-1/18\), \(-7/26\), respectively. With \( \frac{p}{q} \) at \( L_4 \) given by 11.4 we see that \( \left| \frac{pp' - qq'}{1} \right| = 1 \) for any of these values and hence by 11.2 \( d(T) = 1 \). Hence \( \#(I_1) \geq 8 \).

\[
\begin{array}{cccccccc}
-2 & -5 & -1 & -2 & & & & \\
& L_3 & & & & & & \\
-4 & & & & & & & \\
\end{array}
\begin{array}{cccccccc}
-2 & -4 & -1 & -2 & & & & \\
& L_3 & & & & & & \\
-5 & & & & & & & \\
\end{array}
\begin{array}{cccccccc}
-2 & -5 & -1 & -2 & & & & \\
& L_3 & & & & & & \\
-5 & & & & & & & \\
\end{array}
\]

Figure 24.

In fact \( \#(I_1) = 8 \), \( \#(T_3) = 9 \), so that both \( L_1 \) and \( L_2 \) are tips of \( T \). In particular it follows that \( \{L_1^*, L_2^*\} = \{-4, -5\} \), by 6.1. We can now apply 11.1(b) with \( Y = X_1^* \), \( C_1 = L_1^* \), \( C_2 = L_2^* \), and \( C_3 = L_4^* \). Along with 4.1, this yields that \( I_1' - \{L_4\} \), has a branch \( \Gamma_2 \) which is non-rational.

Now, if \( \#(I_2) = 6 \), then \( \Gamma_2 \) is as in (iii) of Figure 2 with \( d(I_2) = 4 \) and there is a tip of \( T \) which is joined to \( L_4 \) and which is a \((-2)\)-curve, contradicting 6.1. Hence \( \#(I_2) = 7 \). Since \( \lambda(I_2) = 1 \), \( I_2 \) is determined by 11.4, whereas \( \Gamma_2 = T - \Gamma_2 \) is the configuration in Figure 25 with the diagonal entry \( \frac{p'}{q'} \) at \( L_4 \) being \(+5/2\). With the value of \( \frac{p}{q} \) at \( w \in \Gamma_2 \) given by 11.4, we see that \( \left| \frac{pp' - qq'}{1} \right| = 1 \) if and only if \( T \) is the configuration shown in Figure 26. But then one easily sees that

\[
\begin{array}{cccc}
-5 & -1 & -4 & -2 \\
& L_1 & & \\
-2 & & & \\
\end{array}
\]

Figure 25.
Rationality of 2-cells

\[ \text{bk}(C) = \text{bk}(T_1) + \text{bk}(T_2) + \text{bk}(T_3) = -(1 + 4/5 + 8/13 + 1/2 + 1/2 + 2/3) < -4 \]

16.3. We may now assume that \( T_1 \) is non-rational; (the case \( T_2 \) being non-rational is similar). As before, one first concludes that \( T_1 - \{ L_1 \} \) has a branch \( I_1 \) which is non-rational, \#(I_1) \geq 6; \#(r(I_1)) \leq 1. Further, if \#(r(I_1)) = 6, then \( I' \) is as in (iii), Figure 2. Clearly \#(T_1) \leq 8. If \#(T_1) = 8, then it follows that there is a \((-2)\)-curve in \( T \), which is a tip of \( T \) and is joined to \( L_1 \), contradicting 6.1, since \( d(I_1) \) is even. If \#(T_1) = 7 one easily determines that \( I' = T - I_1 \) is one of the trees shown in Figure 27 with \( p'/q' \) at \( L_1 \) taking values 15/4, 13/8 and 12/7 respectively. With \( p/q \) at \( w \in I_1 \) equal to \(-4/9\), we see that \(|pp' - qq'| \neq 1\) and hence \#(I_1) \geq 7.

Indeed \#(I_1) = 7, \#(I_1) \geq 1 and it is given by 11.4, where as \( I_1 \) is now given by Figure 28 with \( p'/q' \) at \( L_1 \) taking the value 5/3. Again, with \( p/q \) being given by 11.4, we see that \(|pp' - qq'| = 1\) if and only if \( p'/q' = 5/3\), and \( p/q = 4/7\). This yields the unique possibility for \( T \) as shown in Figure 29. But then again one easily computes that \( \text{bk}(C) < -4 \), as claimed in 16.1.

16.4. Thus we can now assume that \( X_1 \) is not minimal. It follows that one of the images of \( L_1 \), \( L_2 \) and \( L_4 \) is a \((-1)\)-curve on \( X_1 \). Indeed, we claim that the image of \( L_4 \) is exceptional. For, suppose that the image of \( L_4 \) is exceptional. Then, (since \( \sigma = 0 \)), clearly \( X_1 = X' \), and \( \{ L_1 \} = e_3 \), so that after
blowing down $L'_1$, we obtain the minimal surface $X''$. $L'_2$ is a cuspidal curve on $X''$. One easily computes that $K\cdot D \geq 8$ whereas we know that $K\cdot D = \beta_2 - 6 = 7$, which is absurd. The case when the image of $L_2$ is exceptional on $X_1$ is symmetrical. Hence the image of $L_i$ is exceptional on $X_i$.

16.5. Blow-down $L_0$, $L_3$, and $L_4$ successively, to obtain a surface $X_2$. Then clearly $X_2 = X'$. Indeed we claim $X_2 = X' = X''$. For, to begin with since $L_4$ is in $R_2$, $L_4$ is a tip of $D$. $L'_i$ and $L'_4$ meet in a single point $x$, such that $L'_i \cdot L'_4 = 3$, and through $x$, no other component of $D'$ passes. Hence, if at all $X'$ is not minimal, then either $L'_i$ or $L'_4$ is exceptional on $X'$. But then we see that $z > 0$ which contradicts our earlier observation that $z = 0$. Hence $X_2 = X' = X''$. In particular, $\beta_2 = 13$. Clearly at least, one of $\{L'_1, L'_2\}$ is horizontal and hence $\{L'_1 \cap L'_2\} = \{2, -3\}$ or $\{3, -3\}$, so that $\{L'_1, L'_2\} = \{-5, -6\}$ or $\{-6, -6\}$, $L'_3 = -2$, $L'_4 = -2$; $d(T_3) = 3$.

As before, it follows that $T_1$ or $T_2$ is non-rational; say, $T_1$ is non-rational; $\#(T_1) \geq 6$. And then $T_1 - \{L_1\}$ itself has a branch $\Gamma$, which is non-rational; $\#(\Gamma) \geq 6$, $\lambda(\Gamma) = 1$.

16.6. If $\#(\Gamma_1) = 6$, then by 11.3, $\Gamma_1$ is (iii) of Figure 2, with $\lambda(\Gamma_1) = 1$ and hence $\{L_1, L'_2\} = \{-5, -6\}$. It follows that $bk(T_1) \leq -(1/6 + 4/3) = -3/2$, and $bk(T_2) \leq -(1/6 + 1/2) = -2/3$. Since $bk(T_3) = -2$, we have shown that $bk(C) < -4$ as required.

If $\#(\Gamma_1) = 7$, then by 11.4, we have $bk(T_1) \leq -(1/6 + 16/15)$. Now $\#(T_2) = 1$, or 2, so that $T_2 = \{5\}$, $\{5, 2\}$, or $\{6, 2\}$. ($T_2 \neq \{6\}$, by 6.1), and hence $bk(T_2) \leq -4/5$ and hence again we obtain $bk(C) < -4$.

16.7. Finally suppose $\#(\Gamma_1) = 8$. Then $\#(T_3) = 1$ and hence $T_2 = \{5\}$, $L_1 = -6$, $Q(T_1) = \{-6, -3, -2, -2, -2, -2\}$. We should now show that $bk(T_1) < -6/5$. This is clear if $T_1$ has more than three tips. Hence we may assume that $T_1$ has precisely three tips. (Clearly $L_1$ is one of them.) Let $A_1, A_2, A_3$ be the three maximal twigs of $T_1$ with $L_1 \in A_1$. Since $\Gamma_1$ is non-rational it follows that $T_1$ should have one of the configurations shown in Figure 30.

In (i) we may assume that $A_1 = \{2, 2\}$. Then $A_3 = \{2, 3\}$ or $\{3, 2\}$ and $A_1 = \{6, 2, 2\}$. Hence

$$bk(T_1) = bk(A_1) + bk(A_2) + bk(A_3) \leq -(4/21 + 2/3 + 2/5) = -44/35 < -6/5.$$ 

In (ii), if $A_1$ (or $A_3$) is $\{2, 2, 2\}$ then clearly $bk(T_1) \leq -(1/6 + 1/3 + 3/4) < -6/5$. So, we may assume that $A_2$ has a (-3)-curve and hence $A_1 = \{6, 2, 2\}$, $A_3 = \{2, 2\}$, and $A_2 = \{2, 2, 3\}$, $\{2, 3, 2\}$ or $\{3, 2, 2\}$. So, in any case $bk(T_1) \leq -(3/16 + 2/3 + 3/7) < -6/5$. In (iii) we may assume that $A_2 = \{2, 2, 2\}$ and so we are done. In (iv), if $A_3 = \{2, 2, 2\}$ we are done. So we may assume that $A_3 = \{2, 2, 2\}$ and so we are done.
In the first two cases $\text{bk}(A_2) \leq -5/8$ and so $\text{bk}(T_1) \leq -(-4/21 + 5/8 + 1/2) < -6/5$. In the third case, we do not get $\text{bk}(T_1) < -6/5$ as needed. However, the tree $T$ is now as shown in Figure 31. One easily checks that this is not unimodular and so this case does not exist. Thus we have shown that $\text{bk}(C) < -4$, in all cases, and hence the two components of $R_3$ are disjoint, as claimed in 16.0.

Let now, $\{L_{i,j}\}_{1 \leq j \leq 3}$ be the other components of $D$ meeting $L_{i,0}$, i.e., $L_{i,0}$, $L_{i,j} = 1$ ($i = 1, 2$), and $L_i = \bigcup_{j=0}^{3} L_{i,j}$ and $L_3 = \bigcup_{j=0}^{3} L_{2,j}$. Since $D$ is simply connected, it follows that $L_i \cap L_3$ is either empty or a single point or an irreducible curve. In the last case we shall choose the labeling so that $L_i \cap L_3 = L_{1,1} = L_{2,1}$.

\section{17. Completion of the proof of the Theorem 8.2.}

17.0. So far, we have proved that $r_4 = c_1 = \tau = 0$, $b_0 = 1$, $\lambda = 2$, $D$ is unimodular, $K \cdot D = \beta_2 - 6$, $r_3 = 2$ and the two components of $R_3$ are disjoint. We shall denote these two components by $L_{1,0}$ and $L_{2,0}$. Now for the curve $C = D - L_{0,1} - L_{0,2}$ we have $b_0(C) = 5$, $b_1(C) = 0$, $b_2(C) = \beta_2 - 2$, and $K \cdot C = \beta_2 - 4$. Hence by 1.3 and 10.5, $\text{bk}(C) \geq -4$. As before, here also, we shall estimate $\text{bk}(C)$ directly, and show that $\text{bk}(C) < -4$, which will complete the proof of the theorem.
Thus \( L_{1,1} \neq L_{2,1} \) means \( L_{1,j} \neq L_{2,k} \) for any \( j \) and \( k \). Let \( C = T_1 \sqcup T_2 \sqcup T_3 \sqcup T_4 \sqcup T_5 \), \(#(T_5) = t_i\). Then \( bk(C) = 1_{s-1}bk(T_s) \).

17.1. Let \( X_1 \) be the surface obtained by contracting \( L_{1,0} \) and \( L_{2,0} \). Assume, first that \( X_1 \) is minimal, \( X_1 = X' = X'' \). Then, \( \beta_3 = 12, \ L_{1,j} \leq -3, 1 \leq j \leq 3 \). We shall make two subcases, viz., (i) \( L_{1,1} = L_{2,1} \), (ii) \( L_{1,1} \neq L_{2,1} \).

Consider the case \( L_{1,1} = L_{2,1} \). Then, it follows that \( L_{1,1} = -4 \) or \(-5\). Then using 9.7, it follows that the dual graph of the curve \( L = L_1 \cup L_2 \) is one of the three trees shown in Figure 32. Note also that the five components of \( L - L_{1,0} - L_{2,0} \) belong to distinct \( T_5 \). In (a) and (b) all components of \( D - L \) are \((-2)\)-curves whereas in (c), \( D - L \) has one \((-3)\)-curve and all other \((-2)\)-curves.

\[
\begin{array}{ccccccccc}
-4 & -1 & -4 & -1 & -4 & -1 & -5 & -1 & -3 \\
\text{(-3)} & \text{(-3)} & \text{(-3)} & \text{(-3)} & \text{(-3)} & \text{(-3)} & \text{(-3)} & \text{(-3)} & \text{(-3)}
\end{array}
\]

(a) \hspace{2cm} (b) \hspace{2cm} (c)

Figure 32.

Thus if \( t_i \leq 2 \), then \( bk(T_s) \leq -1 \) unless \( T_s = [5] \) or \([5, 3]\). Hence, if \( t_i \leq 2 \) for more than three \( s \in \{1, 2, 3, 4, 5\} \), say \( t_i \leq 2 \) for \( s \leq 4 \), then clearly \( 1_{s-1}bk(T_s) \leq -(1 + 1 + 1 + 5/7) \). On the other hand \( bk(T_5) < -1/2 \) and hence \( 1_{s-1}bk(T_s) < -4 \).

So we may assume \( t_i \geq 3 \), and \( t_i \geq 3 \) and \( t_3 + t_4 + t_5 \leq 4 \). Here again \( bk(T_i) + bk(T_s) < -1 \) and so if \( T_s \neq [5] \) or \([5, 3]\) for any \( s = 3, 4, 5 \), then we are done. So assume that \( T_s = [5] \) or \([5, 3]\). Then it follows that \( Q(T_i \cup T_5) \) does not contain \(-5\) and so it follows that \( bk(T_i) + bk(T_5) < -(1/4 + 1/2 + 1/3 + 1/2) \), (by 10.4, (iii)) and hence \( 1_{s-1}bk(T_s) < -(1/4 + 1/2 + 1/3 + 1/2 + 1 + 5/7) < -4 \) as required.

17.2. Now consider the case \( L_{1,1} \neq L_{2,1} \). Now using 9.7, we conclude that \( Q(L_i) = Q(L_2) = \{-1, -3, -3, -4\} \) and \( Q(C) = \{-4, -4, -3, -3, -3, -3, -2, -2, -2\} \). Note that there is a unique branch \( T_s \) of \( C \) say, \( T_1 \) which meets both \( L_{1,0}, L_{2,0} \), and \( t_i \geq 2 \). Now if \( t_i \leq 2 \), then \( bk(T_i) \leq -1 \) and hence as in 17.1, we may assume, that \( t_i \geq 3 \) for at least two \( s \in \{1, 2, 3, 4, 5\} \). So let \( t_i \geq 3 \). Further if \( t_3, t_4, t_5 \), say, \( t_3 \geq 3 \), then \( t_4 = t_5 = 1, t_3 = 2, t_5 = t_3 = 3 \). Clearly \( T_1 = [4, 4], [4, 3] \) or \([3, 3]\) and hence \( bk(T_1) \leq -10/15 \). Also \( bk(T_2) \leq -(1/4 + 1/2), bk(T_3) \leq -(1/4 + 1/2) \) where as \( bk(T_4) \leq -1, bk(T_5) \leq -1 \), and hence \( 1_{s-1}bk(T_s) \leq -4 \). Finally, consider the case where \( t_i \leq 2 \) for \( s = 3, 4, 5 \). Then \( bk(T_i) \leq -1 \) and in any case \( bk(T_i) < -1/2, bk(T_s) < -1/2 \) so that \( 1_{s-1}bk(T_s) < -4 \).

Thus we have shown that \( bk(C) < -4 \) if \( X_1 \) is minimal.

17.3. Now suppose \( X_1 \) is not minimal so that \( L_{1,j} = -2 \) at least for one
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(i, j). Since \( r_4 = 0 \), if \( L_{1,1} = L_{2,1} \) then \( L_{1,1} \) is never in \( E_1 \) and so we may assume that \( L_{1,2} = -2 \). Contract the image of \( L_{1,2} \) on \( X_1 \), to obtain a surface \( X_2 \). We shall now assume that \( X_2 \) is minimal, i.e. \( X_2 = X' = X'' \), and show that \( bk(C) < -4 \).

It follows that \( \beta_2 = 13 \), and \( L_{1,2} \) is a tip of \( D \) as \( r_4 = 2 \). As before we now make two subcases (i) \( L_{1,1} = L_{2,1} \) and (ii) \( L_{1,1} \neq L_{2,1} \).

17.4. Suppose \( L_{1,1} = L_{2,1} \). It follows that \( L \) has one of the configurations shown in Figure 33, with \( D - L \) consisting of \((-2)\)-curves in (a), (b) and (c), and having one \((-3)\)-curve and other \((-2)\)-curves in (d).

\[
\begin{array}{cccc}
-5 & -1 & -6 & -1 & -3 \\
\text{L}_{1,1} & -3 \\
\text{L}_{1,2} & -3 \\
\text{(a)} & \text{(b)}
\end{array}
\]

Figure 33.

We set \( T_5 = \{ L_{1,2} \} = \{ 2 \} \), so that \( bk(T_5) = -2 \). Thus we have to show that \( \Sigma \beta_{i-1} bk(T_i) < -2 \). If \( t_i = 1 \), then clearly \( bk(T_i) \leq -4/6 \). If \( t_i \geq 2 \), then by 10.4, it easily follows that \( bk(T_i) \leq -(1/6 + 1/3) = -1/2 \) and, for at least one of the \( s \), \( bk(T_s) < -1/2 \). Hence \( \Sigma \beta_{i-1} bk(T_i) < -2 \) as required.

17.5. Now consider the case \( L_{1,1} \neq L_{2,1} \). As before, it follows that \( \Omega(L_1) = \{-5, -4, -2, -1\} \) and \( \Omega(L_2) = \{-4, -3, -3, -1\} \). So that \( \Omega(C) = \{-5, -4, -4, -3, -3, -2, -2, -2, -2, -2\} \). Taking \( T_5 = \{ L_{1,1} \} \), here also we have to show that \( \Sigma \beta_{i-1} bk(T_i) < -2 \). As in 17.2, there is a unique \( T_s \), say \( T_1 \), which meets both \( L_{1,0} \) and \( L_{2,0} \). Then for \( T_2, T_3, T_4 \), it easily follows that \( \Sigma \beta_{i-1} bk(T_i) < -2 \) as required.

17.6. We may now assume that \( X_2 \) is not minimal. This means that the image of \( L_{1,1}, L_{1,2} \) or one of \( L_{2,1} \) on \( X_2 \) is a \((-1)\)-curve. However, using the fact that \( K \cdot D = 3 \), we can see that if \( L_{1,1} = L_{2,1} \) then its image on \( X_2 \) cannot be a \((-1)\)-curve (for, the contraction of the image of \( L_{1,1} \) gives \( X'' \) by 3.1, \( K \cdot D = 8 \). This is a contradiction). So we may assume that \( L_{1,2} \) is a \((-1)\)-curve or \( L_{2,2} \) is a \((-1)\)-curve. In the latter case, i.e. if \( L_{2,2} \) is a \((-1)\)-curve then it follows (by \( r_4 = 0 \)) that \( L_{2,2} \) is a tip of \( D \), \( L_{1,2} = -2 \), and hence clearly \( bk(C) < bk(\{ L_{1,1} \}) + bk(\{ L_{2,2} \}) = -4 \). So we may assume that \( L_{2,2} \leq -3 \).
Hence, the image of $L_{1,3}$ on $X_2$ is a $(-1)$-curve. Again using the fact $K \cdot D = \beta_3 - 6$, it follows that $L_{1,3}$ is also a tip of $D$ as above, $L_{1,3}^2 = -3$. Thus taking $T_1 = \{L_{1,3}\}$, we have to show that $\sum_{i=1}^{4} \text{bk}(T_i) < -2/3$.

Contract the image of $L_{1,3}$ on $X_2$ to obtain a surface $X_3$. It is easily seen that $X_3$ is minimal by 3.1, i.e. $X_3 = X^\pi$. Suppose $L_{1,1} \in T_1$. Then it follows easily that for any component $D_i$ of $T_2$ or $T_3$, we have $D_i \geq -4$ and hence $\text{bk}(T_2) \leq -1/2$, $\text{bk}(T_3) \leq -1/2$ proving thereby that $\sum_{i=1}^{4} \text{bk}(T_i) < -2/3$ as required.

This completes the proof of 8.2.

References


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