Determination of the modulus of quadrilaterals
by finite element methods

By Heihachiro HARA and Hisao MIZUMOTO

(Received March 28, 1989)

Introduction.

In the present paper we aim to establish a method of finite element approximations by which we can determine the modulus of quadrilaterals on Riemann surfaces (cf. Mizumoto and Hara [15] for other treatment). Our method matches the abstract definition of Riemann surfaces, and also will offer a new technique of high practical use in numerical calculation not only for the case of Riemann surfaces but also for the case of plane domains.

Let $\Omega$ be a simply connected subdomain of a Riemann surface $W$ whose closure $\overline{\Omega}$ is a compact bordered subregion of $W$. We assume that the boundary $\partial\Omega$ of $\Omega$ is a piecewise analytic curve. We assign four points $p_1, p_2, p_3$ and $p_4$ on $\partial\Omega$ (in positive orientation w. r. t. $\Omega$), and the two opposite arcs $C_0$ (from $p_1$ to $p_2$) and $C_1$ (from $p_3$ to $p_4$). Then we say that a quadrilateral $Q$ with opposite sides $C_0$ and $C_1$ is given.

We can conformally map the domain $\Omega$ onto a rectangular domain $\Omega = \{ w | 0 < \operatorname{Re} w < 1, 0 < \operatorname{Im} w < M \}$ by a function $w = \varphi(p)$ so that $p_1, p_2, p_3$ and $p_4$ are mapped to $iM$, $0$, $1$ and $1+iM$ respectively. Let $\mathcal{F}$ be the class of all continuous functions $v$ on $\overline{\Omega}$ with $v=0$ on $C_0$ and $v=1$ on $C_1$ which satisfy some restricted conditions (see §2.1). Then the modulus $M(Q)=M$ of the quadrilateral $Q$ is uniquely determined by $Q$, and is given by

$$M(Q) = D(u) = \min_{u=\operatorname{Re}\varphi(p)} D(v)$$

where by $D(v)$ we denote the Dirichlet integral of $v$. Next we assign the two opposite arcs $\tilde{C}_0$ (from $p_2$ to $p_3$) and $\tilde{C}_1$ (from $p_4$ to $p_1$) on $\partial\Omega$. Then a quadrilateral $\tilde{Q}$ with the opposite sides $\tilde{C}_0$ and $\tilde{C}_1$ is defined. We can easily see that $M(Q) = 1/M(\tilde{Q})$. By making use of this relation Gaier [9] presented a method to obtain upper and lower bounds for the modulus $M(Q)$ in the case of some restricted domain $Q$ (e. g. a lattice domain, etc.) by the finite difference and

This research was partially supported by Grant-in-Aid for Scientific Research (No. 60540110), Ministry of Education, Science and Culture, Japan.
finite element approximations, which originates from Opfer [16], [17]. We shall present a method to obtain fairly good upper and lower bounds for $M(Q)$ by our finite element approximation even in the case of a domain $Q$ with curvilinear boundary arcs, and with inner and corner singularities of high order.

It is characteristic of our method that we adopt ordinary triangular meshes and linear elements on a subregion of every fixed parametric disk, our approximating functions of $u=\text{Re}f(\theta)$ satisfy the boundary conditions exactly even in the case of curvilinear boundary arcs, and express singular property exactly near inner and corner singularities. Hence the approximations of high precision of $u$ are obtained, and the fairly good upper and lower bounds to $M(Q)$ can be evaluated exactly. It should be noted that we do not adopt any so-called refined or curvilinear mesh near singularities.

§ 1 is devoted to construction of triangulations $K$ and $K'$ of two kinds. $K$ is a triangulation of $\tilde{Q}$ and $K'$ is a modification of $K$.

In § 2, we introduce and investigate two classes of element functions on $K$ and $K'$: the comparable class $S=S(K)$ (with $u$) and the computable class $S'=S'(K')$. $S\subseteq S'$ and $S'$ is a collection of modifications $v_h=F(v_h)$ of $v_h\in S$, where $F$ defines a one-to-one mapping of $S$ onto $S'$. $D(v_h)$ can be numerically calculated. We shall investigate estimates of differences of $D(v_h)$ and $D(v'_h)$ (see Lemma 2.2).

The finite element approximations $w_h$ and $u'_h$ of $u$ in $S$ and $S'$ respectively are defined by the minimalities:

$$D(w_h) = \min_{v_h\in S} D(v_h) \quad \text{and} \quad D(u'_h) = \min_{v'_h\in S'} D(v'_h)$$

respectively. $u'_h$ can be obtained by solving a system of linear equations. § 3 is devoted to error estimates of $w_h$ and $u_h$ for $u$. In Theorems 3.1 and 3.2, we obtain error estimates:

$$D(w_h-u) \leq C h^2 \quad \text{and} \quad D(u_h-u) \leq C' h^2 \quad \text{resp.,}$$

where $C$ and $C'$ are constants which depend only on the function $u$ and the smallest value of interior angles of triangles. Further, in Theorem 3.2, we obtain an estimate for $D(u)$:

$$D(u) \leq D(u'_h) + \varepsilon(u'_h),$$

where $\varepsilon(u'_h)$ is a quantity of $O(h^2)$ which can be numerically calculated.

Finally, in § 4 we apply our results to numerical calculation of the modulus of quadrilaterals, and we shall show that calculation results for some concrete quadrilaterals are fairly good. With respect to the problems of this type, there have been some investigations by means of finite difference or finite element methods (cf. Gaier [9], Mizumoto [10], [11], [12], and Opfer [16], [17]).

With respect to treatment at boundary singularities, there have been some
investigations (cf. Akin [1], Babuška [2], Babuška and Rosenzweig [3], Babuška, Szabo and Katz [4], Barnhill and Whiteman [5], Blackburn [6], Craig, Zhu and Zienkiewicz [8], Opfer and Puni [18], Rivara [19], Schatz and Wahlbin [20], [21], Thatcher [23], Tsamasphyros [24], Weisel [25], Whiteman and Akin [26], and Yserentant [27]).

§ 1. Triangulation.

1. Collection \( \Phi \) of local mappings. Let \( \Omega \) be a simply connected subdomain of a Riemann surface \( W \) whose closure \( \bar{\Omega} \) is a compact bordered subregion of \( W \). We assume that the boundary \( \partial \Omega \) consists of a finite number of analytic arcs meeting at vertices \( p_k \) \((k=1, \ldots, \kappa)\), and there exist parametric disks \( V_k \) \((k=1, \ldots, \kappa)\) with the centers \( p_k \) and the local parameters \( z=\psi_k(p) \) by which \( V_k \cap \bar{\Omega} \) are mapped onto sectors \( \{ |z| \leq r_k \} \cap \{ 0 \leq \arg z \leq \beta_k \} \) \((0 < p_k < 2\pi, \beta_k \leq \pi)\).

We assign four points \( p_1, p_2, p_3 \) and \( p_4 \) on \( \partial \Omega \) (in the positive orientation with respect to \( \Omega \)), and the two opposite arcs \( C_0 \) (from \( p_1 \) to \( p_2 \)) and \( C_1 \) (from \( p_3 \) to \( p_4 \)). Then we say that a quadrilateral \( Q \) with opposite sides \( C_0 \) and \( C_1 \) is given.

By \( \Phi = \{ z=\varphi_j(p), U_j; j=1, \ldots, m \} \) we denote a finite collection of local parameters \( z=\varphi_j(p) \) \((j=1, \ldots, m)\) and parametric disks \( U_j \) \((j=1, \ldots, m)\) on \( W \) which satisfies the following conditions (i)~(v):

(i) By the mapping \( z=\varphi_j(p) \) \((j=1, \ldots, m)\), \( U_j \) is mapped onto a disk \( |z| < \rho_j \).

(ii) \( \bar{\Omega} \) is covered by \( \{ U_j \} \) \( j=1, \ldots, m \).

(iii) If \( U_j \cap U_k = \emptyset \), then there exists a constant \( L (>1) \) such that for the mapping \( \zeta = \frac{1}{L} < |\partial f/\partial z| < L \) on \( \varphi_j(U_j \cap U_k) \).

Let \( p_k \) \((k=5, \ldots, \nu)\) be the all vertices of \( \partial \Omega \) which are defined as points of \( \{ p_k \} \setminus \{ p_1, p_2, p_3, p_4 \} \).

(iv) Each \( U_j \) \((j=1, \ldots, m)\) contains at most one \( p_k \) \((k=1, \ldots, \nu)\) and if \( p_k = U_j \) then \( \varphi_j(p_k) = 0 \).

(v) If \( U_j \cap \partial \Omega \neq \emptyset \) and \( U_j \) does not contain any \( p_k \) \((k=1, \ldots, \nu)\), then \( \varphi_j(U_j \cap \Omega) \) is a half disk \( \{ |z| < \rho_j \} \cap \{ \text{Im } z > 0 \} \). If \( U_j \) contains some \( p_k \) \((k=1, \ldots, \nu)\), then \( \varphi_j(U_j \cap \Omega) \) is a sector \( \{ |z| < \rho_j \} \cap \{ 0 < \arg z < \alpha_j \} \) \((0 < \alpha_j \leq 2\pi)\).

In the latter case of (v), if \( p_k \neq p_1, p_2, p_3, p_4 \) and \( \alpha_j > \pi/2 \), then by the mapping \( \zeta = (\varphi_j(p))^{\pi/2} \), \( U_j \cap \Omega \) is mapped onto a half disk \( \{ |\zeta| < \rho_j^{\alpha_j} \} \cap \{ \text{Im } \zeta > 0 \} \). If \( U_j \) contains some \( p_k \) \((k=1, 2, 3, 4)\), then by the mapping \( \zeta = (\varphi_j(p))^{\pi/2} \), \( U_j \cap \Omega \) is mapped onto a sector \( \{ |\zeta| < \rho_j^{\alpha_j} \} \cap \{ 0 < \arg \zeta < \pi/2 \} \).

In this case we define anew \( z=\varphi_j(p) \) and \( \rho_j \) by \( \zeta = (\varphi_j(p))^{\pi/2} \) and \( \rho_j^{\alpha_j} \) respectively. Further, if \( U_j \) contains some \( p_k \) \((k=1, 2, 3, 4)\), then by the mapping \( \zeta = (\varphi_j(p))^{\pi/2} \), \( U_j \cap \Omega \) is mapped onto a sector \( \{ |\zeta| < \rho_j^{\alpha_j} \} \cap \{ 0 < \arg \zeta < \pi/2 \} \).

In this case we define anew \( z=\varphi_j(p) \) and \( \rho_j \) by \( \zeta = (\varphi_j(p))^{\pi/2} \) and \( \rho_j^{\alpha_j} \) respectively. Then, the local parameter \( z=\varphi_j(p) \) is no longer conformal at the
2. **Triangulation $K$ associated to $\Phi$.** For the collection $\Phi$ of local parameters and parametric disks defined in §1.1, and for a sufficiently small positive number $h$, we construct a triangulation $K=K^h$ of $\bar{\Omega}$ which satisfies the following conditions (i)~(v). This is called a *triangulation of $\bar{\Omega}$ with width $h$ associated to $\Phi$.*

(i) The points $p_1, \ldots, p_n$ are carriers of some 0-simplices of $K$.

(ii) $K$ is the sum of subtriangulations $K_1, \ldots, K_m$ of $K$ such that each 2-simplex of $K$ belongs to one and only one $K_j (j=1, \ldots, m)$, and the carrier $|s|$ of each 2-simplex $s$ of $K_j$ is contained in $U_j$.

If a 1-simplex $e \in K_j$ does not belong to another $K_k (k \neq j)$, or a 1-simplex $e$ belongs to $K_j \cap K_k (j \neq k)$ and the mapping $\varphi_k \varphi_j^{-1}$ is an affine transformation, then $e$ is said to be linear. If each edge of a 2-simplex $s \in K_j$ is linear and $\varphi_j(s)$ is an ordinary triangle, then $s$ is called a *natural simplex*.

(iii) Each 2-simplex $s \in K_j$ which has not a common edge with any 2-simplex of another $K_k (k \neq j)$, is a natural simplex.

A 2-simplex of $K_\Phi$ which has a common edge with a 2-simplex $s \in K_j (j \neq k)$, is said to be an adjoint (simplex) of $s$ and is denoted by $s'$.

(iv) For each pair of a 2-simplex $s \in K_j$ and its adjoint $s' \in K_k$ with a common edge $e$, either one of the following three cases (a), (b), (c) occurs.

(a) Both $s$ and $s'$ are natural simplices.

(b) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_k(e)$ is a strict concave arc w. r. t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of $s$ and $s'$ except for $e$ are linear (cf. Figure 1). Then $s$ is called a *minor simplex*. The case where $s'$ is a minor simplex and $s$ is its adjoint may also occur.

(c) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_k(e)$ is a strictly convex arc w. r. t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of $s$ and $s'$ except for $e$ are linear (cf. Figure 2). Then $s$ is called a *major simplex*. The case where $s'$ is a major simplex and $s$ is its adjoint may also occur.

---

**Figure 1.** Minor simplex $s$ and its adjoint $s'$.

**Figure 2.** Major simplex $s$ and its adjoint $s'$. 
If $s$ is a minor or major simplex of $K_i$, then it is assumed that $|s'| \subset U_j$ for its adjoint $s'$.

(v) For each 2-simplex $s \in K_j (j = 1, \cdots, m)$, $d(\varphi_j(s)) \leq h$, where throughout the present paper we denote the diameter of a region $G$ by $d(G)$.

Next, we assume that for the fixed $\Phi$ the class of the triangulations $K = K^h$ satisfies the following condition (i') and (ii'):

(i') For each $j = 1, \cdots, m$ the union of carriers of all minor and major simplices of $K_j$, and all their adjoints is contained in a closed subset $R_j$ of $U_j \cap \overline{G}$ which is independent of the individual triangulation $K$.

(ii') The number $N$ of minor and major simplices of $K$ satisfies the inequality:

$$N \leq M \cdot \frac{1}{h},$$

where $M$ is a constant which is independent of the individual triangulation $K$.

3. Normal subdivision of triangulation $K$. For a triangulation $K = K^h$ of $\partial$ with width $h$ associated to $\Phi$ we can construct a subdivision $K'_1 = K^{1/2}$, called the normal subdivision of $K = K^h$ by the following procedure:

(i) $K'_1$ is the sum of the subtriangulations $K'_1, \cdots, K'_m$ which are the subdivisions of $K_1, \cdots, K_m$ respectively which are defined in the following (ii), (iii).

(ii) If $s \in K_j$ is a 2-simplex which is not minor or major, then $s$ is subdivided to four 2-simplices $s_1, s_2, s_3$ and $s_4$ of $K'_j$ so that $\varphi_j(s_1), \varphi_j(s_2), \varphi_j(s_3)$ and $\varphi_j(s_4)$ are mutually congruent ordinary triangles in Figure 3.

(iii) Let $s \in K_j$ and $s' \in K_k$ be a minor (or major) simplex and its adjoint, and let $e_1, e_2$ and $e_3$ be edges of $s$ such that $e_1$ is the common edge of $s$ and $s'$. We subdivide the edges $e_1, e_2$ and $e_3$ to two edges $e_{11}$ and $e_{12}$, $e_{21}$ and $e_{22}$, and $e_{31}$ and $e_{32}$ respectively so that $\varphi_j(e_{11})$ and $\varphi_j(e_{12})$, $\varphi_j(e_{21})$ and $\varphi_j(e_{22})$, and $\varphi_j(e_{31})$ and $\varphi_j(e_{32})$ have the same length respectively. Then we subdivide the simplex $s$ to two minor (or major resp.) simplices $s_1$ and $s_2$ of $K'_j$ and, two natural simplices $s_3$ and $s_4$ of $K'_j$ so that $e_{11}, e_{12}, e_{21}, e_{22}, e_{31}$ and $e_{32}$ are edges.
of $s_1$, $s_2$ and $s_3$ (cf. Figure 4). Here we note that such a subdivision is always possible if $h$ is sufficiently small.

We can see that the normal subdivision $\mathcal{K} = \sum_{i=1}^{n} \mathcal{K}_i$ is a triangulation of $\tilde{\mathcal{K}}$ with width $h/2 + O(h^2)$ associated to $\Phi$ (cf. § 1 of [15]).

4. Naturalized triangulation. For each minor (or major) simplex $s \in \mathcal{K}_j$, we define the naturalized simplex $\hat{s}$ of $s$ as the 2-simplex such that $|s| \supset |\hat{s}|$ ($|\hat{s}| \supset |s|$ resp.) and $\varphi(\hat{s})$ is the ordinary triangle which has two common sides with $\varphi(s)$. Further we define a 2-simplex $b\ell = b\ell(s)$ ($\#\ell = \#\ell(s)$ resp.) with two edges whose carrier is the closed region $|s| - |\hat{s}|$ ($|\hat{s}| - |s|$ resp.). $b\ell(s)$ ($\#\ell(s)$ resp.) is called the deficient (excessive resp.) lune of $s$.

Each triple of a minor (or major) simplex $s \in \mathcal{K}_j$, its ad joint $s' \in \mathcal{K}_k$ and its deficient lune $b\ell$ (excessive lune $\#\ell$ resp.) is denoted by $(s, s', b\ell)$ ($\langle s, s', \#\ell \rangle$ resp.), and is called a triple for a minor (or major) simplex $s$ or for a deficient (excessive resp.) lune $b\ell$ ($\#\ell$ resp.) (cf. Figure 5), where it is always assumed that $|b\ell| \subset |s'|$ for each $(s, s', b\ell)$.

For simplicity of notation, we also denote $b\ell = b\ell(s)$ or $\#\ell = \#\ell(s)$ by $\ell = \ell(s)$. If a minor or major simplex $s$ is in $\mathcal{K}_j$, then we say that $\ell = \ell(s)$ is a lune of
Now we shall define the naturalized triangulation $K'$ associated to $K$.

First, $K'_j (j=1, \ldots, m)$ are defined as triangulations such that the collection of all 2-simplices of $K'_j$ consists of all 2-simplices of $K_j$ which are not minor or major, and of all naturalized simplices of minor and major ones of $K_j$. Then the triangulation $K'$ is defined as the sum of $K'_j (j=1, \ldots, m)$. We should note that $K'$ is no longer a triangulation of $\tilde{\varnothing}$, and also is not an ordinary triangulation.

5. Parametrization of lunar domains. Let $(s, s', \ell)$ be an arbitrary triple for a deficient or excessive lune $\ell$, and let $e_1$ and $e_2$ be two edges of $\ell$ such that $e_1 \supseteq \tilde{\varnothing}s$. Further, let

\begin{equation}
(1.2)\quad z' = (1-t)z_1 + tz_2 \quad (0 \leq t \leq 1)
\end{equation}

and

\begin{equation}
(1.3)\quad \zeta'' = (1-t)\zeta_{s_1} + t\zeta_{s_2} \quad (0 \leq t \leq 1)
\end{equation}

be parameter representations of the oriented segments $\varphi_j(-e_2)$ and $\varphi_k(e_1)$ respectively. The representation (1.3) induces a parameter representation of the curve $\varphi_j(e_1)$:

\begin{equation}
(1.4)\quad z'' = g((1-t)\zeta_{s_1} + t\zeta_{s_2}) \quad (0 \leq t \leq 1),
\end{equation}

where $z = g(\zeta) = \varphi_{j,*}\varphi_{i,*}(\zeta)$. By (1.2) and (1.4) we obtain a parameter representation of the lunar domain $\varphi_j(\ell)$:

\begin{equation}
(1.5)\quad z = z(t, \tau) = (1-\tau)z' + \tau z''
= (1-\tau)((1-t)z_1 + tz_2) + \tau g((1-t)\zeta_{s_1} + t\zeta_{s_2}) \quad (0 \leq t \leq 1, 0 \leq \tau \leq 1).
\end{equation}

6. Area of lune.

**Lemma 1.1.** Let $(s, s', \ell)$ be a triple for an arbitrary deficient or excessive lune $\ell$. Then, the estimate

\begin{equation}
(1.6)\quad A(\varphi_j(\ell)) \leq \frac{h_1}{8} \left( \left| \frac{g''(\zeta_{s_1})}{g''(\zeta_{s})} \right| + O(h_1) \right)
\end{equation}

holds, where throughout the present paper we denote the area of a region $G$ by $A(G)$, $z = g(\zeta) = \varphi_{j,*}\varphi_{i,*}(\zeta)$, $h_1 = d(\varphi_j(\ell))$ and $\zeta_{s_1}$ is one of the vertices of the lunar domain $\varphi_k(\ell)$.

§ 2. Classes of functions.

1. Class \( \mathfrak{H} \). By \( \mathfrak{H} \) we denote the class of all continuous functions \( v \) on \( \bar{\Omega} = \Omega \cup \partial \Omega \) with \( v=0 \) on \( C_0 \) and \( v=1 \) on \( C_1 \), for which the partial derivatives \( \partial v/\partial x \) and \( \partial v/\partial y \) with respect to the local parameter \( z=x+iy \) exist and are continuous on \( \Omega \) at most except for a finite number of rectifiable curves on \( \Omega \), and for which the Dirichlet integral

\[
D(v) = D_\Omega(v) = \int_{\Omega} \left( \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right) dx dy
\]

is finite.

2. Subclass \( S \) of \( \mathfrak{H} \). We define a subclass \( S=S(K) \) of \( \mathfrak{H} \), called the comparable class (with \( u \)), as the class of functions \( v_h \) which satisfy the following conditions (i)~(iv):

(i) \( v_h \in \mathfrak{H} \).
(ii) If \( s \subset K_j \) \( (j=1, \ldots, m) \) is a natural simplex, then

\[
v_h = ax + by + c \quad \text{on} \quad \varphi_j(s) (z=x+iy),
\]

where \( a, b \) and \( c \) are constants.
(iii) Let \( (s, s', b \ell) \) be a triple for a minor simplex \( s \), and let \( e_1 \) and \( e_2 \) be two edges of \( b \ell \) such that \( -e_1 \subset \partial s \). Then

\[
v_h = ax + by + c \quad \text{on} \quad \varphi_j(s),
\]

\[
v_h = \alpha \xi + \beta \eta + \gamma \quad \text{on} \quad \varphi_K(s') - \varphi_K(b \ell),
\]

and \( v_h \) is a harmonic function in \( b \ell \) which satisfies the boundary conditions:

\[
v_h = ax + by + c \quad \text{on} \quad \varphi_K(e_1),
\]

and

\[
v_h = \alpha \xi + \beta \eta + \gamma \quad \text{on} \quad \varphi_K(e_2),
\]

where \( a, b, c, \alpha, \beta \) and \( \gamma \) are constants, and

\[
\zeta = f(z) \equiv \varphi_K \cdot \varphi_j^{-1}(z) \quad (z=x+iy, \zeta = \xi + i\eta).
\]

(iv) Let \( (s, s', \# \ell) \) be a triple for a major simplex \( s \), and let \( e_1 \) and \( e_2 \) be two edges of \( \# \ell \) such that \( e_1 \subset \partial s \). Then

\[
v_h = ax + by + c \quad \text{on} \quad \varphi_j(ts),
\]

\[
v_h = \alpha \xi + \beta \eta + \gamma \quad \text{on} \quad \varphi_K(s'),
\]

and \( v_h \) is a harmonic function in \( \# \ell \) which satisfies the boundary conditions:

\[
v_h = ax + by + c \quad \text{on} \quad \varphi_j(e_2)
\]

and
where $a, b, c, \alpha, \beta$ and $\gamma$ are constants, and $\zeta = \xi + i\eta$ is as in (iii).

3. **Class $S'$ of functions.** Let $K'$ be the naturalized triangulation associated to $K$. For each function $v_h \in S$, we define the function $v'_h$ on $K'$ associated to $v_h$ as the function $v'_h$ which satisfies the following conditions (i)-(iv):

(i) For each 2-simplex $s \in K'_j (j=1, \ldots, m)$
\[
v'_h = ax + by + c \quad \text{on } \phi_h(s),
\]
where $a, b$ and $c$ are constants.

(ii) If $s \in K$ is a natural simplex, then
\[
v'_h = v_h \quad \text{on } |s|.
\]

(iii) If $(s, s', \partial \ell)$ is a triple for a minor simplex $s$, then
\[
v'_h = v_h \quad \text{on } |s| \cup |s'| - |\partial \ell|.
\]

(iv) If $(s, s', \# \ell)$ is a triple for a major simplex $s$, then
\[
v'_h = v_h \quad \text{on } |ts| \cup |s'|.
\]

We should note that the function $v'_h$ is defined just twice on each deficient lune $\partial \ell$, while it is never defined on any excessive lune $\# \ell$. In the former case, for each triple $(s, s', \partial \ell)$ we shall denote the function $v'_h$ on $t \in K'_j$ and $s' \in K'_k$ by $v'_{h, s}$ and $v'_{h, s'}$ respectively.

The class of all functions $v'_h$ associated to $v_h \in S$ is denoted by $S' = S'(K')$ and called the computable class. Let $v'_h$ and $\phi'_h$ be two functions of $S'$. Then the mixed Dirichlet integral $D_{K'}(v'_h, \phi'_h)$ of $v'_h$ and $\phi'_h$ is defined by
\[
D(v'_h, \phi'_h) = \sum \int_{s \in K'} (\frac{\partial v'_h}{\partial x} \frac{\partial \phi'_h}{\partial x} + \frac{\partial v'_h}{\partial y} \frac{\partial \phi'_h}{\partial y}) dx dy,
\]
and the Dirichlet integral $D_{K'}(v'_h)$ of $v'_h$ is defined by
\[
D(v'_h) = D_{K'}(v'_h) = D_{K'}(v'_h, v'_h),
\]
where $D(v'_h)$ can be numerically calculated.

We see that $v'_h = F(v_h)$ defines a one-to-one mapping of $S$ onto $S'$.

4. **Finite element interpolations.** Let $v$ be a function of $S$. We define the finite element interpolation $\hat{v}$ of $v$ in the class $S$ as the function uniquely determined by the following conditions (i) and (ii):

1) We shall use the common notations $D(\ ,\ )$ and $D(\ )$ for both mixed and ordinary Dirichlet integrals of functions of the classes $S$ and $S'$. 

---

$\begin{align*}
v_h &= \alpha \xi + \beta \eta + \gamma \quad \text{on } \phi_h(e),
\end{align*}$
5. Harmonic functions on a lune.

**Lemma 2.1.** Let $\ell = \ell(s)$ be a deficient or excessive lune of $K_j$, let $e_1$ and $e_2$ be two edges of $\ell$, and let $q_1$ and $q_2$ be two vertices of $\ell$. Let $v_1$ and $v_2$ be the functions in the class $C^1$ on $\ell$ which satisfy the condition

$$v_1(q_j) = v_2(q_j) \quad (j = 1, 2).$$

Further, let $\phi$ be the harmonic function in $\ell$ which satisfies the boundary conditions

$$\phi = v_i \quad \text{on} \ e_i \quad (i = 1, 2).$$

Then the inequalities

$$D(\phi) \leq \int_{\ell} \max \left( \left( \frac{\partial v_1}{\partial x} \right)^2 + \left( \frac{\partial v_1}{\partial y} \right)^2, \left( \frac{\partial v_2}{\partial x} \right)^2 + \left( \frac{\partial v_2}{\partial y} \right)^2 \right) dx dy$$

$$\leq D(v_1) + D(v_2)$$

hold.

If we set $\sigma_1 = dv_1$ and $\sigma_2 = dv_2$, then the proof is reduced to one of Lemma 2.1 of [15].

6. Difference of Dirichlet integrals of $v_h$ and $v'_h$.

**Lemma 2.2.** Let $v_h$ be an arbitrary function of the class $S$ and let $v'_h = F(v_h)$.

(i) The inequalities

$$D(v_h) \leq D(v'_h) + \sum_{\# e \in K} D_{e}(v_h)$$

$$\leq D(v'_h) + \sum_{j=1}^{m} \sum_{\# e \in K_j} A(\phi_j(\# e)) \cdot \frac{(v'_h(q_j) - v_h(q_j))^2}{|\phi_j(q_j) - \phi_h(q_j)|^2}$$

$$\cdot \max \left\{ 1, \frac{|\phi_j(q_j) - \phi_h(q_j)|^2}{|\phi_h(q_j) - \phi_h(q_j)|^2} \cdot \max_{\phi_j(\# e)} |f'(z)|^2 \right\}$$

hold, where

$$\frac{|\phi_j(q_j) - \phi_h(q_j)|^2}{|\phi_h(q_j) - \phi_h(q_j)|^2} \cdot \max_{\phi_j(\# e)} |f'(z)|^2 \leq 1 + \kappa h,$$

$q_j$ and $q_2$ are the vertices of $\# e$, and $\kappa$ is a constant which depends only on the transformations $f(z) = \phi_h \circ \phi_j(z)$.

(ii)

$$D(v'_h) \leq D(v_h) + \sum_{\# e \in K} (D_{e}(v'_h, v_h) + D_{e}(v'_h, v'_h))$$

$$= D(v_h) + \sum_{j=1}^{m} \sum_{\# e \in K_j} (A(\phi_j(\# e)) \cdot (a^2 + b^2) + A(\phi_h(\# e)) \cdot (a^2 + b^2),$$
where for each triple \((s, s', b)\) the notations in (iii) of § 2.2 are preserved.

If we set \(\sigma_h = dv_h\) and \(\sigma'_h = dv'_h\), then the proof is reduced to one of Lemma 2.2 of [15].

§ 3. Finite element approximations.

1. Formulation of problems. We can conformally map the domain \(\Omega\) defined in § 1.1 onto a rectangular domain

\[
R = \left\{ w \mid 0 < \text{Re} w < 1, \, 0 < \text{Im} w < M \right\}
\]

by a function \(w = \psi(p)\) so that \(p_1, p_2, p_3\) and \(p_4\) are mapped to \(iM, 0, 1\) and \(1+iM\) respectively. Then the modulus of the quadrilateral \(Q\):

\[
M(Q) = M
\]

is uniquely determined by \(Q\). Our aim is to determine \(M(Q)\) by finite element method.

Now we assign the two opposite arcs \(\tilde{C}_0\) (from \(p_2\) to \(p_3\)) and \(\tilde{C}_1\) (from \(p_4\) to \(p_1\)) on \(\partial \Omega\). Then a quadrilateral \(\tilde{Q}\) with opposite sides \(\tilde{C}_0\) and \(\tilde{C}_1\) is defined. We see that the domain \(\Omega\) can be conformally mapped onto a rectangular domain

\[
\tilde{R} = \left\{ w \mid 0 < \text{Re} w < 1, \, 0 < \text{Im} w < 1/M \right\}
\]

by a function \(w = \tilde{\psi}(p)\) so that \(p_2, p_3, p_4\) and \(p_1\) are mapped to \(i/M, 0, 1\) and \(1+i/M\) respectively. Hence

\[
(3.1) \quad M(\tilde{Q}) = \frac{1}{M(Q)}.
\]

We characterize \(M(Q)\) by a minimal property.

LEMMA 3.1. Let \(u = \text{Re} f\). Then the equalities

\[
(3.2) \quad M(Q) = D(u) = \min_{v \in \mathbb{F}} D(v)
\]

hold. The minimum of the right hand side of (3.2) is attained if and only if \(v = u\).

PROOF. By \(*du\) we denote the conjugate differential of \(du\). Then

\[
*du = 0 \quad \text{along} \quad \partial \Omega - C_0 \cup C_1, \quad \text{and}
\]

\[
v - u = 0 \quad \text{on} \quad C_0 \cup C_1 \quad \text{for each} \quad v \in \mathbb{F}.
\]

Hence

\[
(3.3) \quad D(v-u, u) = \int_{\partial \Omega} (v-u)*du = 0.
\]

This equality implies that
\[ D(v) = D(u) + D(v - u) \geq D(u). \]

In the last inequality, the equality holds if and only if \( v = u \).

The first equality of (3.2) follows from the equalities

\[ D(u) = \int_{a}^{b} u \cdot du = \int_{a}^{b} \phi \cdot du = M. \]

We call \( u \) the harmonic solution in \( \Omega \). Our aim is to obtain finite element approximations of \( u \) in the classes \( S \) and \( S' \), and error estimates of them for \( u \).

2. Finite element approximation \( \omega_h \) in \( S \). By \( \omega_h \) we denote the function of \( S \) such that

\[ D(\omega_h) = \min_{v_h \in S} D(v_h). \]

Since \( S \subseteq \Omega \), we see that

\[ D(u) \leq D(\omega_h). \]

We call \( \omega_h \) the finite element approximation of \( u \) in \( S \).

**Lemma 3.2.** (i) The function \( \omega_h \) has the minimal property

\[ D(\omega_h - u) = \min_{v_h \in S} D(v_h - u), \]

where the minimum is attained if and only if \( v_h = \omega_h \).

(ii) The equality

\[ D(\omega_h - u) = D(\omega_h) - D(u) \]

holds.

**Proof.** (i) First, by a method similar to (3.3), it is shown that

\[ D(v_h - \omega_h, u) = 0 \quad \text{for each } v_h \in S. \]

By (3.4), standard arguments imply that

\[ D(\omega_h, v_h - \omega_h) = 0 \quad \text{for each } v_h \in S. \]

From (3.8) and (3.9), it follows that

\[ D(u - v_h) = D(u - \omega_h) + D(v_h - \omega_h) \geq D(u - \omega_h). \]

In the last inequality, the equality holds if and only if \( v_h = \omega_h \).

(ii) By (3.3) \( D(\omega_h - u, u) = 0 \) and thus (3.7) is obtained.

From (3.9) the following lemma immediately follows.

**Lemma 3.3.** The equality
3. **Finite element approximation** $u'_h$ in $S'$. By $u'_h$ we denote the function of $S'$ such that

\[ D(u'_h) = \min_{v'h \in S'} D(v'h). \]

We call $u'_h$ the **finite element approximation of $u$ in $S'$**. $u'_h$ can be obtained by solving a system of linear equations.

**Lemma 3.4.** The equality

\[ D(v'_h - u'_h) = D(v'_h) - D(u'_h) \]

holds for each $v'_h \in S'$.

**Proof.** By (3.11), standard arguments imply that

\[ D(u'_h, v'h - u'_h) \leq 0 \text{ for each } v'h \in S'. \]

This implies (3.12).

4. **Lemma of Bramble and Zlámal.** The following lemma is due to J. H. Bramble and M. Zlámal (cf. [7]).

**Lemma 3.5.** Let $\Delta$ be a closed triangle on the $z$-plane $(z=x+iy)$ with $d(\Delta) \leq h$ and let $v$ be a function of the class $C^2$ defined on $\Delta$ such that $v=0$ at each vertex of $\Delta$. Then, the inequality

\[ \int_{\Delta} \left( \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right) dx dy \leq \frac{B}{\sin^2 \theta} \int_{\Delta} \left( \left( \frac{\partial^2 v}{\partial x^2} \right)^2 + \left( \frac{\partial^2 v}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 v}{\partial y^2} \right)^2 \right) dx dy \]

holds, where $B$ is an absolute constant and $\theta$ is the smallest interior angle of the triangle $\Delta$.

5. **Pointwise estimate.**

**Lemma 3.6.** Let $\Delta$ be a closed curvilinear triangle on the $z$-plane $(z=x+iy)$ with $d(\Delta) \leq h$ which is the image of some 2-simplex $s \in K_j (j=1, \cdots, m)$ by $z=\varphi(j)$, and let $v$ be a function of the class $C^2$ defined on $\Delta$ such that $v=0$ at each vertex of $\Delta$. Then,

\[ \left| \frac{\partial v}{\partial x} \right|, \left| \frac{\partial v}{\partial y} \right| \leq h \cdot \frac{4}{\sin \theta} \max_{i,j} \left( \left| \frac{\partial^2 v}{\partial x^2} \right| + 2 \left| \frac{\partial^2 v}{\partial x \partial y} \right| + \left| \frac{\partial^2 v}{\partial y^2} \right| \right) (1+\kappa h) \]
on $A$, where $\theta$ is the smallest interior angle of the ordinary triangle which has
common vertices with $A$, and $\kappa$ is a constant which depends only on the trans-
formations $f(z) = \varphi_k \circ \varphi_j^{-1}(z)$.

See Lemma 3.6 of [15] for the proof, and also refer to Theorem 3.1 of
Strang and Fix [22].

6. Smoothness of $u$ on $\Omega$.

**Lemma 3.7.** Let $u$ be the harmonic solution in $\Omega$. Then $u \circ \varphi_j^{-1}$ ($j = 1, \ldots, m$)
are of the class $C^2$ on $\varphi_j(U_j \cap \Omega)$ respectively.

**Proof.** (i) The case where $U_j$ contains some $p_k$ ($k = 1, 2, 3, 4$).

Let us assume that $U_j$ contains $p_j$. The other cases are also similar. Then,
$\varphi_j(p_j) = 0$, $\varphi_j(U_j \cap \Omega) = \{ |z| < \rho_j \} \cap \{ 0 \leq \arg z \leq \pi/2 \}$,

$$u \circ \varphi_j^{-1} = 0 \quad \text{on} \quad \{ |z| \Im z = 0, 0 \leq \Re z \leq \rho_j \},$$
and

$$\frac{\partial}{\partial n} u \circ \varphi_j^{-1} = 0 \quad \text{on} \quad \{ |z| \Re z = 0, 0 < \Im z \leq \rho_j \},$$

where by $\partial/\partial n$ we denote the inner normal derivative. By (3.15) and (3.16) we
see that $u \circ \varphi_j^{-1}$ can be harmonically continued to $\varphi_j(U_j \cap \Omega)$ and thus
especially is of the class $C^2$ on $\varphi_j(U_j \cap \Omega)$.

(ii) The case where $\varphi_j(U_j \cap \Omega) = \{ |z| < \rho_j \} \cap \{ \alpha_j \leq \arg z \leq \pi/2 \}$.

Let $g$ be the function defined on $D = \{ \Im z > 0 \} \cap \{ |z| < \rho_j^{(\alpha)} \}$ by
$g(z) = \varphi_j^{-1}(z^{\alpha_j/2})$. Since $\Re g = \text{const.}$ or $\Im g = \text{const.}$ on
$\{ \Im z = 0 \} \cap \{ |z| < \rho_j^{(\alpha)} \}$, $g$ is
analytic on the closure $\overline{D}$. Then

$$\frac{d}{dz} u \circ \varphi_j^{-1}(z) = \frac{d}{dz} \varphi_j^{-1}(z^{\alpha_j/2}) \cdot \frac{\pi}{\alpha_j} z^{\alpha_j - 1}$$
and

$$\frac{d^2}{dz^2} u \circ \varphi_j^{-1}(z) = \frac{d^2}{dz^2} \varphi_j^{-1}(z^{\alpha_j/2}) \cdot \left( \frac{\pi}{\alpha_j} \right)^2 z^{2(\alpha_j - 1)} + \frac{d}{dz} \varphi_j^{-1}(z^{\alpha_j/2}) \cdot \frac{\pi}{\alpha_j} \left( \frac{\pi}{\alpha_j - 1} \right) z^{\alpha_j - 2}$$
on $\varphi_j(U_j \cap \Omega)$. Hence, $\alpha_j \leq \pi/2$ implies that $d^2 u \circ \varphi_j^{-1}/dz^2$ is continuous on
$\varphi_j(U_j \cap \Omega)$ and thus $u \circ \varphi_j^{-1} = \Re \varphi_j^{-1}$ is of the class $C^2$ on $\varphi_j(U_j \cap \Omega)$.

(iii) The cases except (i) and (ii).

Since $u \circ \varphi_j^{-1} = \text{const.}$ or $\partial u \circ \varphi_j^{-1}/\partial n = 0$ on $\varphi_j(U_j \cap C) = \{ |z| < \rho_j \} \cap \{ \Im z = 0 \}$, or
$\varphi_j(U_j \cap C) = \emptyset$, $u \circ \varphi_j^{-1}$ is harmonic on $\varphi_j(U_j \cap \Omega)$.

7. Approximation by $\omega_k$.

**Theorem 3.1.** Let $u$ be the harmonic solution in $\Omega$ defined in § 3.1 and let
$\omega_h$ be the finite element approximation of $u$ in $S$. Then,

\begin{equation}
D(\omega_h - u) \leq \frac{h^2}{\sin^2 \theta} \left( B \sum_{j=1}^{m} \left[ \int_{\varphi_j(K_j)} \left( \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 u}{\partial y^2} \right)^2 \right) dxdy \right) + C h^2 \max_{j=1}^{m} \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 u}{\partial y^2} \right)^2 \right),
\end{equation}

where $B$ and $C$ are constants independent of the triangulation $K$ and the function $u$, $\theta$ is the smallest value of interior angles of all triangles $\varphi_j(s) \in K_j$; $j=1, \ldots, m$, by $\varphi_j(K_j)$ we denote the image set by $\varphi_j$ of the carrier of $K_j$, and $R_j$ ($j=1, \ldots, m$) are the closed subsets of $U_j \cap \bar{Q}$ defined in (i') of §1.2.

**Proof.** First, by (i) of Lemma 3.2,

\begin{equation}
D(\omega_h - u) \leq D(\bar{u} - u).
\end{equation}

Hence it is sufficient to estimate $D(\bar{u} - u)$.

We have

\begin{equation}
D(\bar{u} - u) = \sum_{j=1}^{m} \sum_{s \in K_j} D_s(\bar{u} - u),
\end{equation}

Here we note that by the Lemma 3.7 $u \circ \varphi_j^{-1} (j=1, \ldots, m)$ is of the class $C^2$ on $\varphi_j(U_j \cap \bar{Q})$. Then, by Lemma 3.5,

\begin{equation}
D_s(\bar{u} - u) \leq \frac{B}{\sin^2 \theta} h^2 \int_{\varphi_j(s)} \left( \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 u}{\partial y^2} \right)^2 \right) dxdy
\end{equation}

for each natural simplex $s$ of $K_j$. For simplicity, we denote the right hand side of (3.20) by $I[\varphi_j(s)]$.

For a triple $(s, s', \ell)$ for a minor simplex $s$, we denote the function $\bar{u}'$ on $\ell s \in K_j$ and $s' \in K_{j'}$ by $\bar{u}'_s$ and $\bar{u}'_{s'}$ respectively. Then, by Lemma 2.1,

\begin{equation}
D_s(\bar{u} - u) \leq D_s(\bar{u}'_s - u) + D_s(\bar{u}'_{s'} - u).
\end{equation}

This inequality and Lemma 3.5 imply that

\begin{equation}
D_s(\bar{u} - u) \leq I[\varphi_j(s)] + I[\varphi_{k}(s')].
\end{equation}

Let $(s, s', \ell)$ be a triple for a major simplex $s$. Then, by Lemma 3.5

\begin{equation}
D_s(\bar{u} - u) \leq I[\varphi_j(s)] + D_s(\bar{u} - u)
\end{equation}

and

\begin{equation}
D_{s'}(\bar{u} - u) \leq I[\varphi_{k}(s')].
\end{equation}

Let

\begin{align*}
\bar{u} &= ax + by + c \quad \text{on } \varphi_j(s), \quad \text{and} \\
\bar{u} &= \alpha \xi + \beta \eta + \gamma \quad \text{on } \varphi_{k}(s').
\end{align*}
where $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ are constants. Then we define functions $\tilde{u}_s$ and $\tilde{u}_{s'+\ell}$ on $s$ and $s'+\ell$ respectively by

$$
\tilde{u}_s = ax + by + c \quad \text{on} \quad \varphi(s), \quad \text{and}
$$

$$
\tilde{u}_{s'+\ell} = a\xi + b\eta + \gamma \quad \text{on} \quad \varphi(s'+\ell).
$$

Then, by Lemma 2.1

(3.25) \quad \Delta t(\tilde{u}_s - u) \leq \Delta t(\tilde{u}_{s'} - u) + \Delta t(\tilde{u}_{s'+\ell} - u).

Further, by Lemma 3.6

(3.26) \quad \Delta t(\tilde{u}_s - u) \leq A(\varphi(\ell)) \cdot \frac{32h^2}{\sin^2 \theta} \max_{\varphi_j(\ell)} \left( \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + 2 \left| \frac{\partial^2 u}{\partial x \partial y} \right|^2 + \left| \frac{\partial^2 u}{\partial y^2} \right|^2 \right)^{(1+\kappa h)^2}

and

(3.27) \quad \Delta t(\tilde{u}_{s'+\ell} - u) \leq A(\varphi(\ell')) \cdot \frac{32h^2}{\sin^2 \theta} \max_{\varphi_k(\ell'+\ell)} \left( \left( \frac{\partial^2 u}{\partial \xi^2} \right)^2 + 2 \left| \frac{\partial^2 u}{\partial \xi \partial \eta} \right|^2 + \left| \frac{\partial^2 u}{\partial \eta^2} \right|^2 \right)^{(1+\kappa h)^2}.

By (3.18)~(3.27), Lemma 1.1 and (1.1), the estimate (3.17) is obtained.

8. Approximation by $u_h$.

**Theorem 3.2.** Let $u$ be the harmonic solution in $\mathcal{G}$ defined in §3.1, let $u_h'$ be the finite element approximation of $u$ in $S'$ and let $u_h = F^{-1}(u_h)$.

(i) The estimate

(3.28) \quad \Delta (u_h - u) \leq \frac{h^2}{\sin^2 \theta} \left( A' \sum_{j=1}^m \int_{\varphi_j(K_j)} \left( \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 u}{\partial y^2} \right)^2 \right) dxdy \right.

\[ + B'h^2 \sum_{j=1}^m \max_{\varphi_j(\ell)} \left( \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 u}{\partial y^2} \right)^2 \right) \]

\[ + C'h^2 \sum_{j=1}^m \max_{\varphi_j(\ell)} \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \]

holds, where $A'$, $B'$ and $C'$ are constants independent of the triangulation $K$ and the function and other notations are the same as in Theorem 3.1.

(ii) The estimate

(3.29) \quad \Delta (u) \leq \Delta (u_h) + \varepsilon(u_h)

holds with

$$
\varepsilon(u_h) \equiv \sum_{j=1}^m \sum_{\varphi \in K_j} A(\varphi(\ell)) \cdot \frac{(u_h(q_2) - u_h(q_1))^2}{|\varphi(q_2) - \varphi(q_1)|^2} \cdot \max \left\{ 1, \frac{|\varphi(q_2) - \varphi(q_1)|^2}{|\varphi(q_2) - \varphi(q_1)|^2} \right\} \max \{|f'(z)|^2\},
$$

where $q_1$ and $q_2$ are the vertices of $\# \ell$, $f(z) \equiv \varphi_k \cdot \varphi_j^{-1}(z)$, and $\varepsilon(u_h)$ is a quantity of
$O(h^2)$ which can be numerically calculated.

**Proof.** (i) First, note that

$$D(u_h-u) \leq 2D(\omega_h-u)+2D(u_h-\omega_h).$$

From Lemmas 2.1, 2.2 and 3.3, and (3.11), it follows that

$$D(u_h-\omega_h) = D(u_h)-D(\omega_h)$$

$$\leq D(u'_h)-D(\omega_h)+\sum_{s\in K} D_{s}(u_h) \leq D(\omega'_h)-D(\omega_h)+\sum_{s\in K} D_{s}(u_h)$$

$$\leq \sum_{j=1}^{m} \sum_{b \in K_{j}} (A(\varphi_{j}(\varphi_{j})(a^2+b^2)) + A(\varphi_{j}(\varphi_{j})(a^2+b^2))$$

$$+ \sum_{j=1}^{m} \sum_{b \in K_{j}} (A(\varphi_{j}(\varphi_{j})(a^2+b^2)) + A(\varphi_{j}(\varphi_{j})(a^2+b^2)),$$

where for each triple $(s, s', b)$ for $b \in K_{j}$

$$\omega'_h = a'x+b'y+c' \quad \text{on} \ \varphi_{j}(t_{s}) \ \text{and}$$

$$\omega'_h = \alpha'_x+\beta'_y+\gamma' \quad \text{on} \ \varphi_{h}(s'),$$

and for each triple $(s, s', b)$ for $b \in K_{j}$

$$u_h = ax+by+c \quad \text{on} \ \varphi_{j}(t_{s}) \ \text{and}$$

$$u_h = a\xi+\beta\eta+\gamma \quad \text{on} \ \varphi_{h}(s'),$$

with constants $a'$, $b'$, $c'$, $\alpha'$, $\beta'$, $\gamma'$, $a$, $b$, $c$, $\alpha$ and $\gamma$.

In the inequality (3.31), we have

$$A(\varphi_{j}(\varphi_{j})(a^2+b^2)) = A(\varphi_{j}(\varphi_{j})(a^2+b^2))$$

$$\leq 2(A(\varphi_{j}(\varphi_{j})(a^2+b^2)) + A(\varphi_{j}(\varphi_{j})(a^2+b^2))$$

$$\leq 2A(\varphi_{j}(\varphi_{j})(a^2+b^2)) + 2A(\varphi_{j}(\varphi_{j})(a^2+b^2))$$

$$\leq 2A(\varphi_{j}(\varphi_{j})(a^2+b^2)) + 2A(\varphi_{j}(\varphi_{j})(a^2+b^2)).$$

Since we can easily verify that

$$A(\varphi_{j}(\varphi_{j})) > \frac{h^2}{4} \sin \theta \quad (h_1 = d(\varphi_{j}(\varphi_{j}))),$$

by Lemma 1.1 we have

$$A(\varphi_{j}(\varphi_{j})) = A(\varphi_{j}(\varphi_{j}))$$

$$\leq \frac{h}{2 \sin \theta} \left( \left| \frac{g''(\zeta_{j})}{g'(\zeta_{j})} \right| + O(h) \right)$$

with the notations in Lemma 1.1. (3.32) and (3.33) imply

$$\sum_{j=1}^{m} \sum_{b \in K_{j}} A(\varphi_{j}(\varphi_{j})(a^2+b^2))$$

$$\leq \frac{C}{\sin \theta} \sum_{j=1}^{m} \sum_{b \in K_{j}} D(\omega_h-u) + 2 \sum_{j=1}^{m} \sum_{b \in K_{j}} A(\varphi_{j}(\varphi_{j})(a^2+b^2)) \max_{\varphi_{j}(\varphi_{j})} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right).$$
where $C$ is a constant depending only on the transformations of local parameters. Since similar estimates for other terms of the right hand side of (3.31) are obtained, from (3.31) it follows that

\begin{equation}
D(u_h - \omega_h) \leq \frac{Ch}{\sin \theta} D(u_h - u) + \frac{Ch}{\sin \theta} D(\omega_h - u)
+ 2 \sum_{j=1}^{m} \sum_{i \in K_j} (A(\varphi_j(\ell)) \cdot \max_{\psi_j(\ell)} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right))
+ A(\varphi_k(\ell)) \cdot \max_{\psi_k(\ell)} \left( \left( \frac{\partial u}{\partial \xi} \right)^2 + \left( \frac{\partial u}{\partial \eta} \right)^2 \right).
\end{equation}

(3.30), (3.35), Theorem 3.1, Lemma 1.1 and (1.1) imply the estimate (3.28).

(ii) Lemmas 3.2 (ii), 3.3 and 2.2 (i) imply the inequalities

\[ D(u) \leq D(\omega_h) \leq D(u_h) \]
\[ \leq D(u_h) + \sum_{j=1}^{m} \sum_{i \in K_j} A(\varphi_j(\ell)) \left( \frac{u_h(q_j) - u_h(q_i)}{|\varphi_j(q_j) - \varphi_j(q_i)|} \right)^2 \]
\[ \cdot \max \left\{ 1, \frac{|\varphi_j(q_2) - \varphi_j(q_1)|}{|\varphi_k(q_2) - \varphi_k(q_1)|} \frac{1}{t} \frac{\max |f'(x)|^2}{f'(x)} \right\}. \]

9. **Estimate of** $D(u_h - \bar{u}').$

**COROLLARY 3.1.** Let $u$ and $u_h$ be the same as in Theorem 3.2, $\bar{u}$ be the finite element interpolation of $u$ in the class $S$, and $\bar{u}' = F(\bar{u})$. Then, the estimate

\begin{equation}
D(u_h - \bar{u}') \leq A^* h^2
\end{equation}

holds, where $A^*$ is a constant dependent only on $u$ and $\theta$ in Theorem 3.1.

**PROOF.** First, by Lemma 2.2 (ii) and (3.33) we have

\[ D(u_h - \bar{u}') \leq D(u_h - \bar{u}) + \sum_{i \in K} (D_s(u_h, a) - \bar{u}_a) + D_s(u_h, a', \bar{u}'_a) \]
\[ \leq D(u_h - \bar{u}) + \sum_{j=1}^{m} \sum_{i \in K_j} \left( A(\varphi_j(\ell)) \frac{D_s(u_h - \bar{u})}{A(\varphi_j(\ell))} \right)
+ \frac{A(\varphi_k(\ell))}{A(\varphi_k(\ell'))} D_s(u_h - \bar{u}) \]
\[ \leq D(u_h - \bar{u}) + \frac{Ch}{\sin \theta} \sum_{j=1}^{m} \sum_{i \in K_j} (D_s(u_h - \bar{u}) + D_s(u_h - \bar{u})) \]
\[ \leq \left( 1 + \frac{Ch}{\sin \theta} D(u_h - \bar{u}) \right) \leq 2 \left( 1 + \frac{Ch}{\sin \theta} \right) (D(u_h - u) + D(u - \bar{u})), \]

where $C$ is the same constant as in (3.34). Then, the proof of Theorem 3.1 and Theorem 3.2 imply (3.36).
§ 4. Applications.

1. **Modulus of a quadrilateral.** By (3.1) and Lemma 3.1 the equalities

\[ D(u) = M(Q) = \frac{1}{M(Q)} = \frac{1}{D(\bar{u})} \]  

hold, where \( u = \text{Re} \bar{f} \) and \( \bar{u} = \text{Re} \bar{f} \).

When we replace \( C_0 \) and \( C_1 \) by \( \tilde{C}_0 \) and \( \tilde{C}_1 \) respectively in the definition of the classes \( S, S' \) of functions, we obtain new classes \( \tilde{S}, \tilde{S}' \) corresponding to \( S, S' \) respectively. Let \( u_h' \) and \( \bar{u}_h' \) be the finite element approximations of \( u \) and \( \bar{u} \) in the spaces \( S' \) and \( S' \) respectively. Then by (ii) of Theorem 3.2 we have the estimates

\[ D(u) \leq D(u_h') + \varepsilon(u_h') \]  

and

\[ D(\bar{u}) \leq D(\bar{u}_h') + \varepsilon(\bar{u}_h') \]  

By (4.1), (4.2) and (4.3) we have upper and lower bounds for the modulus \( M(Q) \):

\[ \frac{1}{D(\bar{u}_h') + \varepsilon(\bar{u}_h')} \leq M(Q) \leq D(u_h') + \varepsilon(u_h') \]  

2. **Numerical example 1** (the example of Gaier [9]). Let \( \Omega \) be the simply connected domain on the \( z \)-plane defined by

\[ \Omega = \{ z \mid 0 < x < 1, 0 < y < 1 \} - \left\{ z \mid \frac{1}{2} \leq x < 1, \frac{1}{2} \leq y < 1 \right\}, \]

and let \( C_0 \) and \( C_1 \) be the boundary parts of \( \Omega \) defined by

![Figure 6. Numerical example 1 (Gaier's example).](image_url)
Figure 7. Triangulation of example 1.
Modulus of quadrilaterals

\[ C_0 = \{ z \mid 0 \leq x \leq \frac{1}{2}, \ y = 0 \} \cup \{ z \mid x = 0, \ 0 \leq y \leq 1 \} \cup \{ z \mid 0 \leq x \leq \frac{1}{2}, \ y = 1 \} \]

and

\[ C_1 = \{ z \mid \frac{1}{2} \leq x \leq 1, \ y = \frac{1}{2} \} \]

respectively, where \( z = x + iy \). Let \( Q \) be the quadrilateral with the two opposite sides \( C_0 \) and \( C_1 \) (cf. Figure 6). We aim to obtain good upper and lower approximate values of the modulus of \( Q \).

We construct a triangulation of the closed region \( \bar{Q} \) as in Figure 7. The closed regions \( G_2 \) and \( G_3 \) are mapped onto the regions \( G^*_2 \) and \( G^*_3 \) respectively by the local parameters \( \zeta = \phi_2(z) = a \sqrt{z-1/2} \) and \( \zeta = \phi_3(z) = b \sqrt{z-(1+i)/2} \) (\( a = 1 \) and \( b = e^{-\pi / 4} \)) respectively, where \( a \) and \( b \) are so determined that \( |d\zeta / dz| = 1 \) on \( |z-1/2| = 1/4 \) and \( |z-(1+i)/2| = 1/\sqrt{27} \) respectively. We construct ordinary triangulations \( K^*_2 \) and \( K^*_3 \) of \( G^*_2 \) and \( G^*_3 \) as in Figure 7 respectively. By \( K_2 \) and \( K_3 \) we denote the image triangulations of \( K^*_2 \) and \( K^*_3 \) by the mappings \( \phi_2^{-1} \) and \( \phi_3^{-1} \) respectively. The triangulation \( K_1 \) of the region \( G_1 = \bar{Q} - (G_2 \cup G_3) \) in Figure 7 is so constructed that each 2-simplex \( s \) of \( K_1 \) is natural or minor according as \( |s| \cap |K_2 + K_3| = \emptyset \) or \( |s| \cap |K_2 + K_3| \neq \emptyset \), where if some intersection is a point then it is interpreted to be vacuous, and the local parameter \( \phi_1(z) \) of \( K_1 \) is the identity mapping \( \phi_1(z) = z \).

Let \( u \) and \( \bar{u} \) be the functions on the present \( Q \) defined in § 4.1, and let \( u_\delta^* \) and \( \bar{u}_\delta^* \) be the finite element approximations of \( u \) and \( \bar{u} \) respectively in the classes \( S'(K') \) and \( S'(K') \) respectively, where \( K' \) is the naturalized triangulation associated to the present \( K \). To attain our aim it is sufficient to make numerical calculations of \( u_\delta^* \) and \( \bar{u}_\delta^* \) (cf. Mizumoto and Hara [13], [14] for the calculation method).

Table 1 shows the exact value of the modulus \( M(Q) \) (see Gaier [9] for the calculation method), Gaier’s computation results and the values of our finite element approximations. Furthermore, computation results for the normal subdivision \( K^4 \) (see Figure 8) of the present \( K \) are shown. We note that \( \varepsilon(u_\delta^*) = \varepsilon(\bar{u}_\delta^*) = 0 \) in the present example. It can be said that the both of upper and lower bounds of \( M(Q) \) by our method are much closer to the exact value than those by Gaier.

3. Numerical example 2 (the case of a Riemann surface). Let \( D_1 = \{ z \mid |z| < \infty \} - \{ z \mid 0 \leq x < \infty, \ y = 0 \} \) and \( C_2 \) be the upper boundary part of \( D_1 \) lying on \( \{ z \mid 1 \leq x < \infty, \ y = 0 \} \), where \( z = x + iy \). Let \( D_2 = \{ z \mid |z| < 1 \} - \{ z \mid 0 \leq x < 1, \ y = 0 \} \) and let \( C_1 \) be the boundary part of \( D_2 \) defined by \( C_1 = \{ z \mid |z| = 1, \ y \geq 0 \} \). Let \( Q \) be the simply-connected covering surface obtained by connecting \( C_1 \) and \( C_2 \) crosswise along the segment \( \{ z \mid 0 \leq x < 1, \ y = 0 \} \) (cf. Figure 9). Let \( Q \) be the quadrilateral with the opposite sides \( C_0 \) and \( C_1 \). By symmetricity of \( Q \) we
Figure 8. Normal subdivision of example 1.
Table 1. Modulus $M(Q)$ of example 1 (the example of Gaier [9]).

<table>
<thead>
<tr>
<th>Exact value</th>
<th>$M(Q) = D(u) = 1.279262$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaier's computation results (Gaier [9])</td>
<td></td>
</tr>
<tr>
<td>$h = 2^{-4}$</td>
<td>Upper bound = 1.49435 (0.21509)</td>
</tr>
<tr>
<td>$h = 2^{-7}$</td>
<td>Upper bound = 1.32659 (0.04733)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Original triangulation ($h = 2^{-4}$)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Upper bound</strong></td>
<td>$D(u'_h) + e(u'_h)$</td>
</tr>
<tr>
<td></td>
<td>$= 1.28396 + 0$</td>
</tr>
<tr>
<td><strong>Lower bound</strong></td>
<td>$\frac{1}{D(\bar{u}'_h) + e(\bar{u}'_h)}$</td>
</tr>
<tr>
<td></td>
<td>$= \frac{1}{0.783599 + 0}$</td>
</tr>
<tr>
<td></td>
<td>$= 1.27616$ (−0.00310)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Our computation results</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Upper bound</strong></td>
<td>$D(u'_h) + e(u'_h)$</td>
</tr>
<tr>
<td></td>
<td>$= 1.28046 + 0$</td>
</tr>
<tr>
<td><strong>Lower bound</strong></td>
<td>$\frac{1}{D(\bar{u}'_h) + e(\bar{u}'_h)}$</td>
</tr>
<tr>
<td></td>
<td>$= \frac{1}{0.782185 + 0}$</td>
</tr>
<tr>
<td></td>
<td>$= 1.27847$ (−0.00079)</td>
</tr>
</tbody>
</table>

(): Deviation from exact value.

Figure 9. Numerical example 2 (the case of a Riemann surface).
immediately see that \( M(Q) = 1 \). We aim to obtain good upper and lower approximate values of \( M(Q) \). The present example is one which exhibits remarkable validity of our method. Namely, it is shown that an unbounded covering surface over the \( z \)-plane with many inner and corner singularities of high order, and with a curvilinear boundary is dealt with by our local treatment method without use of any global conformal mapping.

We construct a triangulation of the bordered region \( \tilde{Q} \) as in Figures 10 and 11. In Figure 10, the closed regions \( G_1 \cup G_2 \cup \cdots \cup G_5, G_6 \cup G_7 \) and \( G_8 \) are mapped onto the regions \( G_1^{*} \cup G_2^{*} \cup \cdots \cup G_5^{*}, G_6^{*} \cup G_7^{*} \) and \( G_8^{*} \) respectively by the mappings \( \zeta = \phi_1(z) = (1/4) \cdot \log z, \zeta = \phi_2(z) = 1/z \) and \( \zeta = \phi_3(z) = \sqrt{\zeta} \) respectively. Further, the regions \( G_6^{*}, G_7^{*}, G_8^{*} \) and \( G_4^{*} \) are mapped onto the regions \( G_4^{**}, G_5^{**}, G_6^{**} \) and \( G_7^{**} \) respectively by the mappings \( Z = \phi_4(\zeta) = \sqrt{\zeta}, Z = \phi_5(\zeta) = e^{-\pi i/4}, \sqrt{\zeta} - \pi i/2, Z = \phi_6(\zeta) = e^{-\pi i/4}, \sqrt{\zeta} - 3\pi i/4 \) and \( Z = \phi_7(\zeta) = \sqrt{2} \cdot \sqrt{\zeta} \) respectively. Let \( \phi_4(z) = \phi_4 \cdot \phi_1(z), \phi_5(z) = \phi_5 \cdot \phi_2(z), \phi_6(z) = \phi_6 \cdot \phi_3(z) \) and \( \phi_7(z) = \phi_7 \cdot \phi_4(z) \). We note that \( |d\phi_1/dz| = 1 \) on \( |z| = 1/4 \), \( |d\phi_3/d\zeta| = 1 \) on \( |\zeta| = 1/\sqrt{27}, |d\phi_4/d\zeta| = 1 \) on \( |\zeta - \pi i/2| = 1/\sqrt{27}, |d\phi_4/d\zeta| = 1 \) on \( |\zeta - 3\pi i/4| = 1/4, |d(\phi_5 \cdot \phi_1)/d\zeta| = 1 \) on \( \Re(\zeta) = (1/4) \log 4, |d\phi_7/d\zeta| = 1 \) on \( |\zeta| = 1/4 \) and \( |d\phi_4/dz| = 1 \) on \( |z| = 1/4 \). We construct ordinary triangulations \( K_1^{**}, K_2^{**}, K_3^{**}, K_4^{**}, K_5^{**}, K_6^{**}, K_7^{**}, K_8^{**} \) of \( G_1^{**}, G_2^{**}, G_3^{**}, G_4^{**}, G_5^{**}, G_6^{**}, G_7^{**}, G_8^{**} \) as in Figure 11 respectively. By \( K_1, K_2, K_3, K_4, K_5, K_6, K_7 \) and \( K_8 \) we denote the image triangulations of \( K_1^{**}, K_2^{**}, K_3^{**}, K_4^{**}, K_5^{**}, K_6^{**}, K_7^{**} \) and \( K_8^{**} \) respectively by the mappings \( \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7 \) and \( \phi_8 \) respectively, and the local parameters of \( K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8 \) are \( Z = \phi_1(z), Z = \phi_2(z), Z = \phi_3(z), Z = \phi_4(z), Z = \phi_5(z) \) and \( Z = \phi_8(z) \) respectively. The triangulations \( K_1, K_2, K_3, K_4, K_5, K_6, K_7 \) and \( K_8 \) are constructed as in Figure 11 so that each 2-simplex \( s \) of \( K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8 \) is natural or minor according as \( |s| \cap |K_1 + K_2 + K_3| \neq \emptyset \) or \( |s| \cap |K_1 + K_2 + K_3| = \emptyset \), where the local parameter of \( K_1 + K_2 + K_3 \) is \( \zeta = \phi_1(z) \). Also the triangulation \( K_8 \) of \( G_8 \) is so constructed that each 2-simplex \( s \) of \( K_8 \) is natural, minor or major according as \( |s| \cap |K_1 + K_2 + K_3| = \emptyset \) or \( |s| \cap |K_1 + K_2 + K_3| \neq \emptyset \), where the local parameter of \( K_1 + K_2 + K_3 \) is \( \zeta = \phi_8(z) \). Further, the triangulation \( K_8 \) of \( G_8 \) is so constructed that each 2-simplex \( s \) of \( K_8 \) is natural, minor or major according as \( |s| \cap |K_1 + K_2 + K_3| = \emptyset \) or \( |s| \cap |K_1 + K_2 + K_3| \neq \emptyset \), where the local parameter of \( K_1 + K_2 + K_3 \) is the identity mapping \( \phi_8(z) = z \).

Let \( u \) and \( \tilde{u} \) be the functions on the present \( \tilde{Q} \) defined in § 4.1, and let \( u_h \) and \( \tilde{u}_h \) be the finite element approximations of \( u \) and \( \tilde{u} \) respectively in the classes \( S'(K') \) and \( \tilde{S}'(K') \) respectively, where \( K' \) is the naturalized triangulation associated to the present \( K \). To attain our aim it is sufficient to make numerical calculations of \( u_h \) and \( \tilde{u}_h \).

Now the function \( u \) is obtained by the following procedure. Let \( J \) be the rectangular domain

\[ J = \{ W \mid 0 < \Re W < 1, 0 < \Im W < 1 \}, \]

and let \( \Gamma_0 \) and \( \Gamma_1 \) be the boundary parts of \( J \) defined by
Modulus of quadrilaterals

Figure 10. Local parameters of example 2.
Figure 11. Triangulation of example 2. (Note: The scales are identical in all figures except the lower sheet extended to infinity.)
The conformal map \( W = \psi(p) \) such that \( Q \) is conformally mapped onto \( \mathbb{D} \) so that \( C_0 \) and \( C_1 \) are mapped onto \( \Gamma_0 \) and \( \Gamma_1 \) respectively, is constructed by the composition of the following mappings, and then \( u = \Re\psi(p) \):

(i) \( w = \sqrt{z} \);

(ii) \( \zeta = \left( \frac{w-1}{w+1} \right)^{1/3} \);

(iii) \( \frac{Z-Z_1}{Z-Z_3} \cdot \frac{Z-Z_2}{Z-Z_3} = \frac{\zeta-\zeta_1}{\zeta-\zeta_3} \cdot \frac{\zeta_3-\zeta_2}{\zeta_3-\zeta_1} \)

where \( \zeta_i = 0, \zeta_2 = -1, \zeta_3 = 1, Z_1 = 1, Z_2 = -1 \) and \( Z_3 = 1/k \) with \( 1/k = 3 + 2\sqrt{2} \);

(iv) \( W = -\frac{1}{2K} \left( \int_0^1 \frac{dz}{\sqrt{(1-Z^2)(1-k^2Z^2)}} - (K + iK') \right) \),

where \( K = K(k) \) and \( K' = K'(k) \) are the complete elliptic integrals.

Table 2 shows the values of our finite element approximations. Furthermore,
computation results for the normal subdivision $K^1$ of the present $K$ are shown. It can be said that the both of upper and lower bounds of $M(Q)$ are close to the exact values.

4. Numerical example 3 (the case of an unbounded domain; cf. §4.3 of [15]). Let $Q=\{z|y>0\}$, and let $C_0$ and $C_1$ be the boundary parts of $Q$ defined by $C_0=\{z|\ -3\leq x \leq -1,\ y=0\}$ and $C_1=\{z|1\leq x \leq 3,\ y=0\}$ respectively, where $z=x+iy$. Let $Q$ be the quadrilateral with the two opposite sides $C_0$ and $C_1$ (cf. Figure 12). We obtain good upper and lower approximate values of the modulus of $Q$. See §4.3 of [15] for the details. Table 3 shows the exact value of the modulus $M(Q)$ which can be calculated by making use of a complete elliptic integral, and the values of our finite element approximations.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{Figure 12. Numerical example 3 (the case of an unbounded domain).}
\end{figure}

\begin{table}[h]
\centering
\caption{Modulus $M(Q)$ of example 3 (the case of an unbounded domain).}
\begin{tabular}{|c|c|c|}
\hline
\textbf{Exact value} & $M(Q)=D(u)=0.781701$ & \\
\hline
\textbf{Original triangulation ($h=0.213758$)} & & \\
\hline
\textbf{Upper bound} & $D(u_h)+\varepsilon(u_h)$ & $D(u_h'-\hat{a}')$
\begin{align*}
&=0.782184+0.429347 \times 10^{-3} \\
&=0.782013 \quad (0.000912)
\end{align*}
\hline
\textbf{Lower bound} & $\frac{1}{D(\hat{a}_h')}$ & $D(\hat{a}_h'-\hat{b}')$
\begin{align*}
&=\frac{1}{1.280878+0.150405 \times 10^{-5}} \\
&=0.780714 \quad (-0.000987)
\end{align*}
\hline
\textbf{Finite element approximations} & & \\
\hline
\textbf{Normal subdivision ($h=0.106879$)} & & \\
\hline
\textbf{Upper bound} & $D(u_h)+\varepsilon(u_h)$ & $D(u_h'-\hat{a}')$
\begin{align*}
&=0.781968+0.107413 \times 10^{-3} \\
&=0.782075 \quad (0.000374)
\end{align*}
\hline
\textbf{Lower bound} & $\frac{1}{D(\hat{a}_h')}$ & $D(\hat{a}_h'-\hat{a}')$
\begin{align*}
&=\frac{1}{1.279506+0.381486 \times 10^{-6}} \\
&=0.781551 \quad (-0.000150)
\end{align*}
\hline
\end{tabular}
\end{table}

( ): Deviation from exact value.
5. **Numerical example 4** (the case of a curvilinear domain; cf. § 4.4 of [15]). Let

\[ \Omega = \{ z \mid \frac{x^2}{16} + \frac{y^2}{15} < 1, \ y > 0 \} \]

![Figure 13. Numerical example 4 (the case of a curvilinear domain: quadrilateral \( \Omega \)).](image)

![Figure 14. Numerical example 4 (the case of a curvilinear domain: quadrilateral \( \Omega' \)).](image)

<table>
<thead>
<tr>
<th>Table 4. Modulus ( M(Q) ) of example 4 (the case of a curvilinear domain).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact value</td>
</tr>
<tr>
<td>---------------------------------------------------------------</td>
</tr>
<tr>
<td><strong>Original triangulation (( h = 0.138840 ))</strong></td>
</tr>
<tr>
<td>Upper bound</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Finite element approximation</strong></td>
</tr>
<tr>
<td>Upper bound</td>
</tr>
<tr>
<td>Lower bound</td>
</tr>
</tbody>
</table>

( ) : Deviation from exact value.
and let $C_0$ and $C_1$ be the boundary parts of $Q$ defined by

$$C_0 = \{ z | 3 \leq x \leq 4, y = 0 \} \cup \left\{ z \left| \frac{x^2}{16} + \frac{y^2}{16} = 1, y \geq 0 \right. \right\}$$

and

$$C_1 = \{ z | -1 \leq x \leq 1, y = 0 \}$$

respectively, where $z = x + iy$. Let $Q$ be the quadrilateral with the opposite sides $C_0$ and $C_1$ (cf. Figure 13).

Further, let $C'_0$ and $C'_1$ be the boundary parts of $Q'$ defined by

$$C'_0 = \{ z | 1 \leq x \leq 3, y = 0 \}$$

and

$$C'_1 = \{ z | -4 \leq x \leq -1, y = 0 \} \cup \left\{ z \left| \frac{x^2}{16} + \frac{y^2}{16} = 1, y \geq 0 \right. \right\}$$

respectively, where $z = x + iy$. Let $Q'$ be the quadrilateral with the opposite sides $C'_0$ and $C'_1$ (cf. Figure 14).

We obtain good upper and lower approximate values of the modulus of $Q$ and $Q'$. See § 4.4 of [15] for the details. Tables 4 and 5 show the exact values.

### Table 5. Modulus $M(Q')$ of example 4 (the case of a curvilinear domain).

<table>
<thead>
<tr>
<th>Exact value</th>
<th>$M(Q') - D(u) = -1.839350$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original triangulation ($h=0.138840$)</td>
<td></td>
</tr>
</tbody>
</table>
| Upper bound | $D(u'_h) + \varepsilon(u'_h)$  
= 1.841976 + 0.351532 × 10^{-3}  
= 1.842328  
(0.00298) | $D(u'_h - \hat{u}'')$  
= 5.86445 × 10^{-5} |
| Lower bound | $\frac{1}{D(u'_h) + \varepsilon(u'_h)}$  
= \frac{1}{0.544588 + 0.145580 × 10^{-3}}  
= 1.835760  
(-0.00359) | $D(u'_h - \hat{u}'')$  
= 2.73084 × 10^{-5} |
| Finite element approximation | | |
| Normal subdivision ($h=0.069420$) | | |
| Upper bound | $D(u'_h) + \varepsilon(u'_h)$  
= 1.840016 + 0.875764 × 10^{-4}  
= 1.840104  
(0.00075) | $D(u'_h - \hat{u}'')$  
= 5.22641 × 10^{-6} |
| Lower bound | $\frac{1}{D(u'_h) + \varepsilon(u'_h)}$  
= \frac{1}{0.543904 + 0.361871 × 10^{-4}}  
= 1.838437  
(-0.00091) | $D(u'_h - \hat{u}'')$  
= 3.00439 × 10^{-6} |

( ) : Deviation from exact value.
values of the modulus $M(Q)$ and $M(Q')$ respectively (see § 4.4 of [15] for the calculation method) and the values of our finite element approximations.

References


Heihachiro HARA
Tamano Laboratory
Mitsui E & S Co., Ltd.
Tamano, Okayama 706
Japan

Hisao Mizimoto
Faculty of Integrated Arts
and Sciences
Hiroshima University
Hiroshima 730
Japan