The tightness about sequential fans and combinatorial properties

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1. Introduction.

Let \( \kappa \) be an infinite cardinal. The sequential fan \( S_\kappa \) with \( \kappa \)-many spines is the quotient space obtained from the disjoint union of \( \kappa \)-many convergent sequences by identifying all the limit points to a single point denoted by \( \infty \). To be precise, \( S_\kappa = \{ \infty \} \cup (\kappa \times \omega) \) as a set, every point of \( \kappa \times \omega \) is isolated, and a basic neighborhood of \( \infty \) is of the form

\[
U_\psi = \{ \infty \} \cup \{ \langle \alpha, n \rangle : n \geq \varphi(\alpha) \}
\]

where \( \varphi \in \omega^\kappa \).

For a topological space \( X \), the tightness of \( X \), \( t(X) \), is the smallest cardinal \( \lambda \) such that for every point \( x \in X \) and \( A \subseteq X \), if \( x \in \text{cl} A \) then there exists \( B \subseteq A \) with \( |B| \leq \lambda \) and \( x \in \text{cl} B \).

It follows immediately from the definition that \( t(X) \leq |X| \) and it is easily seen that \( t(S_\kappa) = \omega \) for each \( \kappa \). But the tightness of the product space of two sequential fans is more complicated.

Gruenhage [4] proved that \( t(S_{\omega_1} \times S_{\omega_1}) = \omega_1 \), but it is an open question whether \( t(S_{\omega_1} \times S_\omega) = \omega_2 \) holds in ZFC. Moreover, such a question whether \( t(S_\kappa \times S_\kappa) = \kappa \) or not, is equivalent to another question related to the collectionwise Hausdorff property. (See [3, 8] for details.)

In this paper we shall give a combinatorial characterization of the tightness of \( S_\omega \times S_\kappa \) for an infinite cardinal \( \kappa \). Especially the tightness of \( S_\omega \times S_\omega \) has a natural combinatorial characterization.

To begin with, let us review the definitions of two familiar cardinals with combinatorial characterizations, \( b \) and \( d \).

**Definition 1.1.** For \( f, g \in \omega^\omega \), \( f \leq^* g \) if for all but finitely many \( n \in \omega \) we have \( f(n) \leq g(n) \). A family \( \mathcal{D} \subseteq \omega^\omega \) is unbounded (respectively dominating) if for every \( f \in \omega^\omega \) there exists \( g \in \mathcal{D} \) such that \( g \leq^* f \) (respectively \( f \leq^* g \)). The unbounding number \( b \) is the smallest size of the unbounded family of \( \omega^\omega \), and the dominating number \( d \) is the smallest size of the dominating family of \( \omega^\omega \).
Now we introduce a new cardinal invariant $b^*$, which is defined with the notion of the unbounded family but differs from $b$.

**Definition 1.2.** $b^*$ is the smallest cardinal $\lambda$ such that, for every unbounded family $\mathcal{G} \subseteq \omega^\omega$, there exists a subfamily $\mathcal{G}' \subseteq \mathcal{G}$ such that $|\mathcal{G}'| \leq \lambda$ and $\mathcal{G}'$ is still unbounded.

Using this notion we can state our main results:

**Theorem 1.3.**
1. For $\omega \leq \kappa < b$, $t(S_\omega \times S_\kappa) = \omega$ holds.
2. $t(S_\omega \times S_b) = b$.
3. For $\kappa \geq b^*$, $t(S_\omega \times S_\kappa) = b^*$ holds.

**Theorem 1.4.**
1. $b < b^* < b$.
2. Both $b < b^*$ and $b^* < b$ are consistent with ZFC.

What happens about $t(S_\omega \times S_\kappa)$ for $b < \kappa < b^*$? In fact it is undecidable under ZFC, that is, both $t(S_\omega \times S_\kappa) = \kappa$ and $t(S_\omega \times S_\kappa) < \kappa$ are consistent with ZFC. To prove this, we study Hechler's result about dominating families of $\omega^\omega$ in Section 4.

Our notation is standard and we refer the reader to [7] for undefined notions.

For $f \in \omega^\omega$ and $\varphi \in \omega^\varphi$ we shall use the notation $U_{f, \varphi}$ rather than $U_f \times U_\varphi$ for the neighborhood of $(\infty, \infty)$ determined by $f$ and $\varphi$. We shall also use $\langle k, m, \alpha, n \rangle$ instead of $\langle \langle k, m \rangle, \langle \alpha, n \rangle \rangle$ to denote points of $S_\omega \times S_\kappa$.

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## 2. Characterization of the tightness of $S_\omega \times S_\kappa$

In this section, we shall give a combinatorial characterization of the tightness of $S_\omega \times S_\kappa$. To state the combinatorial characterization, a part of which is due to [1], we generalize a notion in Definition 1.2.

**Definition 2.1.** Let $b(\kappa)$ be the smallest infinite cardinal $\lambda$ satisfying the following: For every unbounded family $\mathcal{G} \subseteq \omega^\omega$ with $|\mathcal{G}| \leq \kappa$ there exists a subfamily $\mathcal{G}' \subseteq \mathcal{G}$ such that $|\mathcal{G}'| \leq \lambda$ and $\mathcal{G}'$ is still unbounded.

Using this notion $b^*$ is defined as $b(2^\omega)$.

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1) After the submission of the first version of this paper, we have had a chance to see a preprint of Brendle and LaBerge [1]. It deals with a closely related topic and gives a nice idea to simplify the proof of Theorem 1.3. Our previous combinatorial characterization was more complicated.
Theorem 2.2. For any infinite cardinal $\kappa$, $t(S_\kappa \times S_\kappa)$ is equal to $b(\kappa)$.

According to this theorem, it is easy to see Theorem 1.3.

Lemma 2.3. Let $\kappa$ and $\lambda$ be infinite cardinals. Then, $t(S_\kappa \times S_\kappa) \geq \lambda$ if there exists an unbounded family $\mathcal{I} = \{ f_\alpha : \alpha < \kappa \}$ such that any subfamily $\mathcal{I}' \subseteq \mathcal{I}$ with $|\mathcal{I}'| < \lambda$ is bounded.

Proof. Let $A = \{ \langle k, f_\alpha(k), k, \alpha \rangle : k < \omega \cap \alpha < \kappa \}$. We show $A$ witnesses $t(S_\omega \times S_\omega) \geq \lambda$. Let $h \in \omega^\omega$, $\varphi \in \kappa^\omega$. Since $\mathcal{I}$ is unbounded, there exists $\alpha < \kappa$ such that $f_\alpha \not\leq_* h$. We can find $k > \varphi(\alpha)$ such that $f_\alpha(k) > h(k)$ and so $\langle k, f_\alpha(k), k, \alpha \rangle \in A \cap U_{h, \varphi}$, which implies $\langle \infty, \infty \rangle \in \text{cl} A$.

Let $\mathcal{X} \subseteq A$ with $|\mathcal{X}| < \lambda$. There exists $I \subseteq \kappa$ such that $|I| < \lambda$ and $\mathcal{X} \subseteq \{ \langle k, f_\alpha(k), k, \alpha \rangle : k < \omega \land \alpha \in I \}$. By the assumption, there exists $h \in \omega^\omega$ such that $f_\alpha \not\leq_* h$ for all $\alpha \in I$. For $\alpha \in I$, we can put $\varphi(\alpha) < \omega$ so that $f_\alpha(k) \leq h(k)$ for any $k \geq \varphi(\alpha)$. Then, $U_{h', \varphi} \cap \mathcal{X} = \emptyset$, where $h'(k) = h(k) + 1$. This completes the proof. $\square$

Lemma 2.4. Suppose that $A \subseteq S_\omega \times S_\kappa$ satisfies that $\langle \infty, \infty \rangle \in \text{cl} A$ and $\langle \infty, \infty \rangle \not\in \text{cl} C$ for any countable $C \subseteq A$. Then, there exists $B \subseteq A$ such that $\langle \infty, \infty \rangle \in \text{cl} B$ and for any $k < \omega$ and $\alpha < \kappa$

1. $\{ n : \langle k, m, \alpha, n \rangle \in B \text{ for some } m < \omega \}$ and
2. $\{ m : \langle k, m, \alpha, n \rangle \in B \text{ for some } n < \omega \}$

are both finite.

Proof. First we prove that for any $k < \omega$ there exists $M < \omega$ such that $\{ n < \omega : \langle k, m, \alpha, n \rangle \in A \text{ for some } m > M \}$ is finite for all $\alpha < \kappa$. Suppose not, then we can take $k < \omega$ and $\alpha_M < \kappa$ for each $M < \omega$ so that $\{ n < \omega : \langle k, m, \alpha_M, n \rangle \in A \text{ for some } m > M \}$ is infinite. Now we claim that $\langle \infty, \infty \rangle \in \text{cl} \{ \langle k, m, \alpha_M, n \rangle \in A : k, m, M, n < \omega \}$, which contradicts the assumption. Fix $h \in \omega^\omega$ and $\psi \in \omega^\kappa$ arbitrarily and let $M = h(k)$. Then, by the choice of $\alpha_M$, we can find $M > M$ so that there exists $n \geq \varphi(\alpha_M)$ with $\langle k, m, \alpha_M, n \rangle \in A$.

Let $f(k)$ be greater than $M$, then $\{ n : \langle k, m, \alpha, n \rangle \in A \text{ for some } m \geq f(k) \}$ is finite. Symmetrically, we get $\varphi(\alpha)$ so that $\{ m : \langle k, m, \alpha, n \rangle \in A \text{ for some } n \geq \varphi(\alpha) \}$ is finite. Then, $B = A \cap U_{f, \psi}$ is the desired one. $\square$

Proof of Theorem 2.2. By Lemma 2.3, it suffices to show $t(S_\omega \times S_\omega) \leq b(\kappa)$. Gruenhage [4, Lemma 1] proved $t(S_\omega \times S_\omega) = \omega$ in case $\kappa < b$, which implies $t(S_\kappa \times S_\kappa) = b(\kappa)$. So, we assume $\kappa \geq b$.

Let $A \subseteq S_\omega \times S_\kappa$ be so that $\langle \infty, \infty \rangle \in \text{cl} A$ and assume that $\langle \infty, \infty \rangle \not\in \text{cl} C$ for any countable $C \subseteq A$. Then, by Lemma 2.4 we get $B \subseteq A$ with the properties in the lemma. Take an unbounded family $\mathcal{G}$ of strictly increasing functions with $|\mathcal{G}| = b$. We define $f_\mathcal{G}(k) = \max \{ 0 \cup \{ m : \exists n (\langle k, m, \alpha, n \rangle \in B \land k \leq g(n) ) \} \}$. 

First, we show \( \{ f^k : \alpha < \kappa \wedge g \in \mathcal{G} \} \) is unbounded.

Suppose \( f^k \leq^* f \) for all \( \alpha < \kappa \) and \( g \in \mathcal{G} \). Since \( \langle \alpha, \omega \rangle \in \text{cl} B \), there exists \( \alpha < \kappa \) such that the set \( \{ n : \exists k, m(f(k) < m \wedge \langle k, m, \alpha, n \rangle \in B \} \) is infinite. For \( n < \omega \) choose \( k_n \) so that \( f(k_n) < m \) and \( \langle k_n, m, \alpha, n' \rangle \in B \) for some \( m \omega, n' \geq n \).

Since \( \mathcal{G} \) is unbounded, there is \( g \in \mathcal{G} \) such that \( k_n \leq g(n) \) for infinitely many \( n \).

By the first property of Lemma 2.4, the correspondence from \( n \) to \( k_n \) is finite-to-one, so we can find \( n < \omega \) such that \( f(k_n) < m \) and \( \langle k_n, m, \alpha, n' \rangle \in B \).

Since \( g(n) \leq g(n') \) and by the definition of \( f(k_n) \), this implies \( f(k_n) \geq m > f(k_n) \), which contradicts \( f(k_n) < f(k_n) \).

We have shown that \( \{ f^k : \alpha < \kappa \wedge g \in \mathcal{G} \} \) is unbounded. There exists \( \mathcal{J} \) such that \( f^\kappa \leq^\kappa b() \) and \( \{ f^k : \alpha< \mathcal{J} \wedge g \in \mathcal{G} \} \) is unbounded. Let \( D = \{ \langle k, m, \alpha, n \rangle : \alpha \in \mathcal{J} \wedge k, m, n < \omega \} \). We claim that \( \langle \omega, \omega \rangle \in \text{cl} D \), which shows \( t(S_\omega \times S_\omega) \leq b(\kappa) \).

Take arbitrary \( h \in \omega^\omega \) and \( \varphi \in \omega^\omega \). Then we can find \( \alpha \in \mathcal{J} \) and \( g \in \mathcal{G} \) so that \( f^\kappa \leq^\kappa h \). By the definition of \( f^\kappa(k) \), \( f^\kappa(k) > 0 \) implies \( \langle k, f^\kappa(k), \alpha, n \rangle \in D \) for some \( n \) with \( k \leq g(n) \). Since \( f^\kappa \leq^\kappa h \), there are infinitely many \( n \) such that \( \langle k, f^\kappa(k), \alpha, n \rangle \in D \) and \( h(k) < f^\kappa(k) \) for some \( k \). So we can find \( n \geq \varphi(\alpha) \) and \( k < \omega \) with \( h(k) < f^\kappa(k) \) so that \( \langle k, f^\kappa(k), \alpha, n \rangle \in D \), i.e., \( U_{h, \varphi} \cap D \neq \emptyset \).

### 3. Relations between \( b, d \) and \( b^* \).

In this section we shall show that \( b^* \) is located between \( b \) and \( d \) but consistently different from both of them.

**Theorem 3.1.** \( b \leq b^* \leq d \).

**Proof.** \( b \leq b^* \) follows immediately from the definition of \( b^* \). To show \( b^* \leq d \), let \( \mathcal{G} \) be any unbounded family and \( \mathcal{D} = \{ g_\beta : \beta < b \} \) a dominating family.

For each \( \beta < b \), we can find \( f_\beta \in \mathcal{G} \) so that \( f_\beta \leq^\kappa g_\beta \). Let \( \mathcal{D} = \{ f_\beta : \beta < b \} \subseteq \mathcal{G} \).

Then, \( | \mathcal{G} | \leq d \) and \( \mathcal{G} \) is still unbounded.

Now we turn to the consistency proofs. Both of the models satisfying \( b^* < d \) and \( b < b^* \) are obtained by the Cohen extensions.

Before proving them, we observe a basic fact on the Cohen forcing. Let \( C_I = P\text{\textit{Fn}}(I, 2, \omega) \) be the canonical Cohen forcing notion for an infinite set \( I \) (see [7, Chapter 7]).

**Lemma 3.2 ([2, Corollary 3.5]).** For any infinite set \( I \), if \( \mathcal{G} \subseteq \omega^\omega \) is an unbounded family, then \( \models_{C_I} \text{"} \mathcal{G} \text{ is unbounded."} \)

**Definition 3.3.** For a forcing notion \( P \), a standard \( P \)-name \( \dot{f} \) for a real is a name uniquely determined by a system \( \{ A_m : m, n < \omega \} \) with the following:

1. \( A_m \subseteq P \) is an antichain of \( P \) and \( n \neq n' \) implies \( A_m \cap A_{m'} = \emptyset \).
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(2) $\bigcup_{n<\omega} A_{mn}$ is a maximal antichain of $P$, and
(3) For each $p \in A_{mn}$, $p \Vdash \neg \text{Pf}(m) = n$.

**Theorem 3.4.** Let $2^\omega = \lambda$. Then, in the Cohen extension by $C_{\kappa}$ for an infinite $\kappa$, any unbounded family $\mathcal{A}$ of $\omega^\omega$ has an unbounded subfamily of size less than or equal to $\lambda$.

**Proof.** For an infinite $I \subseteq \kappa$, let $X(I)$ be the collection of all standard $C_I$-names of reals and let $\mathcal{X} = X(\kappa)$. It suffices to deal with the case $\kappa > \lambda$. Suppose that there are $p_0 \subseteq C_\kappa$ and a collection $\mathcal{A}$ of standard $C_\kappa$-names for reals such that

$p_0 \Vdash \langle \forall \vec{a} \subseteq \mathcal{A} | \lambda \rightarrow \vec{a} \text{ is bounded} \rangle$.

Let $S = \{X(I) : I \in \mathcal{X} \wedge \text{supp}(p_0) \subseteq I\}$, then $S \subseteq [\mathcal{X}]^{\omega_1}$. $S$ is stationary, since it is unbounded and closed under unions of increasing $\omega_1$-sequences. By assumption and using Lemma 3.2, for each $X = X(I) \in S$ we get a standard $C_I$-name $\dot{g}_X$ for a real so that $p_0$ forces $\dot{f} \leq^* \dot{g}_X$ for all $\dot{f} \in \mathcal{A} \cap X$. By Fodor's lemma for $[\mathcal{X}]^{\omega_1}$ (see [6, Theorem 3.2]) there is a stationary set $S' \subseteq S$ such that $\dot{g}_X = \dot{g}$ for all $X \in S'$. Since $S'$ is unbounded in $[\mathcal{X}]^{\omega_1}$, we have $p_0 \Vdash \langle \forall \vec{a} \subseteq \mathcal{A} | \lambda \rightarrow \vec{a} \text{ is bounded} \rangle$, which is a contradiction. $\square$

**Corollary 3.5.** Assume CH. For a cardinal $\kappa$ of uncountable cofinality, $b = b^* = \omega_1$ and $b = \kappa$ hold in the forcing model by $C_\kappa$.

Using Lemma 3.2 and Theorem 3.4, we can easily prove both the consistency of $b < b^* < b$ and that of $b < b^* = b$.

**Proposition 3.6.** Assume MA+$\omega_1 < 2^\omega = \lambda \leq \kappa$ and $\kappa$ has uncountable cofinality. Then, $b = \omega_1$, $b^* = \lambda$ and $b = \kappa$ hold in the forcing model by $C_\kappa$.

**Proof.** Since MA and $2^\omega = \lambda$ hold in the ground model, we can take an unbounded family $\mathcal{A}$ of order type $\lambda$ with respect to $\leq^*$. Then, in the forcing model $\mathcal{A}$ is still unbounded by Lemma 3.2 and every subfamily of $\mathcal{A}$ of size $< \lambda$ must be bounded, since $\lambda$ is regular. This implies $\lambda \leq b^*$. On the other hand, $b^* \leq \lambda$ by Theorem 3.4. As is well-known, $b = \omega_1$ and $b = \kappa$ hold in the forcing model by $C_\kappa$. $\square$

4. More on $b^*$ and the tightness of $S_\omega \times S_\kappa$.

In this section we study Hechler's result about dominating families of $\omega^\omega$ and show that $t(S_\omega \times S_\kappa)$ for $b < \kappa < b^*$ may or may not be equal to $\kappa$.

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2) J. Brendle informed us that LaBerge and Landver [8] proved this same result by another method independently. The paper was published after the submission of the present paper.
To investigate structures of dominating subfamilies of \( \omega^* \), Hechler \([5]\) introduced the so-called Hechler Forcing. However, his paper had been written before the simplified forcing method appeared and consequently it involves some complicated presentation. Here, we introduce a simplified notion in the current presentation. Since our final purpose is to investigate the notions around the cardinals \( b, b^* \) and \( b \), we confine ourselves only to a well-founded partially ordered set \( R \).

**DEFINITION 4.1.** Let \( R \) be a well-founded partially ordered set. We define forcing notions inductively.

A member of a partially ordered set \( H_a \) for \( a \in R \) is of the form \( \langle s_b, \mathcal{A}_b \rangle : b \in F \) with the following:

1. \( F \) is a finite subset of \( \{ b \in R : b \leq a \} \);
2. \( s_b \in \omega^\omega \) for \( b \in F \);
3. For \( b \in F \), \( \mathcal{A}_b \) is a finite subset of standard names for reals such that if \( f \in \mathcal{A}_b \), \( f \) is an He-name for some \( c < b \).

\( \langle t_c, \mathcal{A}_c \rangle : c \in G \) extends \( \langle s_b, \mathcal{A}_b \rangle : b \in F \) if the following hold:

1. \( F \subseteq G \), and \( \mathcal{A}_c \supseteq \mathcal{A}_b \) and \( s_b \supseteq t_b \) for \( b \in F \);
2. For each \( b \in F \), \( c < b \), an \( H_c \)-name \( j \in \mathcal{A}_b \) and \( k \in \text{dom}(t_b) \setminus \text{dom}(s_b) \), we have
   \[
   \langle t_c, \mathcal{A}_c \rangle : d \in G \wedge (d \leq c) \vdash H_c j(k) \leq t_b(k).
   \]

Finally, \( H_R \) is the set \( \bigcup_{a \in R} H_a \) with the ordering \( \bigcup_{a \in R} \leq_a \), where \( \leq_a \) is the ordering of \( H_a \).

Let \( G \) be the canonical name for an \( H_R \)-generic filter, i.e., \( p \vdash \lnot p \in G \) for \( p \in H_R \) and let \( \dot{a}_a \) be the name for \( \bigcup \{ s_a : \langle s_a, \mathcal{A} \rangle \in p \in G \) for some \( p, \mathcal{A} \} \) for each \( a \in R \).

Note that if \( a < b \) we can put \( \dot{a}_a \) in \( \mathcal{A}_b \).

**LEMMA 4.2.** (1) \( H_R \) satisfies c.c.c.

(2) For \( a \leq b \), the inclusion from \( H_a \) to \( H_b \) is a complete embedding and so is the inclusion from \( H_a \) to \( H_R \).

(3) For \( a, b \in R \), \( a \leq b \) implies \( \vdash \dot{a}_a \leq \dot{a}_b \) and \( a \not\leq b \) implies \( \vdash \dot{a}_a \not\leq \dot{a}_b \).

(4) If any countable subset of \( R \) has a strict upper bound in \( R \), \( \vdash \" \{ \dot{a}_a : a \in R \} \) is a dominating family."

Now it is easy to see the following:

**PROPOSITION 4.3.** Let \( R = \omega_1 \times \omega_1 \times \omega_1 \) with the product ordering. Then \( b=\omega_1, b^*=b=\omega_1, \) and \( t(S_{\omega_1 \times S_{\omega_1}}) = \omega_1 \) hold in the forcing model by \( H_R \).

**PROPOSITION 4.4.** Let \( R = \omega_1 \times \omega_3 \) with the product ordering. Then \( b=\omega_1, b^*=b=\omega_1, \) and \( t(S_{\omega_1 \times S_{\omega_1}}) = \omega_1 \) hold in the forcing model by \( H_R \).
PROOF. By Lemma 4.2 there exists a dominating family \( \{d_a : a \in \mathbb{R}\} \) such that \( d_a \leq^* d_b \) iff \( a \leq b \) in the product ordering. Now, the first two statements are clear. To show the last one, let \( \mathcal{S} \) be an unbounded family of size \( \omega_2 \). For \( f \in \mathcal{S} \) and \( \alpha < \omega_1 \), let \( \beta(f, \alpha) < \omega_2 \) such that \( f \leq^* d_{\alpha, \beta(f, \alpha)} \) if such \( \beta(f, \alpha) \) exists and \( \beta(f, \alpha) = 0 \) otherwise. Let \( \beta_0 = \sup \{\beta(f, \alpha) : f \in \mathcal{S} \wedge \alpha < \omega_1\} < \omega_3 \) and take \( \mathcal{G} \subseteq \mathcal{S} \) so that \( |\mathcal{G}| = \omega_1 \) and \( d_{\alpha, \beta_0} \) does not bound \( \mathcal{G} \) for any \( \alpha < \omega_1 \). Then, \( \mathcal{G} \) is unbounded. □

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