On the regularity of homeomorphisms of $E^n$.

By Tatsuo Homma and Shin'ichi Kinoshita

(Received Feb. 16, 1953)

Introduction. Let $X$ be a compact metric space and $h$ a homeomorphism of $X$ onto itself. The homeomorphism $h$ has been called by B. v. Kerekjarto [3] regular at $p \in X$, if $h$ satisfies the following condition: for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $x$ with $d(p, x) < \delta$ and for each integer $m$

$$d(h^m(p), h^m(x)) < \epsilon.$$

One of the purpose of this paper is to prove the following

**Theorem 1.** Let $X$ be a compact metric space and $h$ a homeomorphism of $X$ onto itself. Assume that $X$ and $h$ have the following property: there exist two distinct points $a$ and $b$ such that

(i) for each point $x \in X - b$ the sequence $\{h^m(x)\}$ converges to $a$ and

(ii) for each point $x \in X - a$ the sequence $\{h^{-m}(x)\}$ converges to $b$, where $m = 1, 2, 3, \ldots$.

Then $h$ is regular at every point of $X$ except for $a$ and $b$.

As a corollary of Theorem 1 we have the following

**Theorem 2.** Let $h$ be a homeomorphism of the $n$-dimensional sphere $S^n$ onto itself satisfying the same condition as that of Theorem 1. Then $h$ is regular at every point of $S^n$ except for $a$ and $b$.

Now let $S^n$ be the $n$-dimensional sphere in the $(n+1)$-dimensional Euclidean space $E^{n+1}$ and let $P$ be a point of $S^n$. Let $p(x)$ be the stereographic projection of $S^n - P$ from $P$ onto the $n$-dimensional Euclidean space $E^n$ tangent at the antipode $O$ of $P$, where we assume that $O$ is the origin of $E^n$. Let $h$ be a homeomorphism of $E^n$ onto itself. Put $\bar{h}(x) = p^{-1}hp(x)$ where $x \in S^n - P$ and put $\bar{h}(P) = P$. Then we have a homeomorphism $\bar{h}$ of $S^n$ onto itself. B. v. Kerekjarto [3] called a

1) The numbers in the brackets refer to the references at the end of this paper.
homeomorphism \( h \) of \( E^n \) onto itself regular at \( p \in E^n \), if \( h \) is regular at \( p^{-1}(p) \). By Theorem 2 we have immediately the following

**Theorem 3.** Let \( h \) be a homeomorphism of \( E^n \) onto itself satisfying the following conditions:

(i) for each \( x \in E^n \) the sequence \( \{h^m(x)\} \) converges to the origin \( O \),

(ii) for each \( x \in E^n \) except for \( O \) the sequence \( \{h^{-m}(x)\} \) converges to the point at infinity \( \infty \), where \( m = 1, 2, 3, \ldots \).

Then \( h \) is regular at every point of \( E^n \) except for \( O \).

If \( n = 2 \), in virtue of a theorem of KerékJártó [3], we have immediately the following

**Theorem 4.** Let \( h \) be a homeomorphism of the plane onto itself satisfying the same conditions as that of Theorem 3. If \( h \) is sense-preserving, then \( h \) is topologically equivalent to the transformation

\[
\begin{align*}
x' &= \frac{1}{2} x, \\
y' &= \frac{1}{2} y,
\end{align*}
\]

and if \( h \) is sense-reversing, then \( h \) is topologically equivalent to the transformation

\[
\begin{align*}
x' &= \frac{1}{2} x, \\
y' &= -\frac{1}{2} y,
\end{align*}
\]

in Cartesian coordinates.

Since Theorem 2 follows immediately from Theorem 1, Theorem 3 immediately from Theorem 2, and Theorem 4 immediately from Theorem 3, we shall prove in this paper Theorem 1 only. To this purpose a notion of bulging sequences will be introduced in §1. Then in §2 Theorem 1 will be proved. In §3 we shall give another application of bulging sequences in relation to the works of A. S. Besicovitch [1] [2].

---

### §1. Bulging sequences.

Let \( A \) be a subset of a separable metric space \( X \) and let \( f \) be a continuous mapping of \( X \) into itself. A sequence \( \{f^n(A)\} \) will be said to be a bulging sequence, if for each natural number \( n \)

\[
f^n(A) = \bigcup_{i=0}^{\infty} f^i(A) = 0.
\]
LEMMA 1. Let $A$ be compact. If $\bigcup_{n=0}^{\infty} f^n(A)$ is not compact, then \{f^n(A)\} is a bulging sequence.

Proof. Suppose on the contrary that \{f^n(A)\} is not a bulging sequence and that there exists a natural number $m$ such that

$$f^m(A) \subseteq f(A) \cup f^{m+1}(A) \cup \cdots \cup f^{m-n}(A).$$

Then it is easy to see that for each natural number $i$

$$f^{m+i}(A) \subseteq f(A) \cup f^{m+1}(A) \cup \cdots \cup f^{m-i}(A).$$

Therefore we have

\[
(*) \quad \bigcup_{n=0}^{\infty} f^n(A) = f(A) \cup f^{m+1}(A) \cup \cdots \cup f^{m+n}(A).
\]

Since a continuous image of a compactum is compact and since a finite sum of compacta is also compact, the right hand side of (*) is compact, which is a contradiction.

LEMMA 2. Let \{f^n(A)\} be a bulging sequence and let

$$C_n = A \setminus f^{-n}(f^n(A) \cup \bigcup_{j=0}^{m-1} f^j(A)).$$

for every natural number $n$. Then $C_n \neq 0$ and $C_n \supseteq C_{n+1}$.

Proof. First we prove that $C_n \neq 0$. Since \{f^n(A)\} is a bulging sequence, there exists a point $p \in f^n(A) - \bigcup_{j=0}^{n} f^j(A)$. Then there exists a point $q \in A$ such that $f^n(q) = p$ and then $q \in A \setminus f^{-n}(f^n(A) \cup \bigcup_{j=0}^{m-1} f^j(A)) \neq C_n$. Therefore $C_n \neq 0$.

Now we prove that $C_n \supseteq C_{n+1}$. Let $x$ be a point of $C_{n+1}$ and suppose that $x \in C_n$. Then there exists an $m > n$ such that $f^n(x) \in f^m(A)$. Therefore $f^{m+n}(x) \in f^{m+n+1}(A)$, which contradicts $x \in C_{n+1}$.

LEMMA 3. Let $A$ be compact and let \{f^n(A)\} be a bulging sequence. Then there exists a point $p \in A$ such that for each natural number $n$

$$f^n(p) \cap \text{Int}(A) = 0.$$  

Proof. Let $C_m$ be the same as in Lemma 2. Take $x_m \in C_m$. Since $A$ is compact, there exists a subsequence $\{x_{m_k}\}$ which converges to a point $p \in A$. Then \{f^n(x_{m_k})\} converges to $f^n(p)$ for every $n$. If $m_k > n$, then $f(x_{m_k}) \cap f^n(C_{m_k}) = f^n(C_{n+1})$ by Lemma 2. Since $f^n(C_m) \cap A = 0$ by the definition of $C_m$, $f^n(x_{m_k}) \cap A = 0$ for every $m_k > n$. Then we have $f^n(p) \cap \text{Int}(A) = 0$ for every $n$, and the proof is complete.
§ 2. Proof of Theorem 1.

In § 2 we suppose that $X$ is a non-degenerated compactum. Take two distinct points $a$ and $b$ of $X$ and let $\varphi$ be a continuous real-valued function on $X$ such that

\[
\begin{align*}
-\frac{1}{2} \pi &\leq \varphi(x) \leq \frac{1}{2} \pi \\
\varphi(x) &= \frac{1}{2} \pi \\
\varphi(x) &= -\frac{1}{2} \pi
\end{align*}
\]

for each $x \in X$, if and only if $x = a$, and if and only if $x = b$.

The existence of such a function is obvious. Put

\[\psi(x) = \tan \varphi(x) .\]

For each real number $r$ put

\[
A(r) = \{x \mid \psi(x) \geq r\} \cup a ,
\]

\[
B(r) = \{x \mid \psi(x) \leq r\} \cup b .
\]

It is easy to see that

(i) $A(r)$ and $B(r)$ are compact,

(ii) if $r > r'$, then $A(r) \subset A(r')$ and $B(r) \supset B(r')$,

(iii) if $r$ tends to $+\infty$, then $A(r)$ converges to $a$, and

(iv) if $r$ tends to $-\infty$, then $B(r)$ converges to $b$.

Now we prove the following

**Lemma 4.** Let $f$ be a continuous mapping of $X$ into itself such that for each $x \in X - b$ the sequence $\{f^n(x)\}$ converges to $a$. Then $\bigcup_{n=0}^{\infty} f^n(A(r))$ is compact for every $r$.

**Proof.** Suppose on the contrary that $\bigcup_{n=0}^{\infty} f^n(A(r))$ is not compact. Then by Lemma 1 $\{f^n(A(r))\}$ is a bulging sequence. Therefore by Lemma 3 there exists a point $p \in A(r)$ such that for each $n$

\[f^n(p) \cap \text{Int}(A(r)) = 0 .\]

Then $\{f^n(p)\}$ does not converge to $a$, which is a contradiction.

Hereafter in § 2 we assume that a homeomorphism $h$ of $X$ onto itself satisfies the condition of Theorem 1. Then we have the following
**Lemma 5.** For each $r$ the sequence $\{h^n(A(r))\}$ converges to $a$.

**Proof.** Since $U_\infty \cap A(r)$ is compact by Lemma 4, there exists a real number $r_0$ such that $U_\infty \cap h^n(A(r)) \subset A(r_0)$. Take $x_n \in h^n(A(r))$. It is easy to see that if we prove that the sequence $\{x_n\}$ converges to $a$, then the proof of Lemma 5 is complete.

Since $x_n \in A(r_0)$, the set $U_\infty x_n$ has a limit point. Now we suppose that $U_\infty x_n$ has a limit point $p \in A(r_0)$ different from $a$. Then there exists a subsequence $\{x_{n_k}\}$ which converges to $p$. Then $\{h^{-m}(x_{n_k})\}$ converges to $h^{-m}(p)$ for every natural number $m$. Now put $y_{n_k} = h^{-n_k}(x_{n_k})$, then $y_{n_k} \in A(r)$. If $n_k > m$, then

$$h^{-m}(x_{n_k}) = h^{-m}h^n(y_{n_k}) = h^{-m}h^n(y_{n_k}) \in A(r_0).$$

Therefore $h^{-m}(p) \subset A(r_0)$ for every $m$. Then $\{h^{-m}(p)\}$ does not converge to $b$, which is a contradiction.

Similarly we have the following

**Lemma 6.** For each $r$ the sequence $\{h^{-n}(B(r))\}$ converges to $b$.

**Proof of Theorem 1.** Let $p \in X^a - b$ and let $\varepsilon$ be a given positive real number. Then there exist real numbers $r_1$ and $r_2$ such that

$$p \in \text{Int}(A(r_1)) \quad \text{and} \quad p \in \text{Int}(B(r_2)),$$

respectively. Put

$$U_1 = \left\{ x \mid d(a, x) < \frac{1}{2} \varepsilon \right\}$$

and

$$U_2 = \left\{ x \mid d(b, x) < \frac{1}{2} \varepsilon \right\}.$$

By Lemma 5 and Lemma 6, there exist natural numbers $n_1$ and $n_2$ such that $h^n(A(r_1)) \subset U_1$ for every $n > n_1$ and that $h^{-n}(B(r_2)) \subset U_2$ for every $n > n_2$, respectively. Now let $V_1$ and $V_2$ be neighbourhoods of $p$ such that $\delta(h^n(V_1)) < \varepsilon$ for every $0 \leq n \leq n_1$ and that $\delta(h^{-n}(V_2)) < \varepsilon$ for every $0 \leq n \leq n_2$, respectively. Take $\delta > 0$ such that

$$\{ x \mid d(p, x) < \delta \} \subset V_1 - V_2 - \text{Int}(A(r_1)) - \text{Int}(B(r_2)).$$
Then it is easy to see that for each \( x \in X \) with \( d(p, x) < \delta \) and for each integer \( m \)

\[
d(h^m(p), h^m(x)) < \varepsilon.
\]

Therefore \( h \) is regular at every point of \( X \) except for \( a \) and \( b \), and the proof is complete.

§ 3. Another application of bulging sequences.

Let \( X \) be a separable metric space and let \( f \) be a continuous mapping of \( X \) into itself. For each point \( x \in X \) the set \( \bigcup_{n=1}^{\infty} f^n(x) \) will be said to be a positive half-orbit of \( x \). Let \( P(f) \) be the set of points whose positive half-orbits are everywhere dense in \( X \) and put \( Q(f) = X - P(f) \). It is easy to see that if \( P(f) = \emptyset \) then \( P(f) \) is everywhere dense in \( X \). Now we prove the following.

**Theorem 5.** Let \( X \) be a locally compact, non-compact, separable, metric space and let \( f \) be a continuous mapping of \( X \) into itself. Then \( Q(f) \) is everywhere dense in \( X \).

**Proof.** Suppose on the contrary that \( Q(f) \) is not everywhere dense in \( X \). Then there exist a point \( p \) and a neighbourhood \( U \) of \( p \) such that \( Q(f) \cap U = \emptyset \) (i.e. \( U \subseteq P(f) \)). Since \( X \) is locally compact, there exists a neighbourhood \( V \) of \( p \) with \( \overline{V} \subseteq U \) such that \( \overline{V} \) is compact.

Now we prove that \( \{f^n(\overline{V})\} \) is a bulging sequence. In fact, if \( \{f^n(\overline{V})\} \) is not a bulging sequence, then the set \( W = \bigcup_{n=0}^{\infty} f^n(\overline{V}) \) is compact by Lemma 1. Since \( \overline{V} \subseteq U \subseteq P(f) \), \( W = \overline{W} = X \) is compact, which is a contradiction. Therefore \( \{f^n(\overline{V})\} \) is a bulging sequence.

Then by Lemma 3 there exists a point \( q \in \overline{V} \) such that \( f^n(q) \in V \) for every natural number \( n \). Therefore \( q \in Q(f) \). Since \( q \in \overline{V} \subseteq U \), we have \( q \in P(f) \), which is also a contradiction, and the proof is complete.

**Corollary.** Let \( f \) be a continuous mapping of \( E^n \) into itself. Then \( Q(f) \), i.e. the set of points whose positive half-orbits are not everywhere dense in \( E^n \), is everywhere dense in \( E^n \).

**Remark 1.** A.S. Besicovitch [1] has shown that there exists a homeomorphism of the plane onto itself such that there exists a point whose positive half-orbit by this homeomorphism is everywhere dense.
On the regularity of homeomorphisms of $E^n$

on the plane. His statement that by this homeomorphism the positive half-orbit of every point of the plane except for the origin is everywhere dense on the plane is erroneous, as he has shown in his recent paper [2]. The fault of his assertion can also be seen by the above Corollary.

Remark 2. If $h$ is a homeomorphism of $E^n$ onto itself, then the set $Q(f)$ will be seen to be an $F_\sigma$ without difficulty.

Department of Mathematics, Tokyo Institute of Technology
and
Department of Mathematics, Osaka University

References