The Sylvester’s law of inertia in simple graded Lie algebras

Dedicated to Professor Ichiro Satake on the occasion of his seventieth anniversally birthday

By Soji KANEYUKI

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Introduction.

Let $H_n(R)$ be the vector space of $n \times n$ real symmetric matrices. The group $GL(n, R)^0 (= \text{the identity component of } GL(n, R))$ acts on $H_n(R)$ by the rule: $X \mapsto AX A$, $X \in H_n(R)$, $A \in GL(n, R)^0$. The Sylvester’s law of inertia asserts that, by this action of $GL(n, R)^0$, $X$ is transformed into the canonical form diag$(1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0)$, which is uniquely determined by $X$. The simple Lie algebra $sp(n, R)$ has a unique grading $sp(n, R) = g_{-1} + g_0 + g_1$, where $g_{-1} = H_n(R)$ and $g_0 \simeq gl(n, R)$. The $GL(n, R)^0$-module $H_n(R)$ is imbedded in $sp(n, R)$ as the $G_0$-module $g_{-1}$, where $G_0$ is the analytic subgroup of Aut $g$ generated by $g_0$. The Sylvester’s law of inertia for $H_n(R)$ is no other than obtaining the complete representatives of $G_0$-orbits in $g_{-1}$. As a generalization of this situation, one can pose:

**PROBLEM.** Let $g = \sum_{k=-v}^{v} g_k$ be a real simple graded Lie algebra, $G_0$ the group of grade-preserving automorphisms of $g$ and let $G_0^0$ be the identity component of $G_0$. Find the $G_0^0$-orbit decomposition and the $G_0$-orbit decomposition of $g_{-1}$.

When $v = 1$, this problem is equivalent to the problem of finding the orbits in a compact simple Jordan triple system under the structure group or the identity component of the structure group. Also it is equivalent to finding the orbit decomposition of a tangent space by the linear isotropy group for a symmetric $R$-space.

The purpose of this paper is to settle the above problem for the case $v = 1$ by a unified method. Partial answers have been obtained by Satake [22, 23], Kaneyuki [9, 10] and Takeuchi [27]. In the following we shall describe briefly how to get the two kinds of orbit decompositions of $g_{-1}$. The sections 1 and 2 are preliminary sections. We give a quick review for the followings: classification and construction of gradations in semisimple Lie algebras [13, 12], the root theory in simple graded Lie algebras $g = g_{-1} + g_0 + g_1$ ([13]), the Jordan triple system $\mathfrak{B}$ on $g_{-1}$ (Loos [18]) and the root-theoretic version of a frame (= a maximal system of pairwise orthogonal idempotents) $\{e_1, \ldots, e_r\}$ in $g_{-1}$, and the Jordan algebra structure $\mathfrak{U}_p, (0 \leq p \leq r)$ in $g_{-1}$. In §3, applying a result of Matsumoto [19], we get a set of good representatives of $G_0$ mod $G_0^0$, which allows us to get the $G_0$-orbit decomposition from the $G_0^0$-orbit decomposition. We consider the root system $\Delta^*$ corresponding to a certain symmetric real flag domain $M^*$. It turns out that the Weyl group $W(\Delta^*)$ of $\Delta^*$, viewed as a subgroup of $G_0^0$, acts on the frame $\{e_1, \ldots, e_r\}$ as signed permutations. Then we can choose the candidates $o_{p,q} (0 \leq p, q \leq r, p + q \leq r)$ of representatives of the $G_0^0$-orbits, which are defined in
terms of the frame. Let \( V_k \) \((0 \leq k \leq r)\) be the union of the \( G_0^0 \)-orbits through the points \( o_{p,q} \) with \( p + q = k \). The sets \( V_k \) were introduced by Takeuchi [28] in a different way. Theorem 3.3 (Gindikin-Kaneyuki [6]) shows that each \( V_k \) is \( G_0 \)-stable and that it consists of equi-dimensional \( G_0^0 \)-orbits. Therefore, in order to find the orbit decomposition, we have only to separate the \( G_0^0 \)-orbits in \( V_k \) \((0 \leq k \leq r)\). In the sections 4 and 5, we carry out this procedure, by using the action of \( W(A) \) and the reduced norm of the Jordan algebra \( \mathfrak{A} \). The main results are Theorems 4.1, 4.2, 5.1, 5.2 and 5.5-5.7. In § 6, we give a list of all open \( G_0^0 \)-orbits whose ambient spaces \( g_{-1} \) are simple Jordan algebras. (Partial results have been obtained by D’Atri-Gindikin [4] and Kaneyuki [9].) This provides a classification of \( \omega \)-domains in the sense of Koecher [16] in simple Jordan algebras.

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Notation and Convention: \( G^0 \) or \((G)^0\) denotes the identity component of a Lie group \( G \). \( G_\theta \) or \((G)_\theta\) denotes the subgroup of a group \( G \) consisting of elements left fixed by an involutive automorphism \( \theta \). GLA (resp. JTS) is an abbreviation for “graded Lie algebra” (resp. Jordan triple system). \( E \) denotes a unit matrix.

§ 1. Semisimple graded Lie algebras.

Let

\[
g = \sum_{k=-v}^{v} g_k
\]

be a real semisimple GLA of the \( v \)-th kind (we are assuming that the subspace \( g_{-1} \) is not zero). We assume further that the gradation (1.1) is of type \( a_0 \), that is, \( g^- := \sum_{k<0} g_k \) is generated by \( g_{-1} \). Let \( (g, Z, \tau) \) be the associated graded triple; more precisely, \( Z \in g \) is the characteristic element of the gradation (1.1), i.e., each subspace \( g_k \) is the eigenspace of \( \text{ad} Z \) for the eigenvalue \( k \), and \( \tau \) is a grade-reversing Cartan involution of \( g \). Let

\[
h = \sum_{k \text{ even}} g_k, \quad m = \sum_{k \text{ odd}} g_k.
\]

Then \( g \) is expressed as a \( Z_2 \)-GLA

\[
g = h + m,
\]

which is also the decomposition by the involution \( \sigma := \text{Ad} \exp \pi i Z \), in which case we have \( \sigma|_h = 1 \) and \( \sigma|m = -1 \). Consider the Cartan decomposition by \( \tau \):

\[
g = \mathfrak{f} + \mathfrak{p},
\]

where \( \tau|_{\mathfrak{f}} = 1 \) and \( \tau|_{\mathfrak{p}} = -1 \). Since \( \sigma \) and \( \tau \) commutes, we have the \((\sigma, \tau)\)-decomposition

\[
g = f_0 + m_\tau + p_0 + m_p,
\]

where \( f_0 = h \cap \mathfrak{f}, p_0 = h \cap \mathfrak{p}, m_\tau = m \cap \mathfrak{f} \) and \( m_p = m \cap \mathfrak{p} \). Note that \( Z \in p_0 \). Choose a
maximal abelian subspace \(a\) of \(p\) containing \(Z\). Then \(a\) is contained in \(g_0 \cap p \subset p_0\). Let \(A\) be the root system for the pair \((g, a)\), which is called a root system of \(g\) compatible with the gradation. Let \((, )\) denote the Killing form of \(g\). Then we have a partition of \(A\):

\[
A = \bigoplus_{k=-v}^{v} A_k,
\]

where \(A_k = \{a \in A : (a, Z) = k\}\), and each graded subspace \(g_k\) can be written as

\[
g_0 = c(a) + \sum_{a \in A_0} g^a,
\]

\[
g_k = \sum_{a \in A_k} g^a, \quad k \neq 0,
\]

where \(c(a)\) is the centralizer of \(a\) in \(g\), and \(g^a\) denotes the root space for a root \(a \in A\). Choose a linear order in \(A\) in such a way that

\[
\prod_{k=1}^{v} A_k \subset A^+ \subset \prod_{k=0}^{v} A_k,
\]

where \(A^+\) denotes the set of positive roots with respect to this order. Let \(\Pi\) be the fundamental system for \(A\). Since the gradation is of type \(\alpha_0\), it is known [13] that \(\Pi_k := \Pi \cap A_k = \emptyset\) for \(k \geq 2\), and hence we have a partition of \(\Pi\):

\[
\Pi = \Pi_0 \bigsqcup \Pi_1, \quad \Pi_1 \neq \emptyset.
\]

Let us consider the reverse process. Let \(g\) be a semisimple Lie algebra and \(a\) be a maximal \(R\)-split abelian subalgebra of \(g\), and let \(\Pi = \{a_1, \ldots, a_r\}\) be a fundamental system of the root system \(A\) for the pair \((g, a)\). A root \(a \in A\) can be written as

\[
a = \sum_{i=1}^{r} m_i(a) a_i.
\]

Suppose that we are given a partition \(\Pi = \Pi_0 \bigsqcup \Pi_1\) with \(\Pi_1 \neq \emptyset\). For a root \(a \in A\), we define the height \(h_{\Pi_1}(a)\) of \(a\) relative to \(\Pi_1\) by putting

\[
h_{\Pi_1}(a) = \sum_{a \in \Pi_1} m_i(a).
\]

If we put

\[
A_k = \{a \in A : h_{\Pi_1}(a) = k\},
\]

then we have a partition \(A = \bigsqcup_{k=-v}^{v} A_k\), where \(v\) is equal to the the height \(h_{\Pi_1}(\delta)\) of the highest root \(\delta \in A\). Let us define the subspaces \((g_k)_{-v \leq k \leq v}\) by the equalities (1.7). Then we have a GLA \(g = \bigsqcup_{k=-v}^{v} g_k\) of type \(\alpha_0\) (cf. [13]).

**Theorem 1.1 ([13]).** Let \(g\) be a real semisimple Lie algebra, and \(A\) be a restricted root system of \(g\). Let \(\Pi\) be a fundamental system of \(A\) and \(\delta\) be the highest root of \(A\). Then there exists a bijection between the set of gradations of the \(v\)-th kind of type \(\alpha_0\) in...
The set of subsets of II satisfying \( n, (9) = v \) and the respective isomorphisms.

A gradation of the first kind in \( g \) is trivially of type \( \alpha_0 \); any gradation of the second kind in \( g \) is of type \( \alpha_0 \), provided that \( g \) is simple (Tanaka [28]).

§ 2. Jordan triple systems on \( g_{-1} \).

We retain the notation in § 1. Let

\[ g = g_{-1} + g_0 + g_1 \]

be a simple GLA (of the first kind), and \((g, Z, \tau)\) be the associated graded triple. Let \( \Delta \) be a root system of \( g \) compatible with the gradation. As a special case of (1.6), we have a partition \( \Delta = \Delta_{-1} \coprod \Delta_0 \coprod \Delta_1 \). Choose a linear order in \( \Delta \) satisfying (1.8). As is known in Takeuchi [26], one can choose a maximal system of strongly orthogonal roots \( \Gamma = \{\beta_1, \ldots, \beta_\nu\} \) in \( \Delta_1 \) in such a way that \((\beta_1, \beta_1) = \cdots = (\beta_\nu, \beta_\nu)\). The number \( \nu \) is equal to the split rank of the symmetric triple \((g, g_0, \sigma)\). Choose a root vector \( E_i \in g^\beta_i \subset g_1 \) (\( 1 \leq i \leq \nu \)) in such a way that

\[ [E_i, E_{-i}] = \frac{2}{(\beta_i, \beta_i)} \beta_i, \]

where \( E_{-i} = -\tau E_i \in g^{-\beta_i} \subset g_{-1} \). Let

\[ X_i = E_i + E_{-i} \in m_g. \]

Then the real span \( c \) of \( X_1, \ldots, X_\nu \) is a maximal abelian subspace of \( m_g \). The root system \( \Delta(g, c) \) for the pair \((g, c)\) is the split root system for the symmetric triple \((g, g_0, \sigma)\). It is known (Oshima-Sekiguchi [20]) that \( \Delta(g, c) \) is either of type C or of type BC. Let \( a_0 \) be the subspace of \( a \) spanned by \( \beta_1, \ldots, \beta_\nu \), and \( \varpi \) be the orthogonal projection of \( a \) onto \( a_0 \) with respect to \((, )\). Then, by considering the inverse Cayley transformation ([8]) of \( c \) onto \( a_0 \) and by taking the inner products with \( Z \), we have

\[
\begin{cases}
  \varpi((\Delta_0)^+) - (0) = \left\{ \frac{1}{2} (\beta_i - \beta_j) : 1 \leq i < j \leq \nu \right\}, \\
  \varpi(\Delta_1) = \left\{ \frac{1}{2} (\beta_i + \beta_j) : 1 \leq i < j \leq \nu \right\}, 
\end{cases}
\]

provided that \( \Delta(g, c) \) is of type C, or

\[
\begin{cases}
  \varpi((\Delta_0)^+) - (0) = \left\{ \frac{1}{2} (\beta_i - \beta_j) : 1 \leq i < j \leq \nu; \frac{1}{2} \beta_i (1 \leq i \leq \nu) \right\}, \\
  \varpi(\Delta_1) = \left\{ \frac{1}{2} (\beta_i + \beta_j) : 1 \leq i < j \leq \nu; \frac{1}{2} \beta_i (1 \leq i \leq \nu) \right\}, 
\end{cases}
\]

provided that \( \Delta(g, c) \) is of type BC, where \((\Delta_0)^+ = \Delta_0 \cap \Delta^+\). We put

\[
a_i = \sum_{\alpha \in \Delta_1, \varpi(\alpha) = \beta_i} g^{-\alpha}, \quad i \leq j,
\]

\[
c_i = \sum_{\alpha \in \Delta_1, \varpi(\alpha) = \beta_i} g^{-\alpha}.
\]
Then $g_{-1}$ can be expressed as

$$(2.7) \quad g = \sum_{1 \leq i \leq j \leq r} a_{ij} + \sum_{1 \leq i \leq r} c_i.$$  

If $\mathcal{A}(g, c)$ is of type $C$, then the second term of the right-hand side of (2.7) does not appear. The dimensions $\dim a_{ij}$ ($i < j$), $\dim a_{ii}$ and $\dim c_i$ do not depend on the choice of $i$ and $j$ ([7]).

Let us consider a triple product $B_r$ on $g_{-1}$:

$$(2.8) \quad B_r(X, Y, U) = \frac{1}{2}[[\tau Y, X], U], \quad X, Y, U \in g_{-1}.$$  

It is known (Loos [17], Satake [21]) that the pair $\mathcal{B} = (g_{-1}, B_r)$ is a compact simple JTS and that $g$ is isomorphic to the Kantor-Tits-Koecher construction for $\mathcal{B}$. (These two facts can be obtained in more general setting of a simple GLA of the second kind and the corresponding compact generalized JTS; see [1, 13]). For simplicity we write $e_i$ for $E_i (1 \leq i \leq r)$ and $(X; Y; U)$ for $B_r(X, Y, U)$. As usual, we define the linear operator $L(X, Y)$ on $g_{-1}$ by

$$(2.9) \quad L(X, Y) U = (XYU), \quad U \in g_{-1}.$$  

Let

$$(2.10) \quad o_{p, q} = \sum_{i=1}^{p} e_i - \sum_{j=p+1}^{p+q} e_j, \quad 0 \leq p, q \leq r, \quad p + q \leq r.$$  

By using the facts [6] that $e_i$ ($1 \leq i \leq r$) is an idempotent of the JTS $\mathcal{B}$ and that $L(e_i, e_j) = 0$ ($i \neq j$), we see that $o_{p, q}$ is an idempotent of $\mathcal{B}$ and that

$$(2.11) \quad L(o_{p, r-p}, o_{r-p, r-p}) = L(o_r, 0, o_r), \quad 0 \leq p \leq r.$$  

**LEMMA 2.1.** Let $g_{-1}(\lambda)$ be the eigenspace of $L(o_r, 0, o_r)$ corresponding to the eigenvalue $\lambda$. Then we have $g_{-1} = g_{-1}(1) + g_{-1}(\frac{1}{2})$, and

$$(2.12) \quad g_{-1}(1) = \sum_{1 \leq i \leq j \leq r} a_{ij},$$  

$$(2.13) \quad g_{-1}(\frac{1}{2}) = \sum_{1 \leq i \leq r} c_i.$$  

**PROOF.** Consider the Peirce decomposition (Satake [21]) of $g_{-1}$ with respect to the operator $L(o_{p, r-p}, o_{p, r-p}) = L(o_r, 0, o_r)$:

$$(2.14) \quad g_{-1} = g_{-1}(1) + g_{-1}(\frac{1}{2}) + g_{-1}(0).$$  

Choose a root $\alpha \in \Delta_1$ such that $\varpi(\alpha) = \frac{1}{2}(\beta_i + \beta_j)$, $i < j$. We have $\sum_{k=1}^{r}(\tilde{\beta}_k, \varpi(\alpha)) = \frac{1}{2} \sum_{k=1}^{r}(\tilde{\beta}_k, \beta_i + \beta_j) = 2$. Let $X \in g^{-\alpha}$. Then it follows that

$$L(o_r, 0, o_r)(X) = B_r(o_r, 0, o_r, X) = \frac{1}{2}[[\varpi(o_r, 0), o_r, X]]$$  

$$= \frac{1}{2} \sum_{k=1}^{r}[-E_k, E_k], X] = \frac{1}{2} \sum_{k=1}^{r}[\tilde{\beta}_k, X] = \frac{1}{2} \left( \sum_{k=1}^{r}(\tilde{\beta}_k, \alpha) \right) X = X,$$  

where $\tilde{\beta}_k = \beta_i - \beta_j$, $1 \leq i < j \leq r$. Therefore, $L(o_r, 0, o_r)$ is a subalgebra of $g_{-1}$.
which implies that the right-hand side of (2.12) is contained in $g_{-1}(1)$. Similarly we have that the right-hand side of (2.13) is contained in $g_{-1}(\frac{1}{2})$. Consequently the lemma follows from (2.14) and (2.7).

We introduce a multiplication $\Box_p$ in $g_{-1}$:

$$X \Box_p Y = B_t(X, o_{p,r-p}, Y), \quad X, Y \in g_{-1}, \quad 0 \leq p \leq r.$$ 

As a property of the Peirce decomposition of a JTS ([21]), we know that $g_{-1}(1)$ become a Jordan algebra with unit element $o_{p,r-p}$ with respect to the multiplication $\Box_p$.

**Proposition 2.2.** Let $g = g_{-1} + g_0 + g_1$ be a real simple GLA. Then the pair $(g_{-1}, \Box_p), 0 \leq p \leq r$, is a Jordan algebra with $o_{p,r-p}$ as unit element, if and only if the split root system $\Delta(g, c)$ is of type $C$. In this case the Jordan algebra $(g_{-1}, \Box_p)$ is simple.

**Proof.** Suppose first that $\Delta(g, c)$ is of type $C$. Then we have (2.4). Therefore there are no roots $\alpha \in \Delta$ such that $\omega(\alpha) = \frac{1}{2} \beta_i (1 \leq i \leq r)$, and so we have $g_{-1}(\frac{1}{2}) = (0)$. By Lemma 2.1, we have $g_{-1}(1) = g_{-1}$. Conversely, suppose that $(g_{-1}, \Box_p)$ is a Jordan algebra with unit element $o_{p,r-p}$. Then, for any $X \in g_{-1}$, we have $X = o_{p,r-p} \Box_p X = B_t(o_{p,r-p}, o_{p,r-p}, X) = L(o_{r,0}, o_{r,0})X$, which implies that $g_{-1}(1) = g_{-1}$ and $g_{-1}(\frac{1}{2}) = (0)$. Consequently $\Delta(g, c)$ is of type $C$, by (2.4) and (2.5). To prove the second assertion, consider the involution $^*$ of the Jordan algebra $g_{-1} = g_{-1}(1)$:

$$X^* = B_t(o_{p,r-p}, X, o_{p,r-p}), \quad X \in g_{-1}.$$ 

Then $B_t$ can be reconstructed as follows ([21]):

$$B_t(X, Y, U) = (X \Box_p Y^*) \Box_p U + X \Box_p (Y^* \Box_p U) - Y^* \Box_p (X \Box_p U).$$

Let $W$ be an ideal of the Jordan algebra $g_{-1}$. Then, by using (2.17), we have that $B_t(W, g_{-1}, g_{-1}) + B_t(g_{-1}, g_{-1}, W) \subset W$. This means that $W$ is a $K$-ideal (cf. [13]) of the JTS $\mathcal{B}$. $\mathcal{B}$ is compact simple, and hence by a result of [1], it is $K$-simple. Therefore $W = (0)$ or $W = g_{-1}$. Thus the Jordan algebra $g_{-1}$ is simple. 

The simple Jordan algebra $(g_{-1}, \Box_p)$ is denoted by $\mathfrak{A}_p$.

§ 3. Generals on the orbit decomposition of $g_{-1}$.

We retain the notation in the previous sections. We will consider exclusively a simple GLA (2.1) : $g = g_{-1} + g_0 + g_1$. We denote by $\text{Aut } g$ the automorphism group of the Lie algebra $g$, and denote by $G^0$ the identity component of $\text{Aut } g$. Let $G_0$ be the subgroup of $\text{Aut } g$ consisting of all grade-preserving automorphisms of the GLA $g$. We need the following subgroups of $\text{Aut } g$:

$G := G_0 G^0$, which is an open subgroup of $\text{Aut } g$,

$G'$ the Zariski connected component of $\text{Aut } g$, which is a subgroup of $G$,

$G_0' := G_0 \cap G'$, which is the Zariski connected component of $G_0$,

$G_0^0$ the (topological) identity component of $G_0$,

$K := \{ g \in G : g \tau = \tau g \}$, which is the maximal compact subgroup of $G$ with Lie $K = \mathfrak{k}$.

$K_0 = G_0 \cap K$,

$K_0^0$ the identity component of $K_0$. 


Let $\mathcal{A}$ be a root system of $\mathfrak{g}$ compatible with the gradation and $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a fundamental system of $\mathcal{A}$ with respect to an order satisfying (1.8). Let $\{Z_1, \ldots, Z_\ell\}$ be the basis of a dual to $\Pi$ with respect to $(\ , \ )$. Consider the involutive automorphisms of $\mathfrak{g}$:

\begin{equation}
\varepsilon_k = \text{Ad} \exp \pi i Z_k, \quad 1 \leq k \leq \ell.
\end{equation}

**Lemma 3.1 (Matsumoto [19]).** Let $Q_1$ be the free abelian subgroup of $\text{Aut} \mathfrak{g}$ generated by $\varepsilon_1, \ldots, \varepsilon_\ell$, and let $Q_0 := Q_1 \cap G^0$. Then $Q_1$ is a subgroup of $G'$, and

\begin{equation}
G'/G^0 \cong Q_1/Q_0,
\end{equation}

in particular,

\begin{equation}
G' = Q_1 G^0.
\end{equation}

Since $\varepsilon_k$ is $+1$ or $-1$ on each root space $\mathfrak{g}^\alpha$, $\alpha \in \mathcal{A} \cup \{0\}$, it follows from (1.7) that $\varepsilon_k$ is grade-preserving for any gradation of $\mathfrak{g}$. This implies, in particular, that $Q_1$ is a subgroup of $G_0$, and hence we have

\begin{equation}
Q_1 G^0 \subset G_0.
\end{equation}

Look at the $(\sigma, \tau)$-decomposition (1.5) for the GLA $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. It is easy to see that $\mathfrak{g}^\sigma := \mathfrak{t}_0 + m_p$ is a reductive subalgebra of $\mathfrak{g}$. The center of $\mathfrak{g}^\sigma$ is at most one-dimensional and the semisimple part of $\mathfrak{g}^\sigma$ is simple ([7]). The triple $(\mathfrak{g}^\sigma, \mathfrak{t}_0, \tau)$ is a Riemannian symmetric triple, the noncompact dual of $(\mathfrak{t}, \mathfrak{t}_0, \sigma)$. Let $G^\sigma$ be the connected Lie subgroup of $G$ corresponding to $\mathfrak{g}^\sigma$. Then $K^0_0$ is a maximal compact subgroup of $G^\sigma$. $M^\sigma = G^\sigma/K^0_0$ is the symmetric space corresponding to $(\mathfrak{g}^\sigma, \mathfrak{t}_0, \tau)$. We have the Cartan decomposition

\begin{equation}
G^\sigma = K^0_0 \exp m_p.
\end{equation}

Since $c$ is a maximal abelian subspace of $m_p$, one can consider the root system $\mathcal{A}^\sigma$ for the pair $(\mathfrak{g}^\sigma, c)$ (or for the symmetric space $M^\sigma$). In Table I, we give a list of real simple GLA's of the first kind and the corresponding subset $\Pi_1$ of $\Pi$ ([13, 12, 14, 18]). In Table II, we give the root systems $\mathcal{A}(\mathfrak{g}, c)$ and $\mathcal{A}^\sigma$ for each simple GLA's of the first kind ([20, 25, 18]). The following notations are used in Table I: $H$ the quaternion algebra over $\mathbb{R}, O$ (resp. $O'$) the Cayley (resp. split Cayley) algebra over $\mathbb{R}$, and $O^C = O \otimes \mathbb{R} C$. $M_{p,q}(K)$ the vector space of $p \times q$ matrices with entries in $K$, where $K = \mathbb{R}, C, H, O, O'$ or $O^C$; $H_n(K)$ the vector space of hermitian matrices of degree $n$ with entries in $K$; $SH_n(H)$ the vector space of skew-hermitian quaternion matrices of degree $n$; $Alt_n(K)$ the vector space of skew-symmetric matrices of degree $n$ with entries in $K$; $\text{Sym}_n(C)$ the vector space of complex symmetric matrices of degree $n$. We employ the numbering of simple roots used in Bourbaki [2].

By the property $[\mathfrak{t}_0, m] \subset [\mathfrak{g}_0, m] \subset m$, the group $K^0_0$ acts on $m$ by the adjoint representation. Moreover, since $[\mathfrak{t}_0, m_p] \subset m_p$ and $[\mathfrak{t}_0, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{-1}$, it follows that this $K^0_0$-action on $m$ leaves both $m_p$ and $\mathfrak{g}_{-1}$ stable.
Table I

<table>
<thead>
<tr>
<th>(g, g_0, g_{-1})</th>
<th>Π</th>
<th>Π_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 (sl(n, R), sl(p, R) + sl(n-p, R) + R, M_{p,n-p}(R)),</td>
<td>n ≥ 3, 1 ≤ p ≤ [n/2]</td>
<td>A_{n-1}</td>
</tr>
<tr>
<td>12 (sl(n, H), sl(p, H) + sl(n-p, H) + R, M_{p,n-p}(H)),</td>
<td>n ≥ 3, 1 ≤ p ≤ [n/2]</td>
<td>A_{n-1}</td>
</tr>
<tr>
<td>13 (su(n, n), sl(n, C) + R, H_n(C)),</td>
<td>n ≥ 3</td>
<td>C_n</td>
</tr>
<tr>
<td>14 (sp(n, R), sl(n, R) + R, H_n(R)),</td>
<td>n ≥ 3</td>
<td>C_n</td>
</tr>
<tr>
<td>15 (sp(n, n), sl(n, H) + R, SH_n(H)),</td>
<td>n ≥ 2</td>
<td>C_n</td>
</tr>
<tr>
<td>16 (so(p+1, q+1), so(p, q) + R, M_{1,p+q}(R)),</td>
<td>0 ≤ p &lt; q or 3 ≤ p = q</td>
<td>B_{p+1}(φ &lt; q)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>D_{p+1}(p = q)</td>
</tr>
<tr>
<td>17 (so(4n), sl(n, H) + R, H_n(H)),</td>
<td>n ≥ 3</td>
<td>C_n</td>
</tr>
<tr>
<td>18 (so(n, n), sl(n, R) + R, A_{1,n}(R)),</td>
<td>n ≥ 4</td>
<td>D_n</td>
</tr>
<tr>
<td>19 (E_6, so(5, 5) + R, M_{1,2}(O^r))</td>
<td>E_6</td>
<td>{ φ }</td>
</tr>
<tr>
<td>110 (E_7, so(1, 9) + R, M_{1,2}(O))</td>
<td>E_7</td>
<td>{ φ }</td>
</tr>
<tr>
<td>111 (E_7, so(6) + R, H_3(O^r))</td>
<td>E_7</td>
<td>{ φ }</td>
</tr>
<tr>
<td>112 (E_{7-26}, so(1-26) + R, H_3(O))</td>
<td>C_3</td>
<td>{ φ }</td>
</tr>
<tr>
<td>113 (sl(n, C), sl(p, C) + sl(n-p, C) + C, M_{p,n-p}(C)),</td>
<td>n ≥ 3, 1 ≤ p ≤ [n/2]</td>
<td>A_{n-1}</td>
</tr>
<tr>
<td>114 (sp(n, C), sl(n, C) + C, Sym_n(C))</td>
<td>n ≥ 3</td>
<td>C_n</td>
</tr>
<tr>
<td>115 (so(n+2, C), so(n, C) + C, M_{1,n}(C))</td>
<td>n ≥ 3, n ≠ 4</td>
<td>B_{(n+2)/2}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>D_{(n+2)/2}</td>
</tr>
<tr>
<td>116 (so(2n, C), sl(n, C) + C, A_{1,n}(C))</td>
<td>n ≥ 4</td>
<td>D_n</td>
</tr>
<tr>
<td>117 (E_6^r, so(10, C) + C, M_{1,2}(O^C))</td>
<td>E_7</td>
<td>{ φ }</td>
</tr>
<tr>
<td>118 (E_7^r, E_6 + C, H_3(O^C))</td>
<td>E_7</td>
<td>{ φ }</td>
</tr>
</tbody>
</table>

**Lemma 3.2.** Let us define a linear endomorphism φ on m by

\[
\varphi(X) = \frac{1}{2}(X - IX), \quad X \in m,
\]

where I = ad\_m Z. Then φ is a K_0\_isomorphism of m_p onto g_{-1}.

**Proof.** The inclusion φ(m_p) ⊂ g_{-1} follows from the fact I^2 = 1. Since I interchanges m_p with m_1, φ sends m_p to g_{-1} isomorphically. Since K_0 acts on g as grade-preserving automorphisms, the element Z is left fixed by K_0. Hence we have [Ad\_m K_0, I] = 0, which implies that φ commutes with the K_0-action.

Let α_{-1} := φ(c) ⊂ g_{-1}. Then α_{-1} is spanned by e_1, ..., e_r, since φ(X_i) = e_i. Let W(Δ^+) be the Weyl group for the root system Δ^+ (or, for the symmetric space M^+). Then we have

\[
W(Δ^+) \simeq N_{K_0}(c)/C_{K_0}(c),
\]

where N_{K_0}(c) (resp. C_{K_0}(c)) is the normalizer (resp. centralizer) of c in K_0. W(Δ^+) acts on c as signed permutations:

\[
X_1 \mapsto ± X_ρ(1), \quad ρ \in Σ_r,
\]

where Σ_r is the permutation group of {1, ..., r}. By Lemma 3.2, this action of W(Δ^+) is transferred onto α_{-1} via φ as the signed permutations:

\[
e_i \mapsto ± e_ρ(i), \quad ρ \in Σ_r.
\]
Recall the quadratic representation \( P \) of the compact simple JTS \( \mathcal{B} = (\mathfrak{g}_{-1}, B_t) \):

\[
P(X)Y = (XYX), \quad X, Y \in \mathfrak{g}_{-1}.
\]

The structure group \( \text{Str} \mathcal{B} \) of the JTS \( \mathcal{B} \) is, by definition, the totality of the elements \( g \in \text{GL}(\mathfrak{g}_{-1}) \) satisfying the condition:

\[
g(XYU) = ((gX)(g^*^{-1}Y)(gU)), \quad X, Y, U \in \mathfrak{g}_{-1},
\]

where \( g^* \) is the adjoint operator of \( g \) with respect to the trace form of \( \mathcal{B} \). A computation shows that

\[
\text{Str} \mathcal{B} = \{ g \in \text{GL}(\mathfrak{g}_{-1}) : P(gX) = gP(X)g^*, X \in \mathfrak{g}_{-1} \}.
\]

Noting that the GLA \( \mathfrak{g} \) is isomorphic to the Kantor-Tits-Koecher construction for \( B_r \), we conclude from Satake [21] that the group \( G_0 \) is isomorphic to \( \text{Str} \mathcal{B} \) and that this isomorphism is given by taking the restriction of the \( G_0 \)-action on \( \mathfrak{g} \) to \( \mathfrak{g}_{-1} \). As a result, the rank of the operator \( P(X) \) is constant on each \( G_0 \)-orbit in \( \mathfrak{g}_{-1} \), when \( X \) varies through that orbit. Let \( V_k \) \((0 \leq k \leq r)\) be the union of \( G_0^0 \)-orbits through the points \( o_{p,q} \).
with \( p + q = k \), that is,

\[
V_k = \bigcup_{p+q=k} G_0^0 \cdot o_{p,q} \subset g_{-1}, \quad 0 \leq k \leq r.
\]

**Theorem 3.3** (Gindikin-Kaneyuki [6]). Let \( g = g_{-1} + g_0 + g_1 \) be a real simple GLA and \( r \) be the split rank of the symmetric pair \((g, g_0)\). Then (1) \( V_k \) is expressed as

\[
V_k = \{ X \in g_{-1} : \text{rk} P(X) = i_k \}, \quad 0 \leq k \leq r,
\]

where \( \text{rk} \) denotes the rank and \( i_k = \text{rk} \cdot P(o_k,0) \). The closure \( \overline{V}_k \) of \( V_k \) is given by

\[
\overline{V}_k = \{ X \in g_{-1} : \text{rk} P(X) \leq i_k \}, \quad 0 \leq k \leq r.
\]

(2) Each \( V_k \) is \( G_0 \)-stable and

\[
g_{-1} = V_0 \coprod V_1 \coprod \cdots \coprod V_r.
\]

(3) An orbit \( G_0^0 \cdot o_{p,q} \) is open if and only if it is contained in \( V_r \), or equivalently, \( p + q = r \).

The assertion (2) was obtained also by Takeuchi [27] by a different method.

**Lemma 3.4.** Let \( \text{Aut} \mathcal{B} \) denote the automorphism group of the JTS \( \mathcal{B} \). Then

\[
\text{Aut} \mathcal{B} = K_0.
\]

**Proof.** The trace form \( \gamma_{\mathcal{B}} \) of \( \mathcal{B} \) is positive definite, since \( \mathcal{B} \) is compact. \( \text{Aut} \mathcal{B} \) is, by definition, the subgroup of \( \text{Str} \mathcal{B} = G_0 \) consisting of all elements \( g \in \text{Str} \mathcal{B} \) satisfying the condition

\[
\gamma_{\mathcal{B}}(gX,gY) = \gamma_{\mathcal{B}}(X,Y), \quad X, Y \in g_{-1}.
\]

On the other hand, we have (cf. [1] and Lemma 3.10 [13])

\[
\gamma_{\mathcal{B}}(X,Y) = -\frac{1}{2}(X,\tau Y), \quad X, Y \in g_{-1}.
\]

Now let \( g \in K_0 \). Then, since \( g \) commutes with \( \tau \), we have that \( g \) satisfies (3.18), which implies that \( K_0 \subset \text{Aut} \mathcal{B} \). By the definition, \( \text{Aut} \mathcal{B} \) is a compact subgroup of \( \text{Str} \mathcal{B} \). But \( K_0 \) is a maximal compact subgroup of \( G_0 \). Hence we have that \( K_0 = \text{Aut} \mathcal{B} \). □

**§ 4. The orbit decompositions of \( g_{-1} \).**

**Theorem 4.1.** Let \( g = g_{-1} + g_0 + g_1 \) be a real simple GLA, and \( r \) be the split rank of the symmetric pair \((g, g_0)\). Suppose that \( \Delta^* \) is of type \( A \). Then the orbit decompositions of \( g_{-1} \) under the groups \( G_0^0 \) and \( G_0 \) are given by

\[
g_{-1} = \coprod_{p+q \leq r} G_0^0 \cdot o_{p,q} = \coprod_{p+q \leq r} G_0 \cdot o_{p,q}.
\]

**Proof.** Since \( \Delta^* \) is of type \( A \), it follows (Tables I and II) that \( \mathcal{A}_r = (g_{-1}, \Box_r) \) is a compact simple Jordan algebra. In this case, the JTS \( \mathcal{B} \) comes from the Jordan algebra \( \mathcal{A}_r \). As a result, \( G_0 \), identified with the structure group \( \text{Str} \mathcal{B} \), coincides with the struc-
Sylvester's law of inertia

Therefore the first equality in (4.1) is the one proved by Kaneyuki [9, 10] and Satake [23]. Since \( \mathfrak{H} \) is compact simple, it is known (Koecher [15], Vinberg [29]) that \( V_{r,0} := G_0^0 \cdot o_{r,0} \) is a homogeneous irreducible self-dual convex cone in \( g_{-1} \). Let \( G(V_{r,0}) \) be the automorphism group of the cone \( V_{r,0} \). By Satake [21], we have

\[
(4.2) \quad G_0|_{g_{-1}} = \text{Str} \mathfrak{H} = G(V_{r,0}) \times \{ \pm 1 \}.
\]

As was shown in [10], any \( G(V_{r,0}) \)-orbit in \( g_{-1} \) coincides with a \( G_0^0 \)-orbit in \( g_{-1} \). Therefore the second equality in (4.1) follows from (4.2).

Now let

\[
\Gamma_k = \left\{ \sum_{i=1}^{k} \delta_i e_i \in a_{-1} : \delta_{i_1}, \ldots, \delta_{i_k} = \pm 1, 1 \leq i_1, \ldots, i_k \leq r \right\}, 1 \leq k \leq r,
\]

\[
\Gamma_0 = \{0\}.
\]

Then the Weyl group \( W(\Delta^*) \) acts on \( \Gamma_k \) by (3.9) and we have

\[
(4.4) \quad \Gamma_k = \bigcup_{p+q=k} W(\Delta^*) \cdot o_{p,q}, \quad 0 \leq k \leq r.
\]

Therefore it follows from (3.7) and (3.13) that

\[
(4.5) \quad V_k = G_0^0 \cdot \Gamma_k, \quad 0 \leq k \leq r.
\]

**Theorem 4.2.** Let \( g = g_{-1} + g_0 + g_1 \) and \( r \) be the same as in Theorem 4.1. Suppose that \( \Delta^* \) is of type \( B \), \( BC \) or \( C \). Then the orbit decompositions of \( g_{-1} \) under \( G_0^0 \) and \( G_0 \) are given by

\[
(4.6) \quad g_{-1} = \prod_{k=0}^{r} G_0^0 \cdot o_{k,0} = \prod_{k=0}^{r} G_0 \cdot o_{k,0}.
\]

In particular, there is a single open orbit \( G_0^0 \cdot o_{r,0} = G_0 \cdot o_{r,0} \).

**Proof.** In view of (3.16), it suffices to show that

\[
(4.7) \quad V_k = G_0^0 \cdot o_{k,0} = G_0 \cdot o_{k,0}, \quad 0 \leq k \leq r.
\]

By the assumption for \( \Delta^* \), the Weyl group \( W(\Delta^*) \) consists of all signed permutations of the form (3.9). Consequently, \( W(\Delta^*) \) acts on \( \Gamma_k \) transitively, i.e., \( \Gamma_k = W(\Delta^*) \cdot o_{k,0} \). Hence (4.5) implies the first equality in (4.7). The second equality in (4.7) follows from the fact that \( V_k \) is \( G_0 \)-stable (Theorem 3.3).

**Remark.** The second equality in (4.7) was obtained also by Takeuchi [27].

In the following we will be concerned exclusively with the case where \( \Delta^* \) is of type \( D \).

**Lemma 4.3.** Suppose that \( \Delta^* \) is of type \( D_r \). Then

\[
(4.8) \quad V_r = G_0^0 \cdot o_{r,0} \cup G_0^0 \cdot o_{r-1,1}.
\]

\[
(4.9) \quad V_k = G_0^0 \cdot o_{k,0} = G_0 \cdot o_{k,0}, \quad 0 \leq k \leq r - 1.
\]
PROOF. In view of (4.5), it suffices to prove that
\[
\Gamma_r = W(A^*) \cdot o_{r,0} \prod_{i=1}^{r-1} W(A^*) \cdot o_{r-1,1},
\]
(4.10)
\[
\Gamma_k = W(A^*) \cdot o_{k,0}, \quad 0 \leq k \leq r - 1.
\]
By the assumption for $A^*$, a signed permutation $e_l \mapsto \delta_l e_l$, \( \delta_l = \pm 1 \) \((1 \leq i \leq r) \) lies in $W(A^*)$ if and only if \( \prod_{i=1}^{r} \delta_i = 1 \). Therefore $\alpha_{p,q}$ with $q$ even (resp. odd) is conjugate to $o_{r,0}$ (resp. $o_{r-1,1}$) under $W(A^*)$. Hence (4.10)\(_1\) follows from (4.4). Let us next consider $\alpha_{p,q}$ with $p+q = k$, $0 \leq k \leq r - 1$. If $q$ is even, then $\alpha_{p,q}$ is conjugate to $o_{k,0}$ under $W(A^*)$. Suppose $q$ is odd. Let $\mu$ be the signed permutation defined by $\mu(e_l) = \delta_l e_l$ \((1 \leq \ell \leq r)\), where $\delta_\ell = -1$ for $p + 1 \leq \ell \leq p + q + 1$, otherwise $\delta_\ell = 1$. Then $\mu$ belongs to $W(A^*)$ and $\mu(\alpha_{p,q}) = o_{k,0}$. This implies (4.10)\(_2\). \(\square\)

Back to the situation in § 2, suppose that $A(g,c)$ is of type C, and consider the Jordan algebra $\mathfrak{H}_p = (g_{-1}, \Box_p)$, \(0 \leq p \leq r\). Let $P_p : g_{-1} \rightarrow \text{End} g_{-1}$ be the quadratic representation of $\mathfrak{H}_p$. Then we have

**Lemma 4.4.** Let $0 \leq p \leq r$. Then
\[
P(X) = P_p(X)P(o_{p,r-p}), \quad X \in g_{-1}.
\]
Moreover the operator $P(o_{p,r-p})$ is nondegenerate on $g_{-1}$.

**Proof.** Let $Y \in g_{-1}$. By using (2.16) and (2.17), we have
\[
P(X)Y = (XYX) = (X \Box_p Y^*) \Box_p X + X \Box_p (Y^* \Box_p X) - Y^* \Box_p (X \Box_p X)
\]
\[
= 2X \Box_p (X \Box_p Y^*) - (X \Box_p X) \Box_p Y^*
\]
\[
= P_p(X)Y^* = P_p(X)P(o_{p,r-p})Y.
\]
Since $A(g,c)$ is of type C, we have that $g_{-1}(1) = g_{-1}$ (cf. §2). On the other hand, by Satake [21], $\pm 1$ are the only eigenvalues of $P(o_{p,r-p})$ on $g_{-1}(1)$, which yields the second assertion. \(\square\)

Consider the JTS \((\_)_p\) coming from $\mathfrak{H}_p$ \((0 \leq p \leq r)\):
\[
(XYU)_p = (X \Box_p Y) \Box_p U + X \Box_p (Y \Box_p U) - Y \Box_p (X \Box_p U),
\]
where $X$, $Y$, $U \in g_{-1}$, and define the linear operator $L_p(X, Y)$ by
\[
L_p(X, Y)U = (XYU)_p.
\]

**Lemma 4.5.** Let $X, Y \in g_{-1}$. Then
\[
L_p(X, Y) = L(X, P(o_{p,r-p})Y).
\]

**Proof.** For simplicity we write $f_p$ for $o_{p,r-p}$. By the definition of a JTS, we have
\[
L(X, P(f_p)Y)U = (X(f_p Yf_p)U)
\]
\[
= ((Yf_p X)f_p U) + (Xf_p (Yf_p U)) - (Yf_p (Xf_p U))
\]
\[
= (X \Box_p Y) \Box_p U + X \Box_p (Y \Box_p U) - Y \Box_p (X \Box_p U)
\]
\[
= (XYU)_p = L_p(X, Y)U.
\]
\(\square\)
PROPOSITION 4.6. Suppose that \( \Delta(g, c) \) is of type C. Let \((\text{Str } \mathfrak{A}_p)^0\) and \((\text{Str } \mathfrak{B})^0\) denote the identity components of the structure groups \(\text{Str } \mathfrak{A}_p\) and \(\text{Str } \mathfrak{B}\), respectively. Then we have

\[
(4.17) \quad (\text{Str } \mathfrak{A}_p)^0 = (\text{Str } \mathfrak{B})^0 = G_0^0.
\]

PROOF. \(\text{Lie Str } \mathfrak{A}_p\) (resp. \(\text{Lie Str } \mathfrak{B}\)) is generated by \(L_p(X, Y)\) (resp. \(L(X, Y)\)), when \(X\) and \(Y\) vary through \(g_1\). Therefore the proposition follows from Lemma 4.5 and the non-degeneracy of \(P(o_{p,r-p})\). \(\square\)

Table II tells us that if \(\Delta^*\) is of type \(D_r\), then \(\Delta(g, c)\) is of type C. In this case one has the Jordan algebra \(\mathfrak{A}_r = (g_{-1}, \Box_r)\) (Proposition 2.2).

PROPOSITION 4.7. Let \(g = g_{-1} + g_0 + g_1\) be a real simple GLA. Suppose that \(\Delta^*\) is of type \(D_r\). Let \(N\) be the reduced norm of the Jordan algebra \(\mathfrak{A}_r = (g_{-1}, \Box_r)\). Suppose \(N(o_{r,0})N(o_{r-1,1}) < 0\). Then

\[
(4.18) \quad V_r = G_0^0 \cdot o_{r,0} \mathbin{\coprod} G_0^0 \cdot o_{r-1,1}.
\]

In particular, there are exactly two open \(G_0^0\)-orbits in \(g_{-1}\).

PROOF. By the assumption, \(\Delta(g, c)\) is of type C. Therefore, by Corollary 2.11 [6], we have that \(V_r = \{X \in g_{-1} : \det P(X) \neq 0\}\). Lemma 4.4 implies that \(X \in V_r\) if and only if \(\det P_r(X) \neq 0\) if and only if \(N(X) \neq 0\). We have thus

\[
(4.19) \quad V_r = \{X \in g_{-1} : N(X) \neq 0\}.
\]

Let \(V_r^+\) (resp. \(V_r^-\)) be the totality of elements \(X \in g_{-1}\) satisfying \(N(X) > 0\) (resp. \(< 0\)). Then

\[
(4.20) \quad V_r = V_r^+ \mathbin{\coprod} V_r^-.
\]

Suppose for simplicity that \(N(o_{r,0}) > 0\). Then \(N(o_{r-1,1}) < 0\). We have \(o_{r,0} \in V_r^+\) and \(o_{r-1,1} \in V_r^-\). The reduced norm \(N\) is a relative invariant polynomial on \(g_{-1}\), that is,

\[
(4.21) \quad N(gX) = \chi(g)N(X), \quad X \in g_{-1}, \quad g \in \text{Str } \mathfrak{A}_r,
\]

where \(\chi\) is an \(R^*\)-valued character of \(\text{Str } \mathfrak{A}_r\). Suppose now that \(g \in G_0^0 = (\text{Str } \mathfrak{A}_r)^0\) (cf. Proposition 4.6). Then we have \(N(g o_{r,0}) = \chi(g)N(o_{r,0}) > 0\), and hence \(G_0^0 \cdot o_{r,0} \subset V_r^+\). Similarly \(G_0^0 \cdot o_{r-1,1} \subset V_r^-\). These two imply (4.18). \(\square\)

COROLLARY 4.8. Under the situation in Proposition 4.7, suppose that \(N(o_{r,0}) > 0\) (resp. \(< 0\)) and \(N(o_{r-1,1}) < 0\) (resp. \(> 0\)). Then

\[
(4.22) \quad G_0^0 \cdot o_{r,0} = \{X \in g_{-1} : N(X) > 0\} \quad \text{(resp. \(< 0\))},
\]

\[
G_0^0 \cdot o_{r-1,1} = \{X \in g_{-1} : N(X) < 0\} \quad \text{(resp. \(> 0\))}.
\]

§ 5. The orbit decompositions of \(g_{-1}\) (continued).

In this section we consider the case where \(\Delta^*\) is of type D.
Theorem 5.1. Let \((g, g_0, g_{-1}) = (\mathfrak{sl}(2p, \mathbb{R}), \mathfrak{sl}(p, \mathbb{R}) + \mathfrak{sl}(p, \mathbb{R}) + \mathbb{R}, \mathbb{M}(\mathbb{R}))\). Then the orbit decompositions of \(g_{-1}\) under the groups \(G_0^0\) and \(G_0\) are given by

\[ g_{-1} = \prod_{k=0}^{p-1} G_0^0 \cdot o_{k,0} \prod G_0^0 \cdot o_{p,0} \prod G_0^0 \cdot o_{p-1,1}, \]

There are exactly two open orbits \(G_0^0 \cdot o_{p,0}\) and \(G_0^0 \cdot o_{p-1,1}\) which are mutually diffeomorphic.

Proof. In this case, \(A\) is of type \(A_{2p-1}\) and is given by

\[ A = \{ \pm (\lambda_i - \lambda_j) : 1 \leq i < j \leq 2p \}. \]

The simple root system \(\Pi\) is given by

\[ \Pi = \{ \alpha_i = \lambda_i - \lambda_{i+1} : 1 \leq i \leq 2p - 1 \}. \]

Since \(\Pi_1 = \{ \alpha_1 \}\) (cf. Table I), we have

\[ \Delta_1 = \{ \lambda_i - \lambda_{p+i} : 1 \leq i, j \leq p \}. \]

The corresponding gradation of \(g = \mathfrak{sl}(2p, \mathbb{R})\) is

\[ g = g_{-1} + g_0 + g_1 \]

\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
p \\
\vdots \\
p
\end{pmatrix}
\]

\[ \oplus \left\{ \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
p \\
\vdots \\
p
\end{pmatrix}
\right\} \oplus \left\{ \begin{pmatrix}
* & \cdots & 0 \\
0 & \cdots & *
\end{pmatrix}
\begin{pmatrix}
p \\
\vdots \\
p
\end{pmatrix}
\right\} \oplus \left\{ \begin{pmatrix}
0 & \cdots & *
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\right\}.
\]

Let

\[ \Gamma = \{ \beta_i = \lambda_i - \lambda_{p+i} : 1 \leq i \leq p \}. \]

Then \(\Gamma\) is a maximal system of strongly orthogonal roots in \(\Delta_1\). Let \(E_{ij} \in g_{-1} = \mathbb{M}(\mathbb{R}) (1 \leq i, j \leq p)\) be the matrix whose \((k, \ell)\)-entry is \(\delta_{ik}\delta_{j\ell}\). It can be seen that the root vector \(E_{-i} \in \mathfrak{g}^{-\beta_i} (1 \leq i \leq p)\) is given by the matrix \(E_{ii} \in \mathbb{M}(\mathbb{R}) = g_{-1}\). Therefore

\[ o_{k,0} = \sum_{i=1}^{k} E_{ii} \in \mathbb{M}(\mathbb{R}), \quad 1 \leq k \leq p, \]

\[ o_{p-1,1} = \sum_{i=1}^{p-1} E_{ii} - E_{pp}. \]

The reduced norm \(N\) of the Jordan algebra \(\mathfrak{U}_p = \mathbb{M}(\mathbb{R})\) is given by \(N(X) = \det X, X \in \mathbb{M}(\mathbb{R})\). Hence \(N(o_{p,0}) = 1\) and \(N(o_{p-1,1}) = -1\). Consequently, by Proposition 4.7, we have that \(V_p = G_0^0 \cdot o_{p,0} \prod G_0^0 \cdot o_{p-1,1}\). Combining this with (4.9) and (3.16), we get (5.1).

Let us next consider the \(G_0\)-orbit decomposition of \(g_{-1}\). For \(g = \mathfrak{sl}(2p, \mathbb{R})\), it is known (Matsumoto [19]) that \(Q_1 \mod Q_0\) is generated by \(e_1\). Since \(e_1\) is not in \(G_0^0\), we have \(e_1 \in G_0 - G_0^0\) (cf. (3.4)). Choose the subset \(\Pi'_1 = \{ \alpha_1 \}\) of \(\Pi\). Then
Let
\[ h_{II}(\mathfrak{g}) = 1. \]

(5.9)
\[ \mathfrak{g} = \mathfrak{g}' - 1 + \mathfrak{g}'_0 + \mathfrak{g}'_1 \]
be the gradation of \( \mathfrak{g} \) corresponding to \( II' \) (cf. § 1), and let

(5.10)
\[ A = \prod_{k=1}^{p} A'_k \]
be the corresponding partition of \( A \). Since \( \beta_1 \in A'_1 \) and \( \beta_k \in A'_0 \) for \( k \geq 2 \), we have that \( E_{-1} \) lies in \( \mathfrak{g}' - 1 \) and \( E_{-k} (k \geq 2) \) lies in \( \mathfrak{g}'_0 \). On the other hand \( \varepsilon_1 = 1 \) on \( \mathfrak{g}'_0 \) and \( \varepsilon_1 = -1 \) on \( \mathfrak{g}' - 1 \) (cf. (3.1), (1.7), (1.11), (1.12)). Hence \( \varepsilon_1 \) sends \( E_{-1} \) to \( -E_{-1} \) and leaves \( E_k (k \geq 2) \) fixed. Consequently \( \varepsilon_1 (o_{p,0}) = -E_{-1} + \sum_{i=2}^{p} E_{-i} \). Let \( a \in W(\mathcal{A}'n) \) be the element interchanging \( E_{-1} \) with \( E_{-p} \) and leaving all other \( E_{-k} (k \neq 1, p) \) fixed. Then it follows that \( a \varepsilon_1 (o_{p,0}) = o_{p-1,1} \), and hence \( G_0 \cdot o_{p-1,1} = G_0 o_{p,0} = G_0 \cdot o_{p,0} \), which proves (5.2). Since \( \varepsilon_1 \) normalizes \( G'_0 \), it is easily seen that \( \varepsilon_1 \) sends \( G'_0 \cdot o_{p,0} \) to \( G'_0 \cdot o_{p-1,1} \).

5.2.

Theorem 5.2. Let \( (\mathfrak{g}, o_0, o_{-1}) = (\text{so}(2n, 2n), \mathfrak{gl}(2n, R), \text{Alt}_{2n}(R)) \). Then the orbit decompositions of \( \mathfrak{g}_{-1} \) under the groups \( G_0^0 \) and \( G_0 \) are given by (5.1) and (5.2) with \( p \) replaced by \( n \).

Proof. The Lie algebra \( \mathfrak{g} = \text{so}(2n, 2n) \) is realized as

(5.11)
\[ \text{so}(2n, 2n) = \{ A \in \mathfrak{gl}(4n, R) : AS + SA = 0 \} \]
\[ = \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} : A_1 + A_4 = 0, \quad A_2, A_3 \in \text{Alt}_{2n}(R) \right\}, \]
where \( S = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \). The root system \( \mathcal{A} \) is of type \( D_{2n} \).

(5.12)
\[ \mathcal{A} = \{ (\lambda_i \pm \lambda_j) : 1 \leq i < j \leq 2n \}, \]
\[ \Pi = \{ \alpha_i = \lambda_i - \lambda_{i+1} (1 \leq i \leq 2n - 1), \alpha_{2n} = \lambda_{2n-1} + \lambda_{2n} \}. \]

Since \( \Pi_1 = \{ \alpha_{2n} \} \) (cf. Table I), we have

(5.13)
\[ \mathcal{A}_1 = \{ \lambda_i + \lambda_j : 1 \leq i < j \leq 2n \}. \]

The gradation \( \mathfrak{g} = \mathfrak{g}' - 1 + \mathfrak{g}'_0 + \mathfrak{g}'_1 \) corresponding to \( \Pi_1 \) is given by (5.6) with \( p \) replaced by \( 2n \). Put

(5.14)
\[ \Gamma = \{ \beta_i = \lambda_{2i-1} + \lambda_{2i} : 1 \leq i \leq n \}. \]

Then \( \Gamma \) is a maximal system of strongly orthogonal roots in \( \mathcal{A}_1 \). It can be seen that the root vector \( E_{-i} \in \mathfrak{g}^{-\beta_i} (1 \leq i \leq n) \) is given by the matrix \( -E_{2i-1,2i} + E_{2i,2i-1} \in \text{Alt}_{2n}(R) = \mathfrak{g}_{-1} \). If we denote by \( \text{Pf}(X) \) the Pfaffian of an alternating matrix \( X \), then the above matrix realization of \( E_{-i} \) shows that \( \text{Pf}(o_{n,0}) = (-1)^n \) and \( \text{Pf}(o_{n-1,1}) = (-1)^{n-1} \). Since the Pfaffian is the reduced norm of the Jordan algebra \( \mathfrak{H}_n = \text{Alt}_{2n}(R) \), it follows from Proposition 4.7 that \( V_n = G_0 \cdot o_{n,0} \prod G_0 \cdot o_{n-1,1} \). Therefore we get (5.1) with \( p \) replaced by \( n \).
Let us next study the open $G_0$-orbits. For $g = \mathfrak{so}(2n, 2n)$, it is known (Matsumoto [19]) that $e_1$ is one of representatives of $Q_1 \text{mod} Q_0$. Similarly as before, we have $e_1 \in G_0 - G_0^0$. Choose a subset $H_1 = \{\alpha_1\}$ of $H$. Then $h_{H_1}(g) = 1$.

Consider the gradation (5.9) of $g = \mathfrak{so}(2n, 2n)$ corresponding to $H_1$ and the partition (5.10) of $\Delta$. Since $h_{H_1}(\beta_1) = 1 \neq 0$ and $h_{H_1}(\beta_k) = 0$ for $k \geq 2$, we have that $E_{-1} \in g'_{-1}$ and $E_{-k} \in g'_0$ for $k \geq 2$. On the other hand $e_1 = 1$ on $g'_0$ and $= -1$ on $g'_{-1}$ + $g'_1$. Hence $e_1$ sends $E_{-1}$ to $-E_{-1}$ and leaves $E_{-k}$ ($k \geq 2$) fixed. Let $a \in W(d^*)$ be the element interchanging $E_{-1}$ with $E_{-n}$ and leaving all other elements $E_{-k}$ ($k \neq 1, n$) fixed. Then we have that $ae_1(o_{n,0}) = o_{n-1,1}$, and hence $G_0 \cdot o_{n-1,1} = G_0 \cdot o_{n,0}$, which proves (5.2) with $p$ replaced by $n$. Since $e_1$ normalizes $G_0^0$, we see that $e_1(G_0^0 \cdot o_{n,0}) = G_0^0 \cdot o_{n-1,1}$. □

5.3 Let us now consider the case $(g, g_0, g_1) = (E_7(7), E_6(6) + R, H_3(O'))$. There is only one possibility of gradations of the first kind for $g = E_7(7)$. That gradation corresponds to $H_1 = \{e_7\}$. Let $\Gamma = \{\beta_1, \beta_2, \beta_3\}$, where

$$
\begin{align*}
\beta_1 &= 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \\
\beta_2 &= \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \\
\beta_3 &= \alpha_7.
\end{align*}
$$

It can be checked that $\Gamma$ is a maximal system of strongly orthogonal roots in $\Delta$. As was shown in [6], $\{e_1, e_2, e_3\}, e_i = E_{-i}$, is a frame (= a maximal system of orthogonal primitive idempotents) of $\mathfrak{B}$. In the present case, the triple product $B_t$ of $\mathfrak{B}$ comes from the natural Jordan algebra structure $\mathfrak{A}$ of $g_{-1} = H_3(O')$ (cf. Loos [18]), that is,

$$(5.16) \quad B_t(X, U, Y) = X \circ (U \circ Y) + (X \circ U) \circ Y - U \circ (X \circ Y),$$

where $\circ$ denotes the Jordan multiplication in $\mathfrak{A}$. Therefore the two structure groups coincide:

$$(5.17) \quad \text{Str} \mathfrak{A} = \text{Str} \mathfrak{B}.$$ 

Let $e_{ii}$ ($i = 1, 2, 3$) be the diagonal matrix $\text{diag}(\delta_{1i}, \delta_{2i}, \delta_{3i}) \in H_3(O')$. Then $\{e_{11}, e_{22}, e_{33}\}$ is a frame in $H_3(O')$.

**Lemma 5.3.** $o_{3,0}$ is an invertible element in the Jordan algebra $\mathfrak{A} := H_3(O')$.

**Proof.** Let $P_{\mathfrak{A}}$ be the quadratic representation of $\mathfrak{A}$. Then (5.16) implies that $P_{\mathfrak{A}}(X) = P(X)$ for $X \in g_{-1} = H_3(O')$, and hence $P_{\mathfrak{A}}(o_{3,0}) = P(o_{3,0})$. The operator $P(o_{3,0})$ is nondegenerate, by Lemma 4.4. Therefore $o_{3,0}$ is an invertible element in $\mathfrak{A}$. □

Recall the Jordan algebra $\mathfrak{A}_3 = (g_{-1}, \square_3)$ in §2. By (5.16), $\mathfrak{A}_3$ is a mutant of $\mathfrak{A}$ by the invertible element $o_{3,0}$.

**Lemma 5.4.** $N(o_{3,0})N(o_{2,1}) < 0$.

**Proof.** Let $N_{\mathfrak{A}}$ be the reduced norm of $\mathfrak{A}$. Then we have (Braun-Koecher [3])

$$
(5.18) \quad N(X) = N_{\mathfrak{A}}(X)N_{\mathfrak{A}}(o_{3,0}), \quad X \in g_{-1}.
$$
Since $o_{3,0}$ is invertible in $\mathfrak{U}$, we have $N_{\mathfrak{U}}(o_{3,0}) \neq 0$. Now consider the two frames $\{e_1, e_2, e_3\}$ and $\{e_{11}, e_{22}, e_{33}\}$ in $\mathfrak{B}$. By Proposition 11.8 in Loos [18] and Lemma 3.4 here, there exists an element $k \in K_0$ such that

$$k e_{3,0} = \sum_{i=1}^{3} \delta_i e_{ii},$$

where $\delta_i = \pm 1$. $N_{\mathfrak{U}}$ is a relative invariant polynomial for the group $\text{Str} \, \mathfrak{U}$. Therefore there exists an $R^*$-valued character $\chi$ of $\text{Str} \, \mathfrak{U} = \text{Str} \, \mathfrak{B} = G_0$ such that

$$N_{\mathfrak{U}}(gX) = \chi(g) N_{\mathfrak{U}}(X), \quad X \in g_{-1}, g \in G_0.$$  

Since $K_0$ is contained in the commutator subgroup $[G_0, G_0]$, $\chi(K_0) = 1$. Therefore we have

$$N_{\mathfrak{U}}(o_{3,0}) = N_{\mathfrak{U}}(ko_{3,0}) = N_{\mathfrak{U}} \left( \sum_{i=1}^{3} \delta_i e_{ii} \right) = \delta_1 \delta_2 \delta_3.$$  

Similarly we have $N_{\mathfrak{U}}(o_{2,1}) = -\delta_1 \delta_2 \delta_3$. Therefore, in view of (5.18), we have $N(o_{3,0})N(o_{2,1}) < 0$.

**Theorem 5.5.** Let $(g, g_0, g_{-1}) = (E_{7}(7), E_{6}(6) + R, H_{3}(O'))$. Then the orbit decompositions of $g_{-1}$ under the groups $G_0^0$ and $G_0$ are given by

$$g_{-1} = \prod_{k=0}^{2} G_0^0 \cdot o_{k,0} \prod_{k=0}^{2} G_0^0 \cdot o_{3,0} \prod_{k=0}^{2} G_0^0 \cdot o_{2,1},$$

(5.22)

$$g_{-1} = \prod_{k=0}^{3} G_0 \cdot o_{k,0}.$$  

(5.23)

There are exactly two open orbits $G_0^0 \cdot o_{3,0}$ and $G_0^0 \cdot o_{2,1}$ which are mutually diffeomorphic. There is a single open $G_0$-orbit in $g_{-1}$.

**Proof.** (5.22) follows from Lemmas 4.3 and 4.4 and Proposition 4.7. Let us consider the $G_0$-orbit decomposition of $g_{-1}$. In the present case $g = E_{7}(7)$, $Q_1$ mod $Q_0$ is generated by $e_2$ (Matsumoto [19]), and hence $e_2 \in G_0 - G_0^0$. Consider the subset $\Pi_1' = \{a_2\}$ of $\Pi_1$. Then $h_{\Pi_1'}(a) = 2$. Let $g = \sum_{k=-2}^{2} g_k'$ be the gradation of $g$ corresponding to $\Pi_1'$ and let $A = \prod_{k=-2}^{2} A_k'$ be the corresponding partition of $A$. By the same reason as for $g = \text{sl}(2p, R)$, we have that $e_2 = 1$ on $g_{-2}' + g_0' + g_2'$ and $e_2 = -1$ on $g_{-1}' + g_1'$. On the other hand, we have $\beta_1 \in A_2', \beta_2 \in A_1'$ and $\beta_3 \in A_0'$ (cf. (5.15)). Consequently $e_2(o_{3,0}) = e_1 - e_2 + e_3$. Let $a \in W(A')$ be the element interchanging $e_2$ with $e_3$ and leaving $e_1$ fixed. Then it follows that $ae_2(o_{3,0}) = o_{2,1}$, which implies $e_2(G_0^0 \cdot o_{3,0}) = G_0^0 \cdot o_{2,1}$. This proves (5.23).

**5.4.** Let us consider the final case $(g, g_0, g_{-1}) = (\text{so}(p+1, q+1), \text{so}(p, q) + R, R^{p+q})$, $2 \leq p \leq q$, in which case $r = 2$ (cf. Table II). The root system $A$ of $g$ is of type $B_{p+1}$ or $D_{p+1}$, according as $p < q$ or $p = q$, respectively. $A$ is given by

$$A = \{ \pm (\lambda_i \pm \lambda_j) \ (1 \leq i < j \leq p+1); \lambda_i \ (1 \leq i \leq p+1) \}, \quad p < q,$$

(5.24)
or
\[ \Delta = \{ \pm (\lambda_i \pm \lambda_j) : 1 \leq i < j \leq p+1 \}, \quad p = q. \]
The gradation of \( g \) corresponds to the subset \( \Pi_1 = \{ \alpha_1 = \lambda_1 - \lambda_2 \} \) of \( \Pi \). \( \Delta_1 \) is given by
\[ (5.25) \quad \Delta_1 = \{ \lambda_1 \pm \lambda_i (2 \leq i \leq p+1); \lambda_1 \}, \]
where \( \lambda_1 \) occurs only when \( p < q \). The subset of \( \Delta_1 \)
\[ (5.26) \quad \Gamma = \{ \beta_1 = \lambda_1 + \lambda_2, \beta_2 = \lambda_1 - \lambda_2 \} \]
is a maximal system of strongly orthogonal roots in \( A_1 \). In this situation we get the simple Jordan algebra \( \mathfrak{U}_2 = (g_{-1}, \Box) \) of rank 2 with unit element \( e := o_{2,0} \) (cf. §2). We need some results on simple Jordan algebras of rank 2 due to Braun-Koecher [3]: The reduced norm \( N \) of \( \mathfrak{U}_2 \) is of signature \((p, q)\), and the multiplication \( \Box_2 \) can be expressed as
\[ (5.27) \quad x \Box_2 y = N(e, x)y + N(e, y)x - N(x, y)e, \quad x, y \in g_{-1}, \]
where \( N(x, y) = (1/2)(N(x+y) - N(x) - N(y)) \). From this it follows that
\[ (5.28) \quad N(e_1, e_1) = N(e_2, e_2) = 0, \quad N(e_1, e_2) = \frac{1}{2}. \]

**Theorem 5.6.** Let \((\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1}) = (\mathfrak{so}(p+1, q+1), \mathfrak{so}(p, q) + \mathbb{R}, \mathbb{R}^{p+q})\), \( 2 \leq p \leq q \). Then the \( \mathcal{G}_0 \)-orbit decomposition of \( \mathfrak{g}_{-1} \) is given by
\[ (5.29) \quad g_{-1} = \bigoplus_{k=0}^1 G_0 \cdot o_{k,0} \bigoplus G_0^0 \cdot o_{2,0} \bigoplus G_0 \cdot o_{1,1}. \]

**Proof.** By using (5.28), we see that \( N(o_{2,0}) = 1 \) and \( N(o_{1,1}) = -1 \). Therefore, from Lemma 4.3 and Proposition 4.7, the assertion follows. \( \square \)

**Theorem 5.7.** Under the same assumption as in Theorem 5.6, the \( \mathcal{G}_0 \)-orbit decomposition of \( g_{-1} \) is given as follows:
\[ (5.30) \quad g_{-1} = \bigoplus_{k=0}^2 G_0 \cdot o_{k,0} \quad \text{for } p = q, \]
\[ (5.31) \quad g_{-1} = \bigoplus_{k=0}^1 G_0 \cdot o_{k,0} \bigoplus G_0 \cdot o_{2,0} \bigoplus G_0 \cdot o_{1,1} \quad \text{for } p < q. \]

**Proof.** Suppose first \( p = q \). In this case, one of generators of \( Q_1 \mod Q_0 \) is \( \epsilon_{p+1} \) (Matsumoto [19]). Note that \( \epsilon_{p+1} \in G_0 - G_0^0 \). Choose the subset \( \Pi'_1 = \{ \alpha_{p+1} \} \) of \( \Pi \). Then \( h_{\Pi'_1}(\mathfrak{g}) = 1 \). Let \( \mathfrak{g} = \sum_{k=1} G_k \) be the gradation of \( g \) corresponding to \( \Pi'_1 \), and let \( \Delta = \bigoplus_{k=1}^\infty \Delta_k \) be the corresponding partition of \( \Delta \). We have \( \epsilon_{p+1} = 1 \) on \( G_0 \), and \( \epsilon_{p+1} = -1 \) on \( g_{-1} + G_1' \). We also have \( \beta_1 \in \Delta'_1 \) and \( \beta_2 \in \Delta'_0 \), since \( h_{\Pi'_1}(\beta_1) = 1 \) and \( h_{\Pi'_1}(\beta_2) = 0 \). As a result, \( \epsilon_{p+1}(o_{2,0}) = -e_1 + e_2 \). Choose an element \( a \in W(\Delta^*) \) interchanging \( e_1 \) with \( e_2 \). Then \( a\epsilon_{p+1}(o_{2,0}) = o_{1,1} \), which implies that \( \epsilon_{p+1}(G_0 \cdot o_{2,0}) = G_0 \cdot o_{1,1} \). This, together with Lemma 4.3, proves (5.30).
Next consider the case \( p < q \). Put \( C^+_{pq} = G^0_0 \cdot \sigma_{2,0} \) and \( C^-_{pq} = G^0_0 \cdot \sigma_{1,1} \) for simplicity. Choose a coordinate system \((x_i)\) in \( g_{-1} = \mathbb{R}^{p+q} \) such that the reduced norm \( N(X) \) is expressed as the canonical form \( x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \). Then

\[
C^\pm_{pq} = \left\{(x_i) \in \mathbb{R}^{p+q} : \sum_{i=1}^p x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \geq 0\right\}.
\]

Let \( S^\pm_{pq} \) be the level surfaces of \( N \), that is,

\[
S^\pm_{pq} = \{(x_i) \in C^\pm_{pq} : N(X) = \pm 1\}.
\]

Then \( C^\pm_{pq} \) are diffeomorphic to \( S^\pm_{pq} \times \mathbb{R}^+ \), respectively. An easy argument shows that \( S^+_{pq} \) (resp. \( S^-_{pq} \)) is diffeomorphic to \( S^{p-1} \times \mathbb{R}^q \) (resp. \( S^{q-1} \times \mathbb{R}^p \)), where \( S^k \) denotes a \( k \)-sphere. Consider the \( i \)-th homology groups \( H_i(C^\pm_{pq}, \mathbb{Z}), 0 \leq i \leq p + q \). Then the above argument shows that \( H_i(C^+_{pq}, \mathbb{Z}) \cong H_i(S^{p-1}, \mathbb{Z}) \) and \( H_i(C^-_{pq}, \mathbb{Z}) \cong H_i(S^{q-1}, \mathbb{Z}) \). Suppose that \( C^\pm_{pq} \) are homeomorphic to each other. Then we have \( H_i(S^{p-1}, \mathbb{Z}) \cong H_i(S^{q-1}, \mathbb{Z}) \) for any \( i, 0 \leq i \leq p + q \), which implies \( p = q \). This contradicts the hypothesis \( p < q \). Therefore \( C^+_{pq} \) is not homeomorphic to \( C^-_{pq} \). Suppose now that there exists only one open \( G_0 \)-orbit in \( g_{-1} \). Then there exists \( a \in G_0 - G^0_0 \) such that \( a\sigma_{2,0} = \sigma_{1,1} \). We then have \( a(C^+_{pq}) = C^-_{pq} \), and hence \( C^+_{pq} \) is homeomorphic to \( C^-_{pq} \), which is a contradiction. Therefore there are exactly two open \( G_0 \)-orbits.

6. Open \( G^0_0 \)-orbits

Let \( g = g_{-1} + g_0 + g_1 \) be a real simple GLA. Suppose that the split root system \( \Delta(g, c) \) of the symmetric pair \((g, g_0)\) is of type \( C_r \). Then we have the simple Jordan algebras \( \mathfrak{U}_p = (g_{-1}, \square_p) \) with unit element \( o_{p,r-p}(0 \leq p \leq r) \) (cf. §2). For an element \( g \in \text{Str} \mathfrak{U}_p \), we define

\[
\theta(g) := (g^*)^{-1},
\]

where \( g^* \) is the adjoint operator of \( g \) with respect to the trace form \( \gamma_p \) of \( \mathfrak{U}_p \). Then \( \theta \) is an involutive automorphism of \( \text{Str} \mathfrak{U}_p \). We denote by \( \text{Aut}_{\text{JTS}} \mathfrak{U}_p \) the automorphism group of the JTS (4.13) coming from the Jordan algebra \( \mathfrak{U}_p \), and we denote by \( (\text{Str} \mathfrak{U}_p)_{\theta} \) the subgroup of \( \theta \)-fixed elements of \( \text{Str} \mathfrak{U}_p \). Then, by the definition of \( \text{Aut}_{\text{JTS}} \mathfrak{U}_p \), we have

\[
(\text{Str} \mathfrak{U}_p)_{\theta} = \text{Aut}_{\text{JTS}} \mathfrak{U}_p.
\]

**Proposition 6.1.** Suppose that the split root system \( \Delta(g, c) \) of the symmetric pair \((g, g_0)\) is of type \( C_r \). Then the open orbit \( G^0_0 \cdot o_{p,r-p} \) \((0 \leq p \leq r)\) is expressed as a symmetric coset space:

\[
G^0_0 \cdot o_{p,r-p} = (\text{Str} \mathfrak{U}_p)^0 / (\text{Str} \mathfrak{U}_p)^0 \cap \text{Aut} \mathfrak{U}_p,
\]

where \( \text{Aut} \mathfrak{U}_p \) denotes the automorphism group of the Jordan algebra \( \mathfrak{U}_p \). (Note that \( G^0_0 = (\text{Str} \mathfrak{U}_p)^0 \) by (4.17)).
PROOF. \( \text{Aut}_p \mathfrak{U} \) is an open subgroup of \( \text{Aut}_{\text{JTS}} \mathfrak{U}_p \) (cf. Satake [21]). Consequently, noting (6.2), we have the inclusions

\begin{equation}
(\text{Str}_p)^0 \subset \text{Aut}_p \subset (\text{Str}_p)^0.
\end{equation}

By taking the intersection of each term in (6.4) with \((\text{Str}_p)^0\), it follows that

\begin{equation}
((\text{Str}_p)^0)^0 \subset \text{Aut}_p \subset ((\text{Str}_p)^0)^0,
\end{equation}

which implies that the coset space in the right-hand side of (6.3) is a symmetric coset space. Since \( \text{Aut}_p \mathfrak{U}_p \) is the isotropy subgroup of \( \text{Str}_p \mathfrak{U}_p \) at the unit element \( o_{p,r-p} \), \( G_0^0 \cdot o_{p,r-p} \) has the coset space expression (6.3). 

Every open orbit \( G_0^0 \cdot o_{p,r-p} \) is an \( \omega \)-domain in the sense of Koecher [16], since that orbit is a connected component of \( V_r \) (note that \( V_r \) coincides with the totality of invertible elements in \( \mathfrak{U}_p \), by Lemma 4.4). As a result, open \( G_0^0 \)-orbits exhaust all \( \omega \)-domains in real simple Jordan algebras. The results similar to Proposition 6.1 were obtained also by Faraut-Gindikin [5] and Vinberg [29].

REMARK 6.2. Assuming that \( \mathcal{A}(g,c) \) is of type C, let us consider the quadratic representation \( P(X) \) of the JTS \( \mathfrak{B} \). Then \( P(X) \) is nondegenerate for \( X \in V_r \) ([6]), \( \det P(X) \) has a constant sign on each connected component of \( V_r \). Put

\begin{equation}
\Phi(X) = \log|\det P(X)|, \quad X \in V_r.
\end{equation}

Then, by Koecher [16] together with Lemma 4.4, the Hessian \( \text{Hess}(\Phi(X)) \) is non-degenerate on each open \( G_0^0 \)-orbit. Hence \( \text{Hess}(\Phi(X)) \) is a \( G_0^0 \)-invariant pseudo-riemannian metric on it. As a conclusion, an open \( G_0^0 \)-orbit provides with an example of pseudo-Hessian symmetric space (For the definition of a Hessian symmetric space, see Shima [24]).

In the following, we give the explicit forms of open \( G_0^0 \)-orbits and their coset space expression (6.3) for each simple \( \text{GLA}(g,g_0,g_{-1}) \) with split root system of type C. Partial results have been obtained by Kaneyuki [11] and d’Atri-Gindikin [4].

(11) with \( p = n/2 \),

\( \{X \in M_p(R) : \det X > 0\} \), \( \{X \in M_p(R) : \det X < 0\} \).

Both are expressed as \( GL(p,R)^0 \times GL(p,R)^0 \) diagonal.

(12) with \( p = n/2 \),

\( \{X \in M_p(H) : \det X \neq 0\} = GL(p,H) \times GL(p,H)/\text{diagonal}. \)

(13) \( H_{n-i,i}(C) = GL(n,C)/U(n-i,i), \quad 0 \leq i \leq n. \)

(14) \( H_{n-i,i}(R) = GL(n,R)^0/ SO(n-i,i), \quad 0 \leq i \leq n. \)

(15) \( \{X \in SH_n(H) : \det X \neq 0\} = GL(n,H)/SO^*(2n). \)

(16) i) \( p = 0 \),

\( \{(x_i) \in R^q : x_1^2 + \cdots + x_q^2 \neq 0\} = R^+ \times SO(q)/SO(q-1). \)
ii) $p = 1$,
\[
\{(x_i) \in \mathbb{R}^{q+1} : x_1^2 - x_2^2 - \cdots - x_{q+1}^2 > 0, x_1 > 0\},
\]
\[
\{(x_i) \in \mathbb{R}^{q+1} : x_1^2 - x_2^2 - \cdots - x_{q+1}^2 > 0, x_1 < 0\},
\]
\[
\{(x_i) \in \mathbb{R}^{q+1} : x_1^2 - x_2^2 - \cdots - x_{q+1}^2 < 0\},
\]

The first two are expressed as $\mathbb{R}^+ \times SO(1,q)^0/SO(q)$. The third one is expressed as $\mathbb{R}^+ \times SO(1,q-1)^0$.

iii) $p \geq 2$,
\[
\begin{align*}
\left\{(x_i) \in \mathbb{R}^{q+p} : \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 > 0 \right\} &= \mathbb{R}^+ \times SO(p,q)^0/SO(p-1,q)^0, \\
\left\{(x_i) \in \mathbb{R}^{q+p} : \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 < 0 \right\} &= \mathbb{R}^+ \times SO(p,q)^0/SO(p,q-1)^0.
\end{align*}
\]

(17) $H_{n-i,i}(H) = GL(n,H)/Sp(n-i,i), \quad 0 \leq i \leq n.$

(18) $\{X \in \text{Alt}_n(R) : \text{Pff}(X) > 0\}, \quad \{X \in \text{Alt}_n(R) : \text{Pff}(X) < 0\}.$

Both are expressed as $GL(2n,R)^0/Sp(n,R)$.

(H11) $\{X \in H_3(O') : N(X) > 0\}, \quad \{X \in H_3(O') : N(X) < 0\},$

where $N$ denotes the reduced norm of $H_3(O')$. Both are expressed as $\mathbb{R}^+ \times E_6(6)/F_4(4)$.

(112) $H_{3-i,i}(O)$, $i = 0,1,2,3$.

$H_{3,0}(O)$ and $H_{0,3}(O)$ are expressed as $\mathbb{R}^+ \times E_6(-26)/F_4$.

$H_{2,1}(O)$ and $H_{1,2}(O)$ are expressed as $\mathbb{R}^+ \times E_6(-26)/F_4(-26)$.

(113) with $p = n/2$,
\[
\{X \in M_p(C) : \det X \neq 0\} = GL(p,C) \times GL(p,C)/\text{diagonal}.
\]

(114) $\{X \in \text{Sym}_n(C) : \det X \neq 0\} = GL(p,C)/SO(n,C)$.

(115) $\{(z_i) \in C^n : z_1^2 + \cdots + z_n^2 \neq 0\} = C^* \times SO(n,C)/SO(n-1,C)$.

(116) $\{X \in \text{Alt}_n(C) : \text{Pff}(X) \neq 0\} = GL(2n,C)/Sp(n,C)$.

(118) $\{X \in H_3(O^C) : N(X) \neq 0\} = C^* \times E_{6C}^C/F_4^C$,

where $N$ denotes the reduced norm of the Jordan algebra $H_3(O^C)$.

In the above list, $H_{n-i,i}(K)$ denotes the set of $n \times n$ $K$-hermitian matrices of signature $(n-i,i)$, where $K = R, C, H, O$.

References


Soji Kaneyuki
Department of Mathematics
Sophia University
Chiyoda-ku, Tokyo 102-8554, Japan