Static Closed-Form Solutions for In-Plane Thick Curved Beams with Variable Curvatures*

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Abstract
An analytical method is derived for obtaining the in-plane static closed-form general solutions of thick curved beams with variable curvatures. The strain considering thickness-curvature effect is adopted to develop static governing equations. The governing equations are formulated as functions of the angle of tangent slope by introducing the coordinate system defined by the radius of centroidal axis and the angle of tangent slope. To solve the governing equations, one can define the fundamental geometric properties, such as the first and second moments of the arc length with respect to horizontal and vertical axes. As the radius is given, the fundamental geometric quantities can be calculated to obtain the static closed-form solutions of the axial force, shear force, bending moment, rotation angle, and displacement fields at any cross-section of curved beams. The closed-form solutions of the ellipse, parabola, and exponential spiral beams under various loading cases are presented. The results show the consistency in comparison with existed results.

Key words: Thick Curved Beams, Variable Curvatures, Analytical Method, Ellipse, Parabola

1. Introduction
In existed literature, researchers usually apply approximate methods for the displacement fields to obtain numerical or analytical solutions of curved beams. The approximate methods including finite elements, power series, and trial functions have been applied by many researchers. However, the method for finding closed-form solutions of curved beams has been rarely studied. Although Langhaar has shown static closed-form solutions for circular beams, to date, few researchers have studied closed-form solutions for non-circular beams.

This paper will present an analytical method for obtaining the in-plane static closed-form general solutions of thick curved beams with variable curvatures. The general solutions expressed by fundamental geometric quantities will be applied to the ellipse, parabola, and exponential spiral beams under various loading cases. By calculating the geometric quantities, one will obtain the closed-form solutions including the axial force, shear force, bending moment, rotation angle, and displacement fields at any cross-section of curved beams. The closed-form solutions would be useful for thick curved beams required the exact description of mathematical models in engineering applications, such as the expressions in optimum design for springs, arch bridges, and stiffeners in aircraft structure.
2. General theory of thick curved beams

2.1 Kinematical relationship

Consider an in-plane thick curved beam. Its geometry and positive sign conventions are shown in Fig. 1(a). The displacements of a point on the centroidal axis are defined by the radial displacement $u$ and the circumferential displacement $v$, while the displacements of a point at distance $z$ away from the centroidal axis are defined by $u$ and $v$. Assume that $u$ does not vary on a cross-section, i.e., $u = u_s$. If cross-sections remain plane and normal to centroidal axis after deformation, then $v$ can be expressed as $v = v_s = \frac{v}{R} ds + z (du/ds)$, where $R$ is the radius of centroidal axis at any arc length $s$, i.e., $R = R(s)$. Here the shear deformation is neglected. For a curved element at distance $z$, the strain can be obtained by $\varepsilon = u/(R + z) + d\tilde{v}/ds$ and $dS = (1 + z/R)ds$, in which the term $z/R$ denotes the thickness-curvature effect(9). In terms of arc length $s$, the strain of thick curved beams with variable curvatures is

$$\varepsilon = \frac{dv}{ds} + \left( \frac{\frac{d^2 u}{ds^2} + \frac{v}{R} \frac{dR}{ds}}{R + z} \right).$$

Equation (1) indicates the theory of thick curved beams differs from that of thin curved beams with regard to the through-thickness strain distribution.

![Fig. 1. The geometry and sign conventions of an in-plane thick curved beam.](image)

2.2 Force and moment

The method of evaluating the thickness-curvature effect can be given by Kim et al(9). If the method is incorporated for the terms of $1/(1 + z/R)$ in Eq. (1), then the axial force $N$ can be obtained by integrating Eq. (1) over the cross-sectional area $A$, i.e.,

$$N = E \int \varepsilon dA = EA \left( \frac{dv}{ds} + \frac{u}{R} \right),$$

where $E$ is the Young’s modulus and $I$ is the moment of inertia of the cross-section. Likewise, the bending moment $M$ can be written as

$$M = E \int z \varepsilon dA = -EI \left( \frac{d^2 u}{ds^2} + \frac{u}{R^2} + \frac{v}{R^2} \frac{dR}{ds} \right).$$

Here consider the curved beams having the same cross-section, so $A$ and $I$ are constants in Eqs. (2) and (3). If $R$ is also a constant, then Eqs. (1)–(3) reduce to the results of thick circular beams derived by Langhaar(8) neglecting the shear deformation effect.

Combining Eqs. (2) and (3) yields the coupling of extension-bending:

$$\frac{dv}{ds} + \frac{u}{R} = \frac{1}{EA} (N + \frac{M}{R}),$$

where the quantity of $dv/ds + u/R$ is the strain on the centroidal axis, given by setting $z = 0$ in Eq. (1). The existence of the extension-bending coupling indicates the main character of thick curved beams.
2.3 Governing equations

Fig. 1(b) shows the sign conventions of positive moments ($M$), axial forces ($N$), shear forces ($V$), and distributed loads, acting on a curved beam element. In terms of arc length $s$, the equilibrium equations of the element are

$$\frac{N}{R} \frac{dV}{ds} - q_R = 0, \quad \frac{dN}{ds} + \frac{V}{R} + q_a = 0, \quad \frac{dM}{ds} = V,$$

where $q_R$ is the distributed load per length in the radial direction and $q_a$ is the distributed load per length in the circumferential direction along the centroidal axis.

For convenience, introduce the dimensionless characteristic quantities defined by the following:

$$\tilde{s} = \frac{s}{R_0}, \quad \tilde{x} = \frac{x}{R_0}, \quad \tilde{y} = \frac{y}{R_0}, \quad \tilde{R} = \frac{R}{R_0}, \quad \tilde{u} = \frac{u}{R_0}, \quad \tilde{v} = \frac{v}{R_0},$$

$$\tilde{M} = \frac{MR_0}{EI}, \quad \tilde{N} = \frac{NR_0^2}{EI}, \quad \tilde{V} = \frac{VR_0^2}{EI}, \quad \tilde{q}_a = \frac{q_a R_0^3}{EI}, \quad \tilde{q}_R = \frac{q_R R_0^3}{EI},$$

where $R_0$ is a characteristic radius. In the following, drop the tildes for simplicity.

To derive the general solutions of Eqs. (2), (3), and (5), set the distributed load $q_R = 0$, $q_a = 0$ in Eqs. (5), and then transform the variable $ds$ into $d\alpha$. Equations (5) can be written as

$$\frac{dV(\alpha)}{d\alpha} = N(\alpha), \quad \frac{dN(\alpha)}{d\alpha} + V(\alpha) = 0, \quad \frac{dM(\alpha)}{d\alpha} = R(\alpha) V(\alpha).$$

Combining the first and second of Eqs. (9) yields $V(\alpha)$ and $N(\alpha)$:

$$V(\alpha) = A_1 \cos \alpha + A_2 \sin \alpha,$$

$$N(\alpha) = -A_1 \sin \alpha + A_2 \cos \alpha,$$

where the constants $A_1$ and $A_2$ denote the reaction of shear force and axial force at $\alpha = 0$, respectively. With the help of Eq. (10), integrating the third of Eqs. (9) once obtains the bending moment $M(\alpha)$:

$$M(\alpha) = A_1 x + A_2 y + M_0,$$

where the constant $M_0$ denotes the reaction moment at $\alpha = 0$.

According to the assumption by Brush and Almroth (10), the rotation angle at a point on the centroidal axis during deformation can be

$$\psi = \frac{v}{R} \frac{du}{ds}.$$

Combining Eqs. (2), (3), and the expression of $d\psi/ds$ derived from Eq. (13) yields the rotation angle $\psi(\alpha)$:

$$\psi(\alpha) = \int_0^\alpha M R d\alpha + \lambda^2 \int_0^\alpha (N + \frac{M}{R}) d\alpha + \psi_0,$$

where the dimensionless constant, $\lambda^2$, is defined by

$$\lambda^2 = \frac{1}{AR_0^2}.$$

It appears that the slenderness ratio, $\lambda$, is the radii ratio of $\sqrt{I/A}/R_0$, while $\sqrt{I/A}$ denotes the radius of gyration of cross-sectional area with respect to centroidal axis.
Substituting Eqs. (11) and (12) into Eq. (14) yields the rotation angle $\psi(\alpha)$:

$$
\psi(\alpha) = \hat{\lambda}^2 [A_1 (R_x + \cos \alpha - 1) + A_2 (R_y + \sin \alpha) + M_0 R_x] \\
+ A_4 I_x + A_5 I_y + M_0 \psi + \psi_0,
$$

(16)

where the constant $\psi_0$ denotes the rotation angle at $\alpha = 0$. The geometric quantities

$$
I_x(\alpha) = \int_0^\alpha x(w)R(w)dw = \int_0^\alpha \frac{x(\hat{s})d\hat{s}}{R}, \quad I_y(\alpha) = \int_0^\alpha y(w)R(w)dw = \int_0^\alpha \frac{y(\hat{s})d\hat{s}}{R},
$$

(17)

are the first moments of the arc length with respect to the vertical and horizontal axes, respectively. Moreover, the geometric quantities $R_x$, $R_y$, and $R_y$ are defined by the following:

$$
R_x(\alpha) = \int_0^\alpha x(w)R(w)dw, \quad R_y(\alpha) = \int_0^\alpha y(w)R(w)dw, \quad R_y(\alpha) = \int_0^\alpha \frac{1}{R(w)}dw.
$$

(18)

Combining Eqs. (2), (3), and (13) obtains the differential equation of radial displacement:

$$
\frac{d^2 u}{d\alpha^2} + u = \hat{\lambda}^2 R(N + \frac{M}{R}) - \frac{d}{d\alpha} (R\psi).
$$

(19)

Its analytical solution can be expressed as

$$
u(\alpha) = A_3 \cos \alpha + A_4 \sin \alpha - \hat{\lambda}^2 \int_0^{\alpha} R\psi(\alpha) \cos \alpha d\alpha - \sin \alpha \int_0^{\alpha} R\psi(\alpha) \sin \alpha d\alpha
$$

$$
- \int_0^{\alpha} (R(N + M) \sin \alpha - \sin \alpha \int_0^{\alpha} (R(N + M) \cos \alpha) d\alpha).
$$

(20)

Substituting Eqs. (11), (12) and (16) into Eq. (20) yields the general solution of radial displacement:

$$
u(\alpha) = A_3 \cos \alpha + A_4 \sin \alpha - \hat{\lambda}^2 \cos \alpha \int_0^{\alpha} A_1 (2R_{xc} - R_{sx} + R_{cx} - x - x \cos \alpha) \\
+ A_2 (3R_{xc} + R_{cy} - y \cos \alpha) + M_0 (R_{ci} - \cos \alpha + 1)] - \hat{\lambda}^2 \sin \alpha \int_0^{\alpha} A_1 (3R_{xc}) \\
+ R_{sx} - y - x \sin \alpha) + A_2 (2R_{sx} - R_{xc} + R_{sy} - \sin \alpha) + M_0 (R_{si} - \sin \alpha)] - \hat{\lambda}^2 \sin \alpha \int_0^{\alpha} A_1 (3R_{xc}) \\
$$

(21)

where the geometric quantities

$$
I_{xx}(\alpha) = \int_0^{\alpha} x^2(w)R(w)dw, \quad I_{yy}(\alpha) = \int_0^{\alpha} y^2(w)R(w)dw,
$$

(22)

are the second moments of the arc length with respect to the vertical and horizontal axes. Moreover, the geometric quantities of $R_{xc}$, $R_{sx}$, $R_{sc}$, $R_{cx}$, $R_{sx}$, $R_{cy}$, $R_{sy}$, $R_{cy}$, and $R_{si}$ are defined in the Appendix.

Combining Eqs. (13) and (20) obtains the general solution of circumferential displacement:

$$
v(\alpha) = -A_3 \sin \alpha + A_4 \cos \alpha + \hat{\lambda}^2 \sin \alpha \int_0^{\alpha} A_1 (2R_{xc} - R_{sy} + R_{cx} - x - x \cos \alpha) \\
+ A_2 (3R_{xc} + R_{cy} - y \cos \alpha) + M_0 (R_{ci} - \cos \alpha + 1)] - \hat{\lambda}^2 \cos \alpha \int_0^{\alpha} A_1 (3R_{xc}) \\
+ R_{sx} - y - x \sin \alpha) + A_2 (2R_{sx} - R_{xc} + R_{sy} - \sin \alpha) + M_0 (R_{si} - \sin \alpha)] \\
+ \sin \alpha \int_0^{\alpha} A_1 (3R_{xc}) \\
$$

(23)

where $A_3$ and $A_4$ are constants determined by boundary conditions. The constants $A_3 = u(0)$ and $A_4 = v(0)$ are the radial and circumferential displacement at $\alpha = 0$, respectively.

Equations (10)–(12), (16), (21), and (23) form a set of general equations with $V$, $N$, $M$, $\psi$, $u$, and $v$, having six unknown constants including $A_1$, $A_2$, $M_0$, $\psi_0$, $A_3$, and $A_4$. 


and $A_4$. The six constants can be solved by suitable boundary conditions. The free end boundary condition at $\alpha = 0$ obtains $A_1 = A_2 = M_0 = 0$. The fixed end boundary condition at $\alpha = 0$ implies $A_1 = A_2 = \psi_0 = 0$. The hinge boundary condition at $\alpha = 0$ yields $A_3 = A_4 = M_0 = 0$. In general, one can directly determine each constant from boundary conditions, and can avoid complicated calculation. In the following, the set general solutions will be applied to solve the static problems of the ellipse, parabola, and exponential spiral beams under various loading cases.

![Fig. 2. The curved beam under pure bending.](image)

### 3. Applications

Consider a thick curved beam as shown in Fig. 2. The beam is symmetric to vertical y-axis. A pair of moments $M_0$ is applied at two free ends. At $\alpha = \beta$, the boundary conditions $V(\beta) = 0$, $N(\beta) = 0$, and $M(\beta) = M_0$ yield $A_1 = A_2 = 0$. For symmetry at $\alpha = 0$, the boundary conditions $\psi(0) = 0$, $u(0) = 0$, and $v(0) = 0$ give $\psi_0 = A_3 = A_4 = 0$. Hence Eqs. (10)–(12) can be written as $N(\alpha) = 0$, $F(\alpha) = 0$, and $M(\alpha) = M_0$, and Eqs. (16), (21), and (23) can be simplified to

\[
\psi(\alpha) = M_0[(2\beta R_x + s),
\]

\[
u(\alpha) = \psi_0[(2\beta R_x + s),
\]

\[
u(\alpha) = M_0[(2\beta R_x + s),
\]

As an example, consider an elliptical beam. Its radius of centroidal axis is

\[R(\alpha) = \frac{a^2}{b} \left(1 - \frac{\sin^2 \alpha}{\Delta(\alpha,k)}\right),\]

where $\Delta(\alpha,k) = \sqrt{1-k^2 \sin^2 \alpha}$ and the modulus $k = \sqrt{b^2-a^2/b}$. Combining Eqs. (7) and (25) yields the parametric forms of $x(\alpha) = a^2 \sin \alpha / b \Delta$, $y(\alpha) = b(1 - \cos \alpha / \Delta)$, which is equivalent to $x^2/a^2 + (y-b)^2/b^2 = 1$. From the third of Eqs. (7) and (17), the arc length and first moments for the elliptical beam are

\[s(\alpha) = b[E(\alpha,k) - k^2 \sin \alpha \cos \alpha \Delta],\]

\[I_s(\alpha) = \frac{1}{2} a^2 \left[1 - \frac{\cos \alpha}{\Delta^2} + \frac{1}{kk'} \left(\tan^{-1} \frac{k'}{k} - \tan^{-1} \frac{k}{k'}\right)\right],\]

\[I_s(\alpha) = b s(\alpha) - \frac{1}{2} a^2 \left[\frac{1}{2k} \left|\frac{1 + k \sin \alpha}{1 - k \sin \alpha} + \sin \alpha \Delta^2\right|\right],\]

where the complementary modulus $k'$ is defined by $k'^2 = 1-k^2$ (i.e., $k' = a/b$). From the third of Eqs. (18) and the Appendix, the geometric quantities $R_i$, $R_{ij}$, and $R_{ij}$ for the elliptical beam are

\[R_i = \frac{1}{3bk^2} (2(2-k^2)E(\alpha,k) - k^2 F(\alpha,k) + k^2 \Delta \sin \alpha \cos \alpha),\]
\[ R_y = \frac{1}{3} \{(3 - 2k^2)(\cos \alpha - 1) + \frac{\sin \alpha}{\Delta} [2(2 - k^2)E(\alpha, k) - k^2 F(\alpha, k)]\}, \]  

\[ R_y = \frac{1}{3k^2} \{(3 - k^2) \sin \alpha \frac{\cos \alpha}{\Delta} [2(2 - k^2)E(\alpha, k) - k^2 F(\alpha, k)]\}, \]

where the function \( F(\alpha, k) \) is the elliptic integral of the first kind and \( E(\alpha, k) \) is the elliptic integral of the second kind. Substituting the expressions of \( \psi(\alpha) \) and \( \phi(\alpha) \), Eqs. (26) and (27) into Eqs. (24) yields the closed-form solutions of the elliptical beam under pure bending:

\[ \psi(\alpha) = M_0 \left\{ \frac{2}{3bk} \left[ -k^2 F(\alpha, k) + 2(2 - k^2)E(\alpha, k) + k^2 \Delta \sin \alpha \cos \alpha \right] + b \left[ E(\alpha, k) - k^2 \sin \alpha \cos \alpha \right] \right\}, \]  

\[ u(\alpha) = b^2 M_0 \left\{ \frac{2}{3 b^3 k} \left[ 3 - 3 \cos \alpha - (3 - 2k^2)(\cos \alpha - 1) \cos \alpha \right] - \frac{3 - k^2}{k^2} \sin \alpha + \frac{k^2}{k^2} [2(2 - k^2)E(\alpha, k) - k^2 F(\alpha, k)] \frac{\sin \alpha \cos \alpha}{\Delta} \right\} + \frac{1}{2} \frac{k}{\Delta} E(\alpha, k) \sin \alpha \cos \alpha \right\}] \]  

\[ v(\alpha) = b^2 M_0 \left\{ \frac{2}{3 b^3 k} \left[ 2k^2 k^2 (1 - \cos \alpha) \sin \alpha - 2k^2 \sin \alpha \cos \alpha \right] + \frac{2}{2} (2 - k^2) \Delta E(\alpha, k) - k^2 \Delta F(\alpha, k)] + \Delta E(\alpha, k) - k^2 \sin \alpha \cos \alpha \right\} - \frac{1}{2} \frac{k}{\Delta} \sin \alpha - \frac{1}{k} \tan^{-1} \left( \frac{k \cos \alpha}{k} \right) \sin \alpha \right\} \]  

\[ \frac{1}{4} \frac{k}{\Delta} \cos \alpha \ln \left| \frac{1 + k \sin \alpha}{1 - k \sin \alpha} \right|. \]

At \( \alpha = \beta = \pi/2 \), the tip displacements of the elliptical beam under pure bending are

\[ u(\pi/2) = -b^2 M_0 \left\{ \frac{2}{3 b^3 k} \left[ 2k^2 k^2 + \frac{1}{2} + \frac{k^2}{4k} \right] \frac{1 + k}{1 - k} \right\}, \]  

\[ v(\pi/2) = b^2 M_0 \left\{ \frac{2}{3 b^3 k} \left[ 2k^2 k^2 + 2k^2 (2 - k^2)E(k) - k^2 K(k) \right] \right\} - \frac{k^2}{2} \frac{k}{2} \tan^{-1} \left( \frac{k}{k} \right) + k^2 E(k) \],

in which the constant \( K(k) \) is the complete elliptic integral of the first kind and \( E(k) \) is the complete elliptic integral of the second kind. Note that if letting \( k^2 \to 1 \) (i.e., \( k \to 0 \), \( a \to b \)), solutions of Eqs. (28)–(32) reduce to the results calculated from Eq. (24) for a circular beam.

Table 1 presents the closed-form solutions for exponential spiral and parabola beams under pure bending. Those curves are expressed by the radius of centroidal axis, in terms of the angle of tangent slope. The parametric forms of \( x(\alpha) \), \( y(\alpha) \) are presented as well. These results imply that the general solutions of Eqs. (24) would be valid for thick curved beams under pure bending.

In this paper, the focus is on the analytical method for obtaining the closed-form solutions. In the following, only a few cases of thick curved beams are presented. Fig. 3(a) shows a cantilever curved beam subjected to a tangential load \( P \) at the free end of \( \alpha = \beta \). Its closed-form tip displacements for the ellipse, parabola, and exponential spiral beams are...
listed in Table 2. Moreover, Fig. 3(b) also shows a symmetric ring under a couple of point loads \( P \). For an elliptic ring, the closed-form radial displacement at \( \alpha = 0 \) can be expressed as

\[
u(0) = \frac{P b^2 z}{2} \left( k' \frac{4 k^2 \lambda^2 - 3b^2 k^2}{3b^2 k^2 E(k) + \lambda^2 [2(2 - k^2)E(k) - k^2 K(k)]} \right) \left[ k^2 k' + (2 - k^2)E(k) \right] + \frac{3}{k^2} \left( \frac{6 k^2}{k^2} - \frac{k^2}{k^2 + k^2} \right) E(k) - \frac{k^2}{k^2} (k^2 - 9) K(k) - 2 k^2 K(k) \right] (33)
\]

Table 3 presents the comparisons of the radial displacements at \( \alpha = 0 \) obtained by various theories for a circular ring (Fig. 3(b)). The present result is consistent with the results by Langhaar (8) and Castigliano’s theorem using thick curved beams theory. In the \( \lambda^2 \) terms, the present result is different from the results by Timoshenko & Gere (11) and Lin & Hsieh (12). These differences are caused by the effect of through-thickness strain distribution in this paper for thick curved beams theory, while the result by Timoshenko & Gere (11) is based on the thin beam theory with in-extensional assumption and the result by Lin & Hsieh (12) is derived from the extensional thin beam theory. However, for a slender curved beam, i.e., \( \lambda \to 0 \), all results are the same.

Table 1. The displacement solutions for curved beams under pure bending.

<table>
<thead>
<tr>
<th>Curved types</th>
<th>( R(\alpha) )</th>
<th>Parametric forms and displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential spiral ( e^{\alpha} )</td>
<td>( x(\alpha) = -\frac{1}{\lambda^2} \left[ 1 - e^{\alpha} \left( \cos \alpha + \sin \alpha \right) \right] ), ( y(\alpha) = \frac{1}{\lambda^2} \left[ 1 - e^{\alpha} \left( \cos \alpha - \sin \alpha \right) \right] )</td>
<td>( u(\alpha) = M_0 \left[ \frac{\lambda^2}{2} (2 - \cos \alpha + \sin \alpha - e^{\alpha}) - \frac{1}{10} \left( \cos \alpha - 3 \sin \alpha + 4 e^{2\alpha} - 5 e^{\alpha} \right) \right] )</td>
</tr>
<tr>
<td>Parabola ( \sec^2 \alpha )</td>
<td>( x(\alpha) = \tan \alpha ), ( y(\alpha) = \frac{1}{\lambda^2} \tan^2 \alpha )</td>
<td>( u(\alpha) = M_0 \left[ \frac{1}{3} \lambda^2 (3 - 2 \cos \alpha - \sec^2 \alpha) + \frac{1}{48} \left( 1 - 16 \cos \alpha - 5 \tan^2 \alpha \right) \right] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( v(\alpha) = M_0 \left[ \frac{1}{3} \lambda^2 (2 \sin \alpha + \tan \alpha) + \frac{1}{48} \left( 16 \sin \alpha - 13 \tan \alpha + 2 \tan^3 \alpha \right) \right] + 3(4 \sin \alpha \tan \alpha - \cos \alpha) \ln</td>
</tr>
</tbody>
</table>

4. Conclusions

This paper has presented an analytical method for obtaining the in-plane static closed-form general solutions of thick curved beams with variable curvatures. The strain
has neglected the shear deformation effect to show the consistency in comparison with the results of thick circular beams derived by Langhaar(8) neglecting the shear deformation effect. To derive the analytical method for the general solutions, one can introduce the coordinate system defined by the radius of centroidal axis and the angle of tangent slope. The general solutions expressed by fundamental geometric quantities form a set of equations having six unknown constants. The six constants can be directly determined by suitable boundary conditions. As the radius in terms of the tangent slope angle is given, the fundamental geometric quantities can be calculated to obtain the static closed-form solutions of the axial force, shear force, bending moment, rotation angle, and displacement fields at any cross-section of curved beams. These results of the applications indicate that the closed-form general solutions derived by the analytical method would be valid for in-plane thick curved beams and rings. Thus the analytical method would be useful to engineers attempting to obtain the exact expressions for thick curved beams in engineering applications.

The problems of out-of-plane or composite analysis may be an interesting extension of the present paper.

![Fig. 3. The cantilever curved beam and symmetric ring under point loads P.](image)

Table 2. The solutions of tip displacements of cantilever curved beams under a tangential load \( P \) for the angle \( \beta = \pi / 2 \) or \( \beta = \pi / 4 \).

<table>
<thead>
<tr>
<th>Curved types</th>
<th>( R(\alpha) )</th>
<th>Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipse</td>
<td>( a^2 \frac{1}{b} \Delta(\alpha, k)^3 )</td>
<td>( v(\pi / 2) = \frac{Pb \lambda^2}{3} \left[ 4k^2 k' + \frac{E(k)}{k^2} \left( 5 + 7k'' - 4k^4 \right) \right. )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( + \frac{2k^2 K(k)}{k' \left( k'' - 5 \right)} - P \left[ k'' k + \frac{k^4}{3} \tan^{-1} \frac{k}{k'} \right] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( - \frac{k^2 E(k)}{3k^2} \left( 1 + 4k^2 \right) + \frac{k^4}{3k^2} K(k) )</td>
</tr>
<tr>
<td>Parabola</td>
<td>( \sec^3 \alpha )</td>
<td>( v(\pi / 4) = P \left[ \lambda^2 \left( \frac{7}{24} - \frac{1}{8} \right) - \frac{1}{10} - \frac{73 \sqrt{2}}{1920} - \frac{3}{128} \ln(1 + \sqrt{2}) \right] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( + \frac{1}{10} + \frac{73 \sqrt{2}}{1920} - \frac{3}{128} \ln(1 + \sqrt{2}) )</td>
</tr>
<tr>
<td>Exponential spiral</td>
<td>( e^{\alpha} )</td>
<td>( v(\pi / 2) = P \left[ \lambda^2 \left( -\frac{7}{20} + \frac{1}{4} e^{\pi} - \frac{1}{4} e^{2 \pi} - \frac{7}{156} \right) \right. )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( + \frac{1}{10} e^{\pi} - \frac{1}{4} e^{2 \pi} + \frac{14}{195} e^{3 \pi} )</td>
</tr>
</tbody>
</table>
Table 3. The displacement solutions for a circular ring under a couple of point loads \( P \).

<table>
<thead>
<tr>
<th>Displacements</th>
<th>Present</th>
<th>Langhaar(8)</th>
<th>Castigliano</th>
<th>Timoshenko and Gere(11)</th>
<th>Lin and Hsieh(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(0)/PR_0^2 )</td>
<td>(-1/EI \left( \frac{\lambda^2}{2} + \frac{1}{2 \pi} \right) )</td>
<td>(-1/EI \left( \frac{\lambda^2}{2} + \frac{1}{2 \pi} \right) + \frac{\pi}{8} )</td>
<td>(1 - \frac{\pi}{8} )</td>
<td>(-\frac{\lambda^2}{8} + \frac{1 - \pi}{8} )</td>
<td></td>
</tr>
</tbody>
</table>

Appendix

The geometric quantities for thick curved beams are defined by the following:

\[
R_{ww}(\alpha) = \int_0^\alpha R(w) \cos^2 w \, dw, \quad R_{ww}(\alpha) = \int_0^\alpha R(w) \sin^2 w \, dw,
\]

\[
R_{ww}(\alpha) = \int_0^\alpha R(w) \sin w \cos w \, dw, \quad R_{ww}(\alpha) = \int_0^\alpha R(w) \cos w [R_{xw}(w) - R_{xw}(\alpha)] \, dw,
\]

\[
R_{xx}(\alpha) = \int_0^\alpha [R(w) \sin w] R_{xx}(w) \, dw, \quad R_{xx}(\alpha) = \int_0^\alpha [R(w) \cos w] R_{xx}(w) \, dw,
\]

\[
R_{yy}(\alpha) = \int_0^\alpha [R(w) \sin w] R_{yy}(w) \, dw, \quad R_{yy}(\alpha) = \int_0^\alpha [R(w) \cos w] R_{yy}(w) \, dw,
\]

References


