Implicit Integration by Linearization for High-Temperature Inelastic Constitutive Models*

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Abstract
In this study, a linearization method is used to develop an implicit integration scheme for a class of high-temperature inelastic constitutive models based on non-linear kinematic hardening. A non-unified model is first considered in which the inelastic strain rate is divided into transient and steady parts driven, respectively, by effective stress and applied stress. By discretizing the constitutive relations using the backward Euler method, and by linearizing the resulting discretized relations, a tensor equation is derived to iteratively achieve the implicit integration of constitutive variables. The implicit integration scheme developed is shown to be applicable to a unified constitutive model in which back stress evolves due to static and dynamic recoveries in addition to strain hardening. The integration scheme is then programmed for a subroutine in a finite element code and applied to a lead-free solder joint analysis. It is demonstrated that the integration scheme affords quadratic convergence in the iterations even for considerably large increments, and that the non-unified and unified models give almost the same results in the solder joint analysis.

Key words: Constitutive Equation, Inelasticity, Computational Mechanics, Finite Element Method, Implicit Integration, Lead-Free Solder Joint

1. Introduction

At high temperatures, metallic materials generally exhibit transient states followed by steady states in inelastic deformation, because thermal recovery occurs and eventually balances with strain hardening. Unified inelastic constitutive models therefore assume both thermal recovery and strain hardening in the evolution equations of hardening variables so that viscoplasticity and creep can be simulated in a unified manner(1). This unified approach is consistent with dislocation movements in viscoplasticity and creep but usually suffers from complexity and difficulty in determining the material parameters(2),(3). On the other hand, the classical, non-unified approach based on the decomposition of inelastic strain into plastic and creep strains has the advantage of easily determining the material parameters, though the decomposition of inelastic strain cannot be valid as far as the inelastic deformation driven by dislocation movements is concerned.

For the inelastic finite element methods based on the Newton-Raphson iterations,
implicit stress integrations and consistent tangent moduli of constitutive equations provide the incremental computations with stability and efficiency\(^{(6),(5)}\). In a previous study\(^{(6)}\), therefore, an implicit integration scheme and a consistent tangent modulus expression were shown for a non-unified constitutive model in which the inelastic strain rate was divided into transient and steady parts. This model had the advantage of easily determining the material parameters. The model was, however, different from the classical non-unified model, because rate-dependence was assumed for both the transient and steady parts. Moreover, the model was not classical because the non-linear kinematic hardening models proposed by Armstrong and Frederick\(^{(7)}\) and Ohno and Wang\(^{(8),(9)}\) were considered for the transient part.

In the previous study\(^{(6)}\), the implicit integration of the non-unified constitutive model mentioned above was reduced to solving two non-linear scalar equations. However, the two equations were intricately coupled to each other; hence, they were iteratively solved by introducing a successive updating procedure and there was no quick convergence in the iterations. It was therefore necessary to accelerate the convergence using Aitken’s \(\Delta^2\) method, as was done in the elastoplastic case\(^{(10)}\). As a result, the program developed in the previous study\(^{(6)}\) became fairly complicated.

The implicit integration of inelastic constitutive equations is usually reduced to solving scalar equations\(^{(5)}\). For sophisticated inelastic constitutive equations, however, fairly complicated scalar equations may be derived as a consequence of the implicit integration\(^{(6),(10)-(12)}\). Hence, implicit integration schemes based on linearization and iterations were developed for rate-independent non-linear kinematic hardening models\(^{(13)-(15)}\). The present authors\(^{(16)}\) also developed such an integration scheme for a rate-dependent non-linear kinematic hardening model. That constitutive model was, however, not effective for inelastic deformation at high temperatures, because any effect of thermal recovery was not taken into account in representing the steady states following transient states.

In this paper, first, the non-unified rate-dependent inelastic constitutive model aforementioned is outlined. Then, an implicit integration scheme based on linearization and iterations is developed for this non-unified model. The resulting integration scheme is shown to be applicable to a unified type of inelastic model, if the scheme is simply modified. The implicit integration scheme developed is programmed for a user subroutine in a commercial finite element code, and applied to inelastic analysis of a solder joint in an electronic package. The convergence in iterations in the integration scheme is thus examined. Both the non-unified and unified models are used to discuss how different results are obtained in the solder joint analysis.

Throughout the paper, \((\cdot)\) indicates differentiation with respect to time \(t\), \((:)\) the inner product between second rank tensors, \(\otimes\) the tensor product, and \(\|\|\) the Euclidean norm of second rank tensors, i.e., \(\|x\|=(x:x)^{1/2}\).

2. Constitutive Relations

First, the high-temperature constitutive model considered in the previous study\(^{(6)}\) is briefly described.

It is assumed that the mechanical component \(\varepsilon^m\) of strain \(\varepsilon\) consists of elastic strain \(\varepsilon^e\) and inelastic strain \(\varepsilon^u\), and that \(\varepsilon^e\) obeys Hooke’s law. Stress \(\sigma\) is then expressed as

\[
\sigma = D^e(T) : (\varepsilon^m - \varepsilon^u),
\]

where \(D^e(T)\) indicates elastic stiffness, which is a function of temperature \(T\).

When metallic materials are inelastically deformed at high temperatures, they may exhibit transient states followed by steady states, where inelastic strain rates depend only on applied stress and temperature\(^{(17),(18)}\). To simply model this type of inelastic behavior, it is assumed that inelastic strain rate \(\dot{\varepsilon}^u\) consists of transient rate \(\dot{\varepsilon}^t\) and steady rate \(\dot{\varepsilon}^s\),
and that $\dot{\varepsilon}^{\text{u}}$ and $\dot{\varepsilon}^{\text{m}}$ are driven by effective stress $y$ and deviatoric stress $s$, respectively. Here $y$ is expressed as

$$y = s - a,$$  

where $a$ indicates the deviatoric part of back stress $\alpha$. Then, $\dot{\varepsilon}^{\text{u}}$ has transient variations due to the evolution of kinematic hardening, while $\dot{\varepsilon}^{\text{m}}$ is not influenced by strain hardening at all. Such $\dot{\varepsilon}^{\text{u}}$ and $\dot{\varepsilon}^{\text{m}}$ can be expressed as

$$\dot{\varepsilon}^{\text{u}} = \frac{3}{2} g^{\text{u}}(\bar{y}, T) \frac{y}{\bar{y}},$$  

$$\dot{\varepsilon}^{\text{m}} = \frac{3}{2} g^{\text{m}}(\bar{s}, T) \frac{s}{\bar{s}},$$

where $g^{\text{u}}$ and $g^{\text{m}}$ denote transient and steady functions, and $\bar{y}$ and $\bar{s}$ represent the equivalent effective stress and the equivalent stress defined, respectively, as

$$\bar{y} = \sqrt{\frac{3}{2}} \|y\|,$$  

$$\bar{s} = \sqrt{\frac{3}{2}} \|s\|.$$  

It is noted that including isotropic hardening variables in $g^{\text{u}}$ is possible but is not considered for simplicity in this paper.

It is assumed that back stress $\alpha$ is decomposed into $M$ parts, $\alpha^{(1)}, \alpha^{(2)}, \ldots$ and $\alpha^{(M)}$, and that each part of back stress evolves in a strain hardening/generalized dynamic recovery format:

$$\alpha = \sum_{i=1}^{M} \alpha^{(i)},$$

$$\dot{\alpha}^{(i)} = \frac{2}{3} h^{(i)}(T) \dot{\varepsilon}^{\text{u}} - \tilde{\varepsilon}^{(i)}(\tilde{\alpha}^{(i)}, T) \alpha^{(i)} \dot{p}^{\text{m}} + \frac{d \ln h^{(i)}(T)}{dT} \alpha^{(i)} T,$$

where $h^{(i)}$ and $\tilde{\varepsilon}^{(i)}$ are material functions for the strain hardening and the dynamic recovery, respectively, and $\tilde{\alpha}^{(i)}$ and $\dot{p}^{\text{m}}$ are defined as

$$\tilde{\alpha}^{(i)} = \sqrt{\frac{3}{2}} \|\alpha^{(i)}\|,$$  

$$\dot{p}^{\text{m}} = \sqrt{\frac{2}{3}} \|\dot{\varepsilon}^{\text{m}}\|.$$  

The third term on the right hand side in Eq. (8) represents the change in $\alpha^{(i)}$ due to temperature variations, which is necessary for thermo-inelasticity.

Equations (1) – (4) combined with $\dot{\alpha} = (2/3)h^{\text{u}}$ were used to simulate the multiaxial creep of Inconel617 at 1000°C by Suzuki and Hamanaka.

3. Backward Euler Discretization

Let us consider an interval in which $\varepsilon^m$, $t$ and $T$ change from $\varepsilon^m_n$, $t_n$ and $T_n$ to $\varepsilon^m_{n+1}$, $t_{n+1}$ and $T_{n+1}$, respectively, and let us signify the increments in the interval by a prefix $\Delta$ as follows: $\Delta \varepsilon^m = \varepsilon^m_{n+1} - \varepsilon^m_n$, $\Delta t = t_{n+1} - t_n$ and $\Delta T = T_{n+1} - T_n$. Here and from now on, the subscripts $n$ and $n+1$ indicate the values of constitutive variables at the starting and end points in the interval. Then, based on the Backward Euler method, the constitutive relations described in Section 2 are discretized as

$$\sigma_{n+1} = D(T_{n+1}) : (\varepsilon^m_{n+1} - \varepsilon^m_n),$$

$$\varepsilon^m_{n+1} = \varepsilon^m_n + \Delta \varepsilon^m_n,$$

$$\Delta \varepsilon^m_{n+1} = \Delta \varepsilon^m_n + \Delta \varepsilon^m_n,$$

$$\Delta \varepsilon^m_{n+1} = \frac{3}{2} g^{\text{u}}(\bar{y}_{n+1}, T_{n+1}) \Delta T_n \frac{y_{n+1}}{\bar{y}_{n+1}},$$
\[ \Delta \epsilon_{n+1}^{m} = \frac{3}{2} y_{n+1}^{m} (s_{n+1}, T_{n+1}) \Delta T_{n+1} \frac{s_{n+1}}{s_{n+1}} , \]  
(15)

\[ y_{n+1} = s_{n+1} - a_{n+1} , \]  
(16)

\[ a_{n+1} = \sum_{i} a_{n+1}^{(i)} , \]  
(17)

\[ a_{n+1}^{(i)} = \theta_{n+1}^{(i)} h(T_{n+1}) \left[ \frac{1}{h''(T_{n+1})} a_{n+1}^{(i)} + \frac{2}{3} \Delta \epsilon_{n+1}^{m} \right] , \]  
(18)

\[ \theta_{n+1}^{(i)} = \left[ 1 + \zeta^{(i)}(\bar{T}_{n+1}, T_{n+1}) \Delta p_{n+1}^{u} \right]^{-1} , \]  
(19)

\[ \Delta p_{n+1}^{u} = \sqrt{2/3} || \Delta \epsilon_{n+1}^{m} || . \]  
(20)

4. Implicit Stress Integration

In this section, a scheme to find \( \sigma_{n+1} \) is built by linearizing the discretized constitutive relations (11) – (20). Here it is supposed that increments \( \Delta \epsilon_{n+1}^{m} \), \( \Delta T_{n+1} \) and \( \Delta T_{n+1} \) are given, and that all constitutive variables are known at the starting point of the interval.

4.1 Linearization and local iterations

Substituting Eq. (12) into Eq. (11) gives

\[ \epsilon_{n+1}^{m} = \epsilon_{n+1}^{m} - \Delta \epsilon_{n+1}^{m} , \]  
(21)

where \( \epsilon_{n+1}^{m} \) indicates the so-called elastic tentative stress defined as

\[ \epsilon_{n+1}^{m} = D_{n+1}^{m} : (\epsilon_{n+1}^{m} - \epsilon_{n+1}^{m}) . \]  
(22)

Taking the deviatoric part of Eq. (21), we have

\[ s_{n+1}^{*} = s_{n+1} - I \cdot D_{n+1}^{m} : \Delta \epsilon_{n+1}^{m} , \]  
(23)

where \( s_{n+1}^{*} \) denotes the deviatoric part of \( \epsilon_{n+1}^{m} \), and \( I \) stands for the deviatoric operator defined in terms of the fourth and second rank unit tensors, \( I \) and \( I \), as

\[ I = I - \frac{1}{2} I \otimes I . \]  
(24)

Then, using Eqs. (16) and (23), we derive

\[ y_{n+1} = s_{n+1}^{*} - I \cdot D_{n+1}^{m} : \Delta \epsilon_{n+1}^{m} - a_{n+1} . \]  
(25)

Now it is recalled that \( \Delta \epsilon_{n+1}^{m} \) and \( \Delta T_{n+1} \) are given. Then, \( \epsilon_{n+1}^{m} \) and \( D_{n+1}^{m} = D(T_{n+1}) \) are known, so that linearizing Eq. (25) leads to

\[ y_{n+1} + \Delta y_{n+1} = s_{n+1}^{*} - I \cdot D_{n+1}^{m} : (\Delta \epsilon_{n+1}^{m} + \Delta \epsilon_{n+1}^{m} ) - (a_{n+1} + \Delta a_{n+1}) . \]  
(26)

Here, using Eqs. (13) and (17), \( \Delta \epsilon_{n+1}^{m} \) and \( \Delta a_{n+1} \) are expressed as

\[ \Delta \epsilon_{n+1}^{m} = \Delta \epsilon_{n+1}^{m} + \Delta \epsilon_{n+1}^{m} , \]  
(27)

\[ \Delta a_{n+1} = \sum_{i} \Delta a_{n+1}^{(i)} . \]  
(28)

As seen from Eqs. (14), (15) and (18), \( \Delta \epsilon_{n+1}^{m} \), \( \Delta \epsilon_{n+1}^{m} \) and \( \Delta a_{n+1}^{(i)} \) depend on \( y_{n+1} \), \( s_{n+1}^{*} \) and \( \Delta \epsilon_{n+1}^{m} \), respectively, in addition to \( T_{n+1} \). As a result, when \( T_{n+1} \) is prescribed, \( \Delta \epsilon_{n+1}^{m} \), \( \Delta \epsilon_{n+1}^{m} \) and \( \Delta a_{n+1}^{(i)} \) have the following expressions (see Section 4.3):

\[ \Delta \epsilon_{n+1}^{m} = P_{n+1} : \Delta y_{n+1} , \]  
(29)

\[ \Delta \epsilon_{n+1}^{m} = C_{n+1} : \Delta s_{n+1} , \]  
(30)

\[ \Delta a_{n+1}^{(i)} = H_{n+1}^{(i)} : \Delta \epsilon_{n+1}^{m} , \]  
(31)

where \( P_{n+1} \), \( C_{n+1} \) and \( H_{n+1}^{(i)} \) are fourth rank tensors and represent tangent moduli. Using Eqs. (16), (28), (29) and (31), Eq. (30) is rewritten as
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\[ d\Delta \varepsilon_{n+1} = L_{n+1} : dy_{n+1}, \]  

(32)

where

\[ L_{n+1} = C_{n+1} : (I + H_{n+1} : P_{n+1}), \]  

(33)

\[ H_{n+1} = \sum_{j=1}^{M} H_{n+1}^{(j)}. \]  

(34)

Consequently, Eq. (26) becomes

\[ A_{n+1} : dy_{n+1} = s_{n+1}^{*} - a_{n+1} - y_{n+1} - I_d : D_{n+1}^{e} : \Delta \varepsilon_{n+1}, \]  

(35)

where

\[ A_{n+1} = I + I_d : D_{n+1}^{e} : (P_{n+1} + L_{n+1}) + H_{n+1} : P_{n+1}. \]  

(36)

By solving Eq. (35) for \( dy_{n+1} \), \( y_{n+1} \) is updated to \( y_{n+1} + dy_{n+1} \). Therefore, by performing the iterations shown in Fig. 1, \( y_{n+1}, \Delta \varepsilon_{n+1}, a_{n+1}, s_{n+1} \) and \( \Delta \varepsilon_{n+1} \) can be determined. Then, using Eq. (21), \( \sigma_{n+1} \) can be obtained. The iterations mentioned above need to be done to obtain \( \sigma_{n+1} \) at each integration point in each finite element. The iterations will thus be referred to as \textit{local iterations} from now on.

4.2 Initial values for local iterations

An initial value \( y_{n+1}^{(0)} \) of \( y_{n+1} \) is necessary to perform the local iterations shown in Fig. 1. If \( y_{n+1}^{(0)} \) is not appropriately chosen, the local iterations may not converge well. The choice of \( y_{n+1}^{(0)} \) is made in this study as follows.

If \( \Delta \varepsilon_{n+1}^{e} \) is completely elastic, the elastic tentative stress \( \sigma_{n+1}^{e} \) should be chosen as \( \sigma_{n+1} \). Consequently, using Eqs. (16) – (19), \( y_{n+1}^{(0)} \) is written as

\[ y_{n+1}^{(0)} = s_{n+1}^{*} - \sum_{i=1}^{M} \left[ h^{(i)}(T_{n+1})/h^{(i)}(T_{n}) \right] a_{n}^{(i)}. \]  

(37)

If \( \Delta \varepsilon_{n+1}^{m} \) occupies \( \Delta \varepsilon_{n+1}^{u} \), Eq. (14) provides

\[ \frac{\sqrt{2/3}}{I_d : \Delta \varepsilon_{n+1}^{e}} = g^{u}(T_{n+1}, T_{n}) \Delta T_{n+1}. \]  

(38)

Hence, by solving the above equation for \( y_{n+1}^{(0)} \), \( y_{n+1}^{(0)} \) is expressed as

\[ y_{n+1}^{(0)} = \frac{\sqrt{2/3}}{I_d : \Delta \varepsilon_{n+1}^{e}} I_d : \Delta \varepsilon_{n+1}^{m}. \]  

(39)

If \( \Delta \varepsilon_{n+1}^{m} \) is equal to \( \Delta \varepsilon_{n+1}^{u} \), Eq. (15) gives

\[ \varepsilon_{n+1}^{m}, \varepsilon_{n+1}^{u}, \varepsilon_{n+1}^{s}, a_{n+1}^{(i)}, \sigma_{n+1}, T_{n}; \text{known} \]

\[ \Delta \varepsilon_{n+1}^{m}, \Delta \varepsilon_{n+1}^{u}, \Delta T_{n+1}; \text{given} \]

\[ y_{n+1}^{(0)} \quad (37), (39), (41) \]

\[ \Delta \varepsilon_{n+1}^{u} \quad (14), a_{n+1}^{(i)} \quad (18) \]

\[ s_{n+1} \quad (16), \Delta \varepsilon_{n+1}^{m} \quad (15) \]

\[ P_{n+1} \quad (42), C_{n+1} \quad (43), H_{n+1}^{(j)} \quad (50) \]

\[ y_{n+1} + dy_{n+1} \quad (35) \]

\[ \sigma_{n+1} \quad (21), \varepsilon_{n+1}^{m}, \varepsilon_{n+1}^{u}, a_{n+1}^{(i)} \]

Fig. 1 Computational algorithm for implicit stress integration
\[
\sqrt{\frac{2}{3}} \left\| \mathbf{I}_d : \Delta \mathbf{e}^{(m)}_{\text{ref}} \right\| = g^m(\mathbf{\tau}^{(0)}_{x1}, T_{x1}) \Delta \mathbf{a}_{x1}.
\]
Thus, by solving the above equation for \( \mathbf{\tau}^{(0)}_{x1} \), and by using Eqs. (16) – (19), \( \mathbf{y}^{(0)}_{x1} \) is represented as
\[
\mathbf{y}^{(0)}_{x1} = \frac{2}{3} \mathbf{\tau}^{(0)}_{x1} \mathbf{I}_d : \Delta \mathbf{e}^{(m)}_{\text{ref}} - \frac{\mu}{12} \sum_{i=1}^{n_h} \mathbf{h}^{(i)}(T_{x1}) \mathbf{a}^{(i)}_{x1}.
\]
Then, the three candidates of \( \mathbf{y}^{(0)}_{x1} \) given by Eqs. (37), (39) and (41) are compared to choose \( \mathbf{y}^{(0)}_{x1} \) that has the smallest norm.

### 4.3 Derivation of \( \mathbf{P}_{x1} \), \( \mathbf{C}_{x1} \) and \( \mathbf{H}^{(i)}_{x1} \)

The fourth rank tensors \( \mathbf{P}_{x1} \) and \( \mathbf{C}_{x1} \) in Eqs. (29) and (30) are shown to have the following forms by differentiating Eqs. (14) and (15), respectively:

\[
\mathbf{P}_{x1} = \frac{3}{2} \left[ \frac{\partial g_{x1}}{\partial n_{x1}} \mathbf{n}_{x1} \otimes \mathbf{n}_{x1} + \frac{\partial g_{x1}}{\partial \mathbf{\nu}_{x1}} (\mathbf{I} - \mathbf{n}_{x1} \otimes \mathbf{n}_{x1}) \right] \Delta \mathbf{a}_{x1},
\]

\[
\mathbf{C}_{x1} = \frac{3}{2} \left[ \frac{\partial g_{x1}}{\partial \mathbf{\nu}_{x1}} \mathbf{\nu}_{x1} \otimes \mathbf{\nu}_{x1} + \frac{\partial g_{x1}}{\partial \mathbf{\nu}_{x1}} (\mathbf{I} - \mathbf{\nu}_{x1} \otimes \mathbf{\nu}_{x1}) \right] \Delta \mathbf{a}_{x1},
\]

where \( \mathbf{n}_{x1} \) and \( \mathbf{\nu}_{x1} \) are defined as

\[
\mathbf{n}_{x1} = \frac{1}{\sqrt{2}} \mathbf{y}_{x1} / \mathbf{y}_{x1}, \quad \mathbf{\nu}_{x1} = \frac{1}{\sqrt{2}} \mathbf{s}_{x1} / \mathbf{s}_{x1}.
\]

To derive the fourth rank tensor \( \mathbf{H}^{(i)}_{x1} \) appearing in Eq. (31), Eq. (18) is differentiated to have

\[
d\mathbf{a}^{(i)}_{x1} = \frac{d\mathbf{\theta}^{(i)}_{x1}}{d\mathbf{\nu}^{(i)}_{x1}} \frac{d\mathbf{\nu}^{(i)}_{x1}}{d\mathbf{\sigma}^{(i)}_{x1}} \frac{d\mathbf{\sigma}^{(i)}_{x1}}{d\mathbf{\sigma}^{(i)}_{x1}} \mathbf{a}^{(i)}_{x1} + \frac{2}{3} \mathbf{\theta}^{(i)}_{x1} \mathbf{h}^{(i)}(T_{x1}) \mathbf{d}\Delta \mathbf{e}^{(m)}_{\text{ref}}.
\]

Here, by noting Eq. (19), \( \frac{d\mathbf{\theta}^{(i)}_{x1}}{d\mathbf{\nu}^{(i)}_{x1}} \) is expressed as

\[
\frac{d\mathbf{\theta}^{(i)}_{x1}}{d\mathbf{\nu}^{(i)}_{x1}} = \frac{\partial \mathbf{\theta}^{(i)}_{x1}}{\partial \mathbf{\sigma}^{(i)}_{x1}} \frac{d\mathbf{\sigma}^{(i)}_{x1}}{d\mathbf{\sigma}^{(i)}_{x1}} + \frac{\partial \mathbf{\theta}^{(i)}_{x1}}{\partial \mathbf{\sigma}^{(i)}_{x1}} \frac{d\mathbf{\sigma}^{(i)}_{x1}}{d\mathbf{\sigma}^{(i)}_{x1}},
\]

and by employing Eqs. (9) and (20), \( d\mathbf{\sigma}^{(i)}_{x1} \) and \( d\mathbf{\sigma}^{(i)}_{x1} \) are represented as

\[
d\mathbf{\sigma}^{(i)}_{x1} = (3/2) \mathbf{a}^{(i)}_{x1} : d\mathbf{\sigma}^{(i)}_{x1},
\]

\[
d\mathbf{\sigma}^{(i)}_{x1} = (2/3) \Delta \mathbf{e}^{(m)}_{\text{ref}} : d\mathbf{\sigma}^{(i)}_{x1} / \Delta \mathbf{\sigma}^{(i)}_{x1},
\]

Moreover, Eqs. (14) and (44) provide \( \mathbf{\sigma}^{(i)}_{x1} = \sqrt{\frac{2}{3}} \Delta \mathbf{e}^{(m)}_{\text{ref}} / \Delta \mathbf{\sigma}^{(i)}_{x1} \). Then, by obtaining \( d\mathbf{\theta}^{(i)}_{x1} \) from Eqs. (46) – (49), and by substituting the resulting \( d\mathbf{\theta}^{(i)}_{x1} \) into Eq. (46), it is shown that

\[
\mathbf{H}^{(i)}_{x1} = \frac{2}{3} \left( \mathbf{\theta}^{(i)}_{x1} \mathbf{h}^{(i)}(T_{x1}) \mathbf{I}
\right.
\]

\[
+ \left( \mathbf{\theta}^{(i)}_{x1} - \mathbf{\kappa}^{(i)}_{x1} \mathbf{\sigma}^{(i)}_{x1} \right)^{-1} \left( \frac{\kappa^{(i)}_{x1} \mathbf{\theta}^{(i)}_{x1} \mathbf{h}^{(i)}(T_{x1})}{\mathbf{\sigma}^{(i)}_{x1}} \mathbf{a}^{(i)}_{x1} \otimes \mathbf{a}^{(i)}_{x1} + \frac{\sqrt{2}}{3} \lambda^{(i)}_{x1} \mathbf{a}^{(i)}_{x1} \otimes \mathbf{\nu}^{(i)}_{x1} \right),
\]

where \( \mathbf{\kappa}^{(i)}_{x1} = \frac{\partial \mathbf{\sigma}^{(i)}_{x1}}{\partial \mathbf{\sigma}^{(i)}_{x1}} \) and \( \lambda^{(i)}_{x1} = \frac{\partial \mathbf{\sigma}^{(i)}_{x1}}{\partial \mathbf{\sigma}^{(i)}_{x1}} \).

### 5. Consistent Tangent Modulus

The non-linear equilibrium equations of nodal forces in finite element methods are usually solved using iterations based on the Newton-Raphson method. These iterations, which will be referred to as global iterations henceforth, have quadratic convergences, if we employ the so-called consistent tangent modulus \( \frac{\partial \mathbf{\sigma}^{(i)}_{x1}}{\partial \mathbf{\sigma}^{(i)}_{x1}} \) (i.e., tangent modulus of discretized constitutive relations). In this section, \( \frac{\partial \mathbf{\sigma}^{(i)}_{x1}}{\partial \mathbf{\sigma}^{(i)}_{x1}} \) is more generally derived for the non-unified model described in Section 2 than in the previous study.
which elastic stiffness $\mathbf{D}^\varepsilon$ was assumed to be isotropic.

To evaluate the variation $\partial \sigma_{\varepsilon_{\text{in}}}$ in $\sigma_{\text{in}}$ due to the variation $d\Delta \varepsilon_{\varepsilon_{\text{in}}}^m$ in $\Delta \varepsilon_{\text{in}}^m$, Eqs. (11) – (13) are differentiated to derive

$$d\sigma_{\varepsilon_{\text{in}}} = \mathbf{D}_{\varepsilon_{\text{in}}}^\varepsilon : (d\Delta \varepsilon_{\varepsilon_{\text{in}}}^m - d\Delta \varepsilon_{\text{in}}^m) .$$

(51)

Taking the deviatoric part of the above equation gives

$$ds_{\varepsilon_{\text{in}}} = I_d : \mathbf{D}_{\varepsilon_{\text{in}}}^\varepsilon : (d\Delta \varepsilon_{\varepsilon_{\text{in}}}^m - d\Delta \varepsilon_{\text{in}}^m) .$$

(52)

Using Eqs. (16) and (29), $d\Delta \varepsilon_{\varepsilon_{\text{in}}}^m$ is expressed as

$$d\Delta \varepsilon_{\varepsilon_{\text{in}}}^m = P_{\varepsilon_{\text{in}}} : (ds_{\varepsilon_{\text{in}}} - da_{\varepsilon_{\text{in}}}) .$$

(53)

Then, substituting Eqs. (17), (31) and (33) into Eq. (53), and using Eq. (30), we derive

$$M_{\varepsilon_{\text{in}}} : (d\Delta \varepsilon_{\varepsilon_{\text{in}}}^m + d\Delta \varepsilon_{\text{in}}^m) = (P_{\varepsilon_{\text{in}}} + M_{\varepsilon_{\text{in}}} : C_{\varepsilon_{\text{in}}}) : ds_{\varepsilon_{\text{in}}} ,$$

(54)

where

$$M_{\varepsilon_{\text{in}}} = I + P_{\varepsilon_{\text{in}}} : H_{\varepsilon_{\text{in}}} .$$

(55)

Now, substituting Eq. (52) into Eq. (54), and solving the resulting equation for $d\Delta \varepsilon_{\varepsilon_{\text{in}}}^m + d\Delta \varepsilon_{\text{in}}^m$, we obtain

$$d\Delta \varepsilon_{\varepsilon_{\text{in}}}^m + d\Delta \varepsilon_{\text{in}}^m = (M_{\varepsilon_{\text{in}}} + U_{\varepsilon_{\text{in}}})^{-1} : U_{\varepsilon_{\text{in}}} : d\Delta \varepsilon_{\text{in}}^m ,$$

(56)

where

$$U_{\varepsilon_{\text{in}}} = (P_{\varepsilon_{\text{in}}} + M_{\varepsilon_{\text{in}}} : C_{\varepsilon_{\text{in}}}) : I_d : \mathbf{D}_{\varepsilon_{\text{in}}}^\varepsilon .$$

(57)

It is then shown that Eqs. (51) and (56) provide

$$\frac{\partial \sigma_{\varepsilon_{\text{in}}}}{\partial \Delta \varepsilon_{\varepsilon_{\text{in}}}^m} = \mathbf{D}_{\varepsilon_{\text{in}}}^\varepsilon : \left[ I - (M_{\varepsilon_{\text{in}}} + U_{\varepsilon_{\text{in}}})^{-1} : U_{\varepsilon_{\text{in}}^-1} \right] .$$

(58)

6. Application to a Unified Type of Constitutive Model

The constitutive model described in Section 2 had a decomposition of $\dot{\varepsilon}_{\text{in}}$ into $\dot{\varepsilon}^\text{in}$ and $\dot{\varepsilon}^\text{in}$ to represent the transient and steady states in inelastic deformation. According to the idea of unified constitutive modeling, this decomposition is not assumed, but thermal recovery is explicitly taken into account instead, so that Eqs. (3), (4) and (8) are replaced by

$$\dot{\varepsilon}^\text{in} = \frac{3}{2} \hat{\varepsilon}^\text{in} (\bar{y}, T) \frac{\hat{y}}{\bar{y}} ,$$

(59)

$$\dot{a}^{(i)} = \frac{2}{3} h^{(i)}(T) \dot{\varepsilon}^\text{in} - \zeta^{(i)}(\bar{\sigma}^{(i)}, T) a^{(i)} \hat{p}^\text{in} - \eta^{(i)}(\bar{\sigma}^{(i)}, T) a^{(i)} + \frac{d \ln h^{(i)}(T)}{dT} a^{(i)} \frac{T}{T} ,$$

(60)

where $\hat{\varepsilon}^\text{in}$ denotes an inelastic strain-rate function, and $\hat{p}^\text{in}$ is defined as $\hat{p}^\text{in} = \sqrt{\frac{2}{3} \| \dot{\varepsilon}^\text{in} \|^2}$. The third term on the right hand side in Eq. (60) stands for the thermal recovery of $a^{(i)}$.

Equations (59) and (60) have the following discretized forms based on the backward Euler method:

$$\Delta \varepsilon_{\varepsilon_{\text{in}}}^m = \frac{3}{2} \hat{\varepsilon}^\text{in} (T_{\varepsilon_{\text{in}}^-1}, T_{\varepsilon_{\text{in}}}) \Delta t_{\varepsilon_{\text{in}}} \frac{\hat{y}_{\varepsilon_{\text{in}}}}{y_{\varepsilon_{\text{in}}}} ,$$

(61)

$$a_{\varepsilon_{\text{in}}}^{(i)} = \theta^{(i)}_{\varepsilon_{\text{in}}} h^{(i)}(T_{\varepsilon_{\text{in}}}) \left[ \frac{1}{h^{(i)}(T_{\varepsilon_{\text{in}}})} a_{\varepsilon_{\text{in}}}^{(i)} + \frac{2}{3} \Delta \varepsilon_{\varepsilon_{\text{in}}}^m \right] ,$$

(62)

where

$$\theta^{(i)}_{\varepsilon_{\text{in}}} = \left[ 1 + \zeta^{(i)}(\bar{\sigma}^{(i)}, T_{\varepsilon_{\text{in}}}) \Delta p_{\varepsilon_{\text{in}}}^\text{in} + \eta^{(i)}(\bar{\sigma}^{(i)}, T_{\varepsilon_{\text{in}}}) \Delta \sigma_{\varepsilon_{\text{in}}}^\text{in} \right]^{-1} .$$

(63)

It is seen that Eqs. (61) and (62) with the subscript ‘in’ replaced by ‘tr’ have the same forms as Eqs. (14) and (18), which were derived for $\Delta \varepsilon_{\varepsilon_{\text{in}}}^m$ and $a_{\varepsilon_{\text{in}}}^{(i)}$ in Section 3. It is
also seen that Eq. (63) with \( \Delta p_{n,1}^{\text{tr}} \) replaced by \( \Delta p_{n,1}^{\text{ss}} \) can be incorporated into Eq. (50), only where \( \Theta_{n,1}^{(i)} \) effectively appeared in Sections 4 and 5. Therefore, the implicit integration scheme and the consistent tangent modulus described in Sections 4 and 5 are applicable to the unified constitutive model based on Eqs. (61) – (63) if \( \Delta p_{n,1}^{\text{ss}} \) is regarded as \( \Delta p_{n,1}^\text{tr} \) and \( \Delta p_{n,1}^\text{tr} \) is disregarded.

### 7. Example of Finite Element Analysis

Finite element analysis of a gull-wing lead type solder joint was performed to verify the implicit integration scheme and the consistent tangent modulus described in Sections 4 and 5. Both the non-unified and unified constitutive models outlined in Sections 2 and 6 were employed in the analysis, though the non-unified model is mainly taken up here.

#### Table 1 Elastic constants and coefficients of thermal expansion for solder joint analysis

<table>
<thead>
<tr>
<th></th>
<th>( E ) [GPa]</th>
<th>( \nu )</th>
<th>CTE ([10^{-5}/\degree \text{C}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chip</td>
<td>303.4</td>
<td>0.30</td>
<td>5.8</td>
</tr>
<tr>
<td>Lead, Pad</td>
<td>130.4</td>
<td>0.35</td>
<td>16.7</td>
</tr>
<tr>
<td>Solder</td>
<td>40.0</td>
<td>0.30</td>
<td>24.7</td>
</tr>
<tr>
<td>Board</td>
<td>18.3</td>
<td>0.28</td>
<td>15.9</td>
</tr>
</tbody>
</table>

#### Table 2 Material parameters of Sn-3.5Ag-0.5Cu for non-unified model; stress [MPa], strain [mm/mm], time [s], and temperature [K]

<table>
<thead>
<tr>
<th></th>
<th>( E ) [GPa]</th>
<th>( \nu )</th>
<th>CTE ([10^{-5}/\degree \text{C}])</th>
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<td>Board</td>
<td>18.3</td>
<td>0.28</td>
<td>15.9</td>
</tr>
</tbody>
</table>

#### Transient function:

\[
g^{\nu} = 1.0 \times 10^{-3}(\bar{y}/\sigma_0^{\nu})^{\nu} \\
\sigma_0^{\nu} = \exp(651.9T^{-1} + 0.7134) \\
m^{\nu} = 8.0 \times 10^{-7}T^2 - 8.57 \times 10^{-2}T + 27.94
\]

#### Steady state function:

\[
g^{\nu} = 1.0 \times 10^{-3}(\bar{y}/\sigma_0^{\nu})^{\nu} \\
\sigma_0^{\nu} = \exp(918.4T^{-1} + 0.4358) \\
m^{\nu} = 1.074 \times 10^{-3}T^2 - 0.1151T + 35.76
\]

#### Kinematic hardening parameters:

\[
\zeta = 100 \\
h = \exp(582.3T^{-1} + 5.395)
\]

Fig. 2 Finite element model of solder joint structure
7.1 Model solder joint structure and material properties

The model solder joint structure considered had the same shape and size as that in the study by Pidaparti and Song (25). The model structure had two symmetric planes and was divided into finite elements using 20-node quadratic brick, reduced integration elements, as shown in Fig. 2. The total numbers of elements and nodes were 10,025 and 69,892, respectively.

It was supposed that, except for the solder, the model structure consisted of elastic materials, for which the same material properties as in the study by Pidaparti and Song (25) were assumed. Table 1 shows the values of Young’s modulus $E$ and Poisson’s ratio $\nu$ as well as the coefficients of thermal expansion (CTEs) employed in the present analysis. For the solder, the material functions and parameters of the non-unified model were determined to fit the uniaxial tension and creep curves of a Sn-3.5Ag-0.5Cu solder alloy (26) at 0°C to 125°C as follows. First, the material function $g^u$ was determined using the steady-state relations in the uniaxial tension and creep tests (Fig. 3). Second, by supposing $M = 1$ in Eq. (7), and by assuming a temperature-independent constant for $\zeta$ in Eq. (8), the material function $g^u$ and the material parameters $h$ and $\zeta$ in Eq. (8) were chosen to reproduce the transient states in the uniaxial tension and creep tests. The solid lines in Figs. 4 and 5 depict examples of the tension and creep curves simulated. Table 2 shows the material functions and parameters obtained for the temperatures ranging from 0°C to 125°C.

The model structure was uniformly subjected to two cycles of temperature variations between 0°C to 125°C. The temperature variations consisted of four branches in one cycle as illustrated in Fig. 6. Each branch was divided into either 6 or 30 increments.
Version 6.5 of a finite element program ABAQUS was employed to perform the finite element analysis mentioned above. The implicit integration algorithm illustrated in Fig. 1 and the consistent tangent modulus expressed as Eq. (58) were coded for a user subroutine UMAT in ABAQUS. For the (local) iterations in the implicit integration algorithm, the following condition was used as a convergence criterion:

$$\left\|dy^{n+1}\right\|/\left\|y^{n+1}\right\| < 10^{-8}.$$  \hspace{1cm} (64)

For the (global) iterations in ABAQUS to solve non-linear nodal force equilibrium equations, the default condition of ABAQUS was used as a rule.

### 7.2 Results for the non-unified model

The most significant inelastic strain occurred at the integration point indicated by an arrow in Fig. 7. This point will be referred to as point P_{max} hereafter. The vertical stress versus mechanical strain relation at point P_{max} is shown in Fig. 8. It is seen from the figure that almost the same stress versus strain relation was obtained in the two cases of 6 and 30 increments per step. This means that the finite element computation was successful even when the number of increments per step was as small as 6.

Figure 9 illustrates the number of local iterations required for the convergence condition (64) to be satisfied at point P_{max}. As seen from the figure, the number of local iterations at convergence was 4.5 on average and 6 at most even in the case of 6 increments per step. It is emphasized that quadratic convergence was realized in the local iterations (Fig. 10). The quadratic convergence is due to the fourth rank tensors $P_{n+1}$, $C_{n+1}$ and
\( H^{(1)} \) introduced in Eqs. (29) – (31), which are tangent moduli and hence allow the local iterations in Fig. 1 to be based on the Newton-Raphson method.

The default convergence condition was used for the global iterations, as aforementioned. The number of global iterations required for the convergence was 1.9 on average and 3 at most (Fig. 11). This result verifies the consistent tangent modulus expressed as Eq. (58) and is comparable with that in the previous study(6).

7.3 Results for the unified model

The unified model described in Section 6 was also employed to perform the finite element analysis. The material functions and parameters of the unified model were determined using the same experimental results as for the non-unified model (Table 3), though a substantial process of trial and error was necessary as previously reported(2),(3). The unified model then gave almost the same tension and creep curves as the non-unified model (Figs. 4 and 5).

Using the unified model for the analysis, most significant inelastic strain was found to occur at the same integration point, \( P_{\text{max}} \), as for the non-unified model. The vertical stress versus mechanical strain relation at point \( P_{\text{max}} \) for the unified model is shown in Fig. 12. By comparing Figs. 8 and 12, it is seen that the non-unified and unified models provided almost the same relations as each other. It is also noted that the two models gave almost the same numbers of local and global iterations at convergence (Table 4).

Table 3  Material parameters of Sn-3.5Ag-0.5Cu for unified model; stress [MPa], strain [mm/mm], time [s], and temperature [K]

<table>
<thead>
<tr>
<th>Inelastic function:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g^\alpha = 1.0 \times 10^{-4}(\bar{y}/\sigma_{\text{ref}})^\alpha )</td>
</tr>
<tr>
<td>( \sigma_{\text{ref}} = \exp(648.2T^{-1} + 0.7328) )</td>
</tr>
<tr>
<td>( m^\alpha = 1.475 \times 10^{-4}T^2 - 0.1337T + 36.35 )</td>
</tr>
<tr>
<td>Kinematic hardening parameters:</td>
</tr>
<tr>
<td>( \zeta = 110 )</td>
</tr>
<tr>
<td>( h = \exp(1.071 \times 10^7T^{-1} + 3.734) )</td>
</tr>
<tr>
<td>( \eta = \exp(2.252 \times 10^7T^{-1} + 4.047)(\zeta \eta_{\text{ref}}/h)^\eta )</td>
</tr>
<tr>
<td>( \mu = 3.4 \times 10^{-4}T^2 + 0.3017T + 75.24 )</td>
</tr>
</tbody>
</table>

8. Conclusions

In this study, first, an implicit integration scheme was developed for a non-unified high-temperature inelastic constitutive model using so-called linearization. A consistent tangent modulus was also generally derived for the non-unified model. The implicit integration scheme and the consistent tangent modulus were shown to be applicable to a unified constitutive model. Then, by coding the implicit integration scheme and the consistent tangent modulus for a user subroutine in a finite element program, a gull-wing lead type of model solder joint structure subjected to cyclic thermal loading was analyzed. A lead-free solder Sn-3.5Ag-0.5Cu was supposed for the model structure. It was thus demonstrated that quadratic convergence occurs in the iterations introduced in the implicit integration scheme, and that the non-unified and unified constitutive models give nearly the same results.

References


(24) Suzuki, A. and Hamanaka, J., A creep constitutive equation of Inconel 617 in multiaxial
