SIMP-Based Dynamic Topological Optimum depending on Maximum Eigenfrequency for Reinforcement of Concrete Beams*

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Abstract

Computer aided topology optimization design is a relatively new but rapidly expanding area of structural mechanics. Topology optimization design is used in an increasing rate by for example building and bridge engineering as well as the car, machine and aerospace industries. The reason for this is that it often achieves great savings and design improvements by solving the basic engineering problem of distributing a limited amount of material in a design space, in which a certain objective function has to be optimized. In static problems a common objective is to minimize the compliance such as strain energy and in dynamic problems the first natural eigenfrequency is often maximized in order to evaluate the stiffest structures. The goal of this study is to get a wide use for structural designs in the topology optimization. Therefore with respect to SIMP dynamic topology optimization problems are treated in comparisons with static problems in this research. For the topology optimization problems implemented algorithms can be the optimality criteria method and the method of moving asymptotes (MMA). Numerical applications topologically maximizing the first natural eigenfrequency or minimizing strain energy of plates verify the generality and wide use of topology optimization with respect to structural designs.

Key words: Computer Aided, SIMP, Topological Optimum, Dynamic, Eigenfrequency

1. Introduction

In general structural optimization researches in areas of structural mechanics require perfect understanding of both all mechanical behaviors and computationally effective numerical or mathematical solutions. Therefore the structural optimization theory compiles all the available data and techniques into engineering and optimizers constitute the best engineering expert groups. However contrary to realization for electronics, magnetics, mechanics, ships and airplane fields, practical applications for especially civil structures and architectural buildings in structural optimal designs have not been completed until now\(^1\). It is because the research direction of engineering optimizers was theoretical completion rather than practical application. In substantial most of them disregarded against potential ability and practical worth for structural optimization. Because of these reasons, the optimization design has been neglected from structural designers in spite of the advanced computational age. However the problem will be able to substantially be resolved by engineers carrying out a practical application of the principle problem.

In general optimal design for structural systems is to find values of the variables that
Structural optimization can be conveniently distinguished into size, shape, and topology optimization as shown in Fig. 1 depending on what is varied in the optimization process. In size optimization, the design variables are the segment sizes (e.g., truss cross-sectional areas). Shape optimization considers the change in material location and is numerically implemented using boundary movement (e.g., truss node locations). Topology optimization is where the connectivity is changed (e.g., additional nodes for trusses).

In this study, topology optimization is treated for the optimal design, and this study tried to escape the typical research of solution algorithms. In order to achieve the goal, not academic static but practical dynamic problems are investigated. For a dynamic structural analysis, free vibration problems of linear elastic structures are considered. A density distribution method based on SIMP (Solid Isotropic Microstructure with Penalization) material is used for continuous topology optimization method. This approach substitutes the integers 0 and 1 for continuous values, forcing them to become discrete values using penalization, which provides computational simplicity for the optimal design. In order for topology optimization designs of dynamic problems to be also available for personal computers, a user-based MATLAB program is developed with easy understanding. The dynamic topology optimization method applies consistent or lumped masses. The solutions are compared to optimal solutions of typical static problems.

The outline of this study is as follows: With respect to SIMP formulation, the static and dynamic material topology optimization problems are described in Section 2. In Section 3, a numerical algorithm of the static and dynamic topology optimization methods based on FEM and gradient-oriented optimization method is presented. The comparison between optimal topology extraction results for static and dynamic problems is studied in several numerical applications in Section 4. Section 5 presents conclusions of this study.

2. Material Topology Optimization Problems

In general, the field of material topology optimization conveniently deals with voids(0)-solids(1) material distribution and depends on linear elastostatic problems. The schematic of the two-phase material topology optimization of a solid structure with the specified field and boundary conditions is shown in Fig. 2.
stiffest or least compliant structure using a given fixed load, the possible support conditions, and the restrictions on the volume of material used.

\[ f = \frac{1}{2} \int_{\Omega_x} \delta \epsilon^T C \delta \epsilon \, d\Omega \]  

where according to discretization, the continuous material tensor \( C \) is dependent on the density-stiffness relationship of the typical SIMP approach. The discontinuous Heaviside function is regularized for a smoothed and continuous form near the material boundaries. The function can be included in a strain energy formulation since the original Heaviside function determines the solid and void regions in a design domain.

The inequality optimization constraint is \( 0 \leq \Phi \leq 1 \), which ensures that the density stays within reasonable bounds. Equality constraints are a linear elastostatic equilibrium, which clearly presents the state equation, and an equation controlling the volume of the used material under the volume fraction \( V_{\text{ref}} \) as follows, respectively.

\[ \int_{\Omega_x} \delta \epsilon^T C \delta \epsilon \, d\Omega = \int_{\Omega_x} \delta \mathbf{u}^T B \, d\Omega + \int_{\Gamma_t} \delta \mathbf{u}^T n \, d\Gamma_t \]  

\[ \int_{\Omega_x} d\Omega - V_{\text{ref}} = 0 \]  

2.2 Optimization Formulations for Dynamic Problems

Governing equation for free vibration systems considered in this study can be written as

\[ M \ddot{u} + K u = F = 0 \]  

By using Laplace transformation Eq.(4) can be rewritten as

\[ M \mathbf{U}(s)^2 + K \mathbf{U}(s) = 0 \]  

By substituting \( \omega_i^2 \) for \( s \) into Eq. (5), the final eigenvalue problem is defined as

\[ [K - \omega_i^2 M] \mathbf{u}_i = 0 \]  

where \( K \) and \( M \) are the global stiffness and mass matrix, respectively. \( \omega_i \) is the \( i \)-th eigenfrequency and \( \mathbf{u}_i \) denotes the corresponding eigenvector depending on \( \omega_i \). In order to numerically solve Eq. (6), \( K \) and \( M \) have to be the symmetric and positive definite\(^6\) stiffness and mass matrices of the finite element-based, generalized structural eigenvalue.

Eigenvalue optimization designs are profitable for mechanical structural systems subjected to dynamic loading conditions like earthquakes and wind loads. The dynamic behaviors of structural systems can be estimated by eigenfrequency which describes
structural stiffness. In general maximizing first-order eigenfrequency can be an objective for dynamic topology optimization problems since stiffness of structures also increases when eigenfrequency increases. Problems of topology optimization for maximizing natural eigenfrequencies of vibrating elastostatic structures have been considered in the studies(7),(8),(9).

Assuming that damping can be neglected, such a dynamic optimization design problem can be formulated as follows.

\[
\min \Phi : \omega_k^2(\Phi) = \frac{u_i^T K u_i}{u_i^T M u_i} \quad (7.1)
\]

subject to:

\[
\frac{V(\Phi)}{V_0} \leq g \quad (7.2)
\]

\[
\left[ K - \omega^2_t M \right] \mu_i = 0 \quad (7.3)
\]

\[
0 < \Phi_{\min} \leq \Phi \leq \Phi_{\max} \quad (7.4)
\]

where these discrete formulations for the dynamic problem are equal to continuous formulations, i.e. Eqs. (1)–(3) for the static problem except for objective and governing equation.

### 2.3 Interpolation Scheme by Using SIMP Material

After discretization of the continuous design domain, the material density \( \Phi_i^h \) is constantly assigned to each finite element and is defined by applying a penalty contour to the design variable field, i.e. as in the so-called "power law or SIMP approach". According to the SIMP approach, the material density distribution affects element stiffness and the element stiffness-density relationship may be expressed in terms related to Young’s modulus \( E_i^h \), is associated with the updated element density \( \Phi_i^h \) and it is defined as

\[
E_i^h(\Phi_i^h) = E_o \left( \frac{\Phi_i^h}{\Phi_o} \right)^k, \quad k \geq 1, \ 0 \leq \Phi_i^h \leq 1, \ i = 1 \cdot \cdot \cdot N_e \quad (8)
\]

where \( E_o \) and \( \Phi_o \) denote nominal values of Young’s modulus and material density of elements, respectively, and \( N_e \) is the number of elements.

According to the penalized Young’s module, element stiffness matrix of four-node square elements with eight-DOF used in this study is written as

\[
K_e = \int_\Omega B^T C B \ d\Omega
\]

\[
E_i^h(\Phi_i^h) = \frac{E_i^h(\Phi_i^h)}{1 - \nu^2} \left[ \begin{array}{ccccccccc}
k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 \\
\cdot & k_1 & k_8 & k_7 & k_6 & k_5 & k_4 & k_3 \\
\cdot & \cdot & k_1 & k_6 & k_5 & k_4 & k_3 & k_2 \\
\cdot & \cdot & \cdot & k_1 & k_8 & k_3 & k_2 & k_5 \\
\cdot & \cdot & \cdot & \cdot & k_1 & k_2 & k_3 & k_4 \\
\cdot & \cdot & \cdot & \cdot & \cdot & k_1 & k_8 & k_7 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_1 & k_6 \\
\end{array} \right]_{\text{sym.}}
\]

\[
k_1 = \frac{1}{2} - \frac{\nu}{12} \quad k_5 = -\frac{1}{4} + \frac{\nu}{12}
\]

\[
k_2 = \frac{1}{8} + \frac{\nu}{8} \quad k_6 = -\frac{1}{8} - \frac{\nu}{8}
\]

\[
k_3 = -\frac{1}{4} - \frac{\nu}{12} \quad k_7 = \frac{\nu}{6}
\]

\[
k_4 = -\frac{1}{8} + \frac{3\nu}{8} \quad k_8 = \frac{1}{8} - \frac{3\nu}{8}
\]
Please note that the stiffness formulation is used for both static and dynamic problems in this study. For example, an isotropic material model with a plane stress (such as a wall structure) is used here without loss of generality, so that

$$
\mathbf{C}_i^b = \frac{\mathbf{E}_i^b(\Phi_i^b)}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1}{2} - \nu
\end{bmatrix}
$$

(10)

where \( \mathbf{C}_i^b \) is a material tensor of each finite element \( i \) and includes the updated term of Young’s modulus \( \mathbf{E}_i^b \) which has been defined by the updated element density \( \Phi_i^b \). \( \nu \) is Poisson’s ratio.

According to dynamic topology optimization problems using SIMP material, mass matrix also includes the same penalty formulation such as the stiffness matrix. Therefore it can be written as

$$
\mathbf{M}_c = (\Phi_c)^k \mathbf{M}_0
$$

(11)

For the mass matrix, a lumped mass matrix \( \mathbf{M}_L \), a consistent mass matrix \( \mathbf{M}_C \) or a combination of those two can be used. The lumped and consistent mass matrices are written as respectively in case discretization of eight-node square elements with 8 DOFs.

$$
\mathbf{M}_C^b = \int \phi \mathbf{N}^T \mathbf{N} \, dV

\begin{bmatrix}
4 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\
-4 & 0 & 2 & 0 & 1 & 0 & -2 & 0 \\
-4 & 0 & 2 & 0 & 1 & 0 & -2 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 4 & 0 & 2 & 0 & 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
$$

(12)

where \( \Phi \) and \( A \) denote the material density and area of elements, respectively and \( \mathbf{I} \) is the 8×8 unit matrix.

2.4 Sensitivity Analyses for Static and Dynamic Problems

In general, the sensitivity of the optimization problems such as objective functions or constraints is mainly calculated by analytical methods due to small error. The analytical variational approach is used here since it is numerically more efficient than the discrete method for certain optimization problems. With respect design variables (for instance, material element densities), the total differential form(10),(11) of the objective function is the combination of parts of an explicit partial derivative and an implicit partial derivative as follows.

$$
\nabla_s f = \nabla_s^{\text{ex}} f + \nabla_s^{\Phi} f \nabla_s \mathbf{u}
$$

(14)

According to static topology optimization problem, under the assumptions that external forces \( \mathbf{b}, \mathbf{t} \), the differential matrix \( \mathbf{L} \) and a Jacobi matrix \( \mathbf{J} \) are independent of the design variables, the total partial derivative is written as a simple continuous formulation as

$$
\nabla_s f = \frac{1}{2} \int_{\Omega_s} \varepsilon^T \nabla_s \mathbf{C}(\Phi) : \varepsilon \, d\Omega_x
$$

(15)

According to dynamic topology optimization, the total derivative is written as a simple discrete formulation as follows.
3. Numerical Algorithm for Material Topology Optimization Design

In general, topology optimization processes are composed of a structural analysis, a sensitivity analysis and an optimization method. For structural analyses, a finite element method is used. A variational approach with adjoint method is implemented for sensitivity analyses. MMA\(^{(12)}\) of gradient-based optimizations is used since it can reduce computational costs in case many design variables are used like in this study. The goal of MMA is to linearly approximate the objective function and constraints. The general numerical algorithm for the present design is sketched in Fig. 3. The developed MATLAB code for dynamic topology optimization design is based on MATLAB code\(^{(13)}\) of Sigmund for static designs.

4. Numerical Examples and Discussion

4.1 Static and Dynamic topology optimization designs of Clamped Beams

Structures for the numerical examples are 2D clamped beams as shown in Fig. 4 (a). Figure 4 (a) shows the structure fixed in the left and right sides.
A 4m×2m design domain is discretized using 40×20 square finite elements with four nodes. The material parameters are assumed to be Young’s modulus of concrete $E = 1.0\ \text{GPa}$ and Poisson’s ratio $\nu = 0.3$. For static optimal problem, an applied load of $P = 1\ \text{kN}$ is concentrated at the bottom center of the structures. For both static and dynamic problems the penalty parameter is $k = 3.0$ for the SIMP approach. Objective function is minimal strain energy $(\text{kN·m})$ for static optimal design and maximal fundamental first-order eigenfrequency for dynamic optimal design. The volumes of 30%, 40%, 50%, 60%, and 70% in Fig. 4 (a) are fixed during the entire optimization procedures. A consistent mass, a lumped mass, and a combination of those two are used for eigenvalue analyses into dynamic topology optimization design.

Figure 5 shows the optimal topologies with density distribution contours designed by mass types to dynamic eigenfrequency-based topology optimization. As can be seen, the maximum of first-order eigenfrequency decreases when the volume constraint increases. In addition, when the ratio of the consistent mass increases, the maximal first-order eigenfrequency also increases. It can be also seen that under the same volumes, optimal shapes and topologies are equal regardless of mass types.

Figure 6 shows the optimal topologies with 0.5 density isoline contours designed for static minimal strain energy-based and dynamic eigenfrequency-based topology optimization methods. The mass type for dynamic optimal design is an equal combination between a consistent and a lumped mass and all optimal results are converged to about iterations of 50. As can be seen, the results of the dynamic optimal design are not equal to those of the conventional static optimal design. When the volume constraints are changed, optimal topologies for dynamic design are instable since the connectivity of quasi-members is not equal. However for static design there is no change of connectivity.
4.2 Static and Dynamic topology optimization designs of Cantilever Beams

Structures for the numerical examples are 2D cantilever beams as shown in Fig. 4 (b). Figure 4 (b) shows the structure fixed in the left side and free in the right side. A 3m×2m design domain is coarsely discretized using 30×20 square finite elements with four nodes. Optimization input data are equal to Section 4.1 and the mass type equally combined a consistent with a lumped mass was selected for the dynamic topology design. For the static topology design a concentrated load P=1kN is vertically applied to the center of the free support in the cantilever.

Figure 7 shows the optimal topologies with 0.5 density isoline contours designed for static minimal strain energy-based and dynamic eigenfrequency-based topology optimization methods like Fig. 6. It can be seen that the dynamic results in Fig. 7 are the same as the left half of dynamic topology results seen in Fig. 6.
5. Conclusions

This study presents a numerical comparison of optimal solutions that associates both static and dynamic material topology optimizations. Clearly, the coupling of dynamic as well as static model designs using SIMP topology optimization has proven to be a really promising practical topology optimization design.

Substantially, the originality and significance of this study are related to deal with easy
understanding for users and practical application for topology optimization design. For the first purpose in this study computational MATLAB-based dynamic topology optimization program was developed to extend Sigmund’s static topology optimization MATLAB codes. For the second purpose, this study presents conceptually that both dynamic and static optimal solutions should be considered for practical optimum design, especially, for example architectural and civil structures, i.e. safe space for human-being’s life. The computational strategy and design method extracting both dynamic and static results would be treated in future works. In addition, the so-called performance-based optimal design method would be able to be more practical by dealing with three-dimensional and nonlinear problems in future works.

References