The Dynamics of a Beam-Mass System due to the Interaction between the Initial Curvature of the Beam and the Existence of Parametric Resonance under Primary Resonance*

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Abstract
The objective of this paper is to present an analytical methodology to evaluate the dynamic phenomena produced by the interaction between the initial curvature of a beam and the occurrence of parametric resonance under the condition when the excitation created by the motion of the attached mass is near the fundamental mode of vibration. In the analysis, a mass traveling with constant speed is assumed. The method of multiple time scales is employed to obtain the weak form of the occurrence of parametric resonance of a beam-mass system with an initially curved beam under primary resonance. Result of present study indicates that under certain conditions, the interaction between the geometric imperfection of beam and the parametric resonance plays an important role to the dynamics of the system.

Key words: Geometric Imperfection of Beam, Parametric Resonance, Primary Resonance

1. Introduction

Vibrations of flexible structures with attached moving masses have been the subject of many interests (1)-(9). Generally due to the effect of inertia, the beam is subject to axial and transverse forces. The mathematical model of the problem then reduces to a multi-degree-of-freedom dynamical system with time dependent coefficients.

Chang and Liu (1) considered various effects, such as longitudinal deflections, inertia, and nonlinearities of the beam and the variation of moving masses, to the response due to the motion of moving loads. The result of their study shows that the largest amplitude of response occurs in the linear model.

Kononov and Borst (2) analyzed the occurrence of instability of the response of four different flexible structures under elastic foundation due to the motion of a riding mass moving with constant velocity. Their result indicated that negative damping might occur when the mass velocity exceeded the smallest phase velocity of the waves in the system. This could cause the solutions became unstable.

Wang (3) employed the methods of Newtonian and the multiple time scales to study the growth of small amplitude vibrations into large motion regime of a beam-mass system due to the occurrence of two-component parametric resonance. In a later study (4), he investigated the transient dynamics of multiple accelerating/decelerating masses traveling on an initially curved beam. The result indicates that the initial curvature of a beam can result significant effects to the dynamics of a beam-mass system even if the geometric imperfection of the beam is tiny. The approaches used in Wang (3),(4) are also the basis of

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this study.

Mallik et al. (5) studied the steady state response of an infinite beam that is supported on an elastic foundation due to the motion of a riding mass. They concluded that without considering the damping effect, the steady state solution does not exist if the speed of mass reaches a critical condition.

Nikkhoo et al. (6) considered the dynamics and its control of a beam-mass system. They found that under certain speed range, the effect produced by higher vibrational modes is not negligible.

Although, the dynamics of a beam-mass system due to the happening of parametric resonance has been studied, however, the phenomena generated by the existence of geometric imperfection of the beam to the dynamics caused by the occurrence of parametric resonance under primary resonance has not been studied yet. Hence unlike other papers, in which a perfect straight beam is assumed, in this paper a moving mass traveling on a beam with initial imperfection is considered. The dynamics due to the interaction between the initial curvature of the beam and the occurrence of parametric resonance when the excitation is near the fundamental mode of vibration are investigated.

Result of present study indicates that under certain conditions, the influence produced by the interaction between the initial curvature of a beam and the existence of parametric resonance to the response of a beam-mass system is significant.

2. Model formulation

In this study, a mass traveling on a finite, simple supported, initially curved inextensible beam is considered. The beam rests on a uniform elastic foundation and is of length $\ell$ and initial variation $v_0(s)$. Here $v_0(s)$, $v_0(s) = v_0 \sin(\pi s / \ell)$, is the initial deviation of the beam measured from straight axis with $v_0$ being the amplitude of initial deviation. The static state of the beam is obtained by assuming that the gravity of the beam and the foundation preload are in the state of equilibrium.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{system_configuration.png}
\caption{System configuration and force equilibrium diagram}
\end{figure}

From Fig. 1 and following the procedures outlined in (3,4), the equation governing the motion of the system in the $j$ (transverse) direction and the equation of motion of the mass, in dimensionless form, are \(^{(3,4)}\)

\begin{align}
\ddot{v} + \dddot{v} + k \dddot{v} + M \{ (\dddot{v} + \ddot{v}_0) (\dot{\xi})^2 + 2 \dddot{v} \dot{\xi} + (\dddot{v} + \ddot{v}_0) \dot{\xi} + \dddot{v} - [ f (\dddot{v} + \ddot{v}_0) + g ] \delta(\eta - \xi) \}
&= 0, \quad 0 < \eta < 1, \quad \tau > 0, \\
\dddot{\xi} - [ \mu (\dddot{v} + \ddot{v}_0) (\dot{\xi})^2 - 2 \mu \dddot{v} \dot{\xi} ] = \dddot{f} - \dddot{g}[ \mu - (\dddot{v} + \dddot{v}_0) ] + \mu \dddot{\xi}, \quad \eta = \xi, \quad \tau > 0,
\end{align}

where a superposed prime and a dot denote the $\eta$ and $\tau$ differentiation and $\dddot{v}_0 = \dddot{v}_0 / \ell = (\dddot{v}_0 / \ell) \sin \pi \eta = \dddot{v}_0 \sin \pi \eta$. Other dimensionless quantities in Eqs. (1) and (2) are defined as

\begin{align}
\tau &= \sqrt{\frac{EI}{m \ell^4}}, \quad M = \frac{M}{m \ell}, \quad \dot{f} = \frac{m f}{EI}, \quad \ddot{g} = \frac{g}{EI}, \quad \dot{k} = \frac{k f}{EI}, \quad \dddot{\xi} = \frac{\dddot{\xi}}{\ell}, \quad \dddot{v} = \frac{\dddot{v}}{\ell}, \quad \dddot{v}_0 = \dddot{v}_0, \quad \eta = \frac{s}{\ell}, \quad \xi = \frac{\xi}{\ell}, \quad \dddot{v} = \frac{\dddot{v}}{\ell}, \quad \dddot{v}_0 = \dddot{v}_0, \quad (3)
\end{align}
where \( \tau, \hat{M}, \hat{g}, \hat{k}, \hat{\eta}, \hat{\xi} (\xi = \xi(\tau)) \), and \( \hat{v} (\hat{v} = \hat{v}(\eta, \tau)) \) are, respectively, the dimensionless quantities of time, the mass of the moving mass, the gravity, the foundation stiffness, the arc length of the beam at time \( \tau \), the position of the mass along the arc of the beam at time \( \tau \), and the transverse displacement of the beam from the undeformed state at time \( \tau \). \( \bar{M} \) and \( \mu \) represent the coefficient of friction between the mass and the beam and the foundation stiffness per unit length, respectively. \( E \) is the Young’s modulus and \( I \) is the area moment of inertia of the beam.

Based on the assumption that whenever the mass being propelled by a thrust along the path, the thrust on the mass will be along the tangent to the vibrating path, \( \hat{f} \) denotes the normalized thrust applied on the mass and is a prescribed function of time. For example, \( \hat{f} \) may be a positive constant to increase the speed or a negative constant to decrease the speed of the mass. It is mentioned here that Eqs. (1) and (2) with the inextensibility constraint account for \( \hat{v}(\eta, \tau) \) and \( \xi \) when \( \hat{M}, \mu, \hat{g}, \) and \( \hat{\xi}_0 \) are specified.

In the following, the attached mass is assumed to travel with constant speed on the beam. Therefore, substitution of Eq. (2) into Eq. (1) and neglect of nonlinear terms yields

\[
\ddot{v} + \hat{c}'' + \hat{k}v = \hat{M}[(1 + \mu(\hat{v} + \hat{\xi}_0))(\hat{\xi}^2 + 2\hat{\xi}' + \hat{\xi}'' + \hat{\xi}')]\delta(\eta - \xi), \quad 0 < \eta < 1, \tau > 0,
\]

Examination of the dynamics governed by Eq. (4) is the main purpose in this study.

Representing \( \hat{v} \) as a continuous function and letting

\[
\hat{v} = \sum_{n=1}^{\infty} A_n(\tau) \sin n\pi\eta, \quad 0 < \eta < 1, \tau > 0,
\]

hence, the boundary condition of simple supported beam is satisfied. The approximate solution of the system can be obtained by employing the Galerkin’s method. Using Galerkin’s procedure for minimizing error, one multiplies Eq. (4) by \( \sin j\pi\eta \) and integrates Eq. (4) with respect to \( \eta \) from zero to 1. Thus, the dynamic equation of the beam-mass system is obtained and is given as

\[
\ddot{A}_n(\tau) + \omega_n^2 A_n(\tau) = 2\hat{M} \left[ (\hat{\xi}_n S_n(\xi) + (\mu \hat{g} R_n) + (\xi_n^2 \tilde{S}_n)\tilde{v}_n) + \mu \hat{g} \sum_{n=1}^{\infty} R_{\mu n}(\xi) A_n(\tau) \right] - \left[ \sum_{n=1}^{\infty} (\hat{\xi}_n^2 S_{\mu n}(\xi) A_n(\tau) + 2\hat{g} R_{\mu n}(\xi) A_n(\tau) - \tilde{S}_{\mu n}(\xi) \tilde{A}_n(\tau)) \right], \quad 0 < \eta < 1, \tau > 0
\]

where \( \omega_n = ((j\pi)^4 + \hat{k}) \), \( \tilde{v}_n = \tilde{v}_n / \ell \), \( R_{\mu n}(\xi) = (n\pi) \cos n\pi\xi \sin j\pi\xi \), \( \tilde{S}_n(\xi) = \sin n\pi\xi \), \( \tilde{S}_{\mu n}(\xi) = \sin n\pi\xi \sin j\pi\xi \) and \( S_{\mu n}(\xi) = (n\pi)^2 \tilde{S}_{\mu n}(\xi) \).

To analyze the system governed by Eq. (6), one allows the response of the system to be small but finite. Thus, the method of multiple time scales can be used to predict the responses of the system. According to this method \(^7\), it is assumed that the amplitude \( A_n(\tau) \) has the expansion \( A_n(\tau; \varepsilon) = \varepsilon A_{1n}(\tau, \tau_0, \tau_1, \tau_2, \ldots) + \varepsilon^2 A_{2n}(\tau, \tau_0, \tau_1, \tau_2, \ldots) + \ldots \) with \( \tau_n = \varepsilon^n \tau, n = 0, 1, 2, \ldots \), where \( \varepsilon \) is a measure of the amplitude of the response and is small compared to unity. In addition, \( d/d\tau = \partial / \partial \tau_0 + \varepsilon \partial / \partial \tau_1 + \varepsilon^2 \partial / \partial \tau_2 + \ldots = D_0 + \varepsilon D_1 + \ldots \) and \( d^2/d\tau^2 = D_0^2 + 2\varepsilon D_0 D_1 + \ldots \)

For the purpose of studying the parametric resonance of the non-autonomous differential equations, one sets \( \hat{M} = \varepsilon \hat{M}, \hat{\xi} = \varepsilon \hat{\xi} \) and \( \hat{\xi}_0 = \varepsilon \hat{\xi}_0 \). After manipulating these equations and then equating coefficients of equal power of \( \varepsilon \), one obtains to order one and two:
\[ \varepsilon^1 : D_0^2 A_{ij} + \omega_j^2 A_{ij} = 2\bar{M}g \hat{S}_{ij} = 2\bar{M}g \sin(j\pi \xi \tau_0) \]  

(7)

\[ \varepsilon^2 : D_0^2 A_{ij} + \omega_j^2 A_{ij} = -2D_0 D_i A_{ij} + 2\bar{M}(V_{\xi}^2)^2 S_{ij} v_0^2 + 2\mu \bar{M}g R_{ji} v_0 + 2\mu \bar{M}g \sum_{n=1}^{\infty} R_{jn} A_{in} \]

\[ + 2\bar{M} \sum_{n=1}^{\infty} [(V_{\xi}^2)^2 A_{in} - R_{jn} D_i A_{in} + \hat{S}_{jn} D_i A_{in}] \]  

(8)

It is shown in Eq. (7) that unbounded oscillation occurs when the frequency \( \omega_j \) is near \( (j\pi \xi) \). Hence, in the following, the conditions considered are related to the cases when the natural frequency \( \omega_j \) is away from \( (j\pi \xi) \).

From Eq. (7), it is seen that the amplitude, \( A_{ij} \), is harmonic in \( \tau_0 \), and its solution can be represented as

\[ A_{ij} = a_j \cos(\omega_j \tau_0 + \phi_j) + \frac{2\bar{M}g}{(\omega_j^2 - (j\pi \xi)^2)} \sin(j\pi \xi \tau_0) = a_j \cos \beta_j + 2\bar{M}g \Lambda_j \sin(j\pi \xi \tau_0) \]  

(9)

where \( a_j = a_j(\tau_1, \tau_2, \ldots) \) is the amplitude of response; \( \phi_j = \phi_j(\tau_1, \tau_2, \ldots) \) is the phase angle and \( \Lambda_j = \frac{1}{(\omega_j^2 - (j\pi \xi)^2)} \). Here, for convenience, rewriting Eq. (9) as

\[ A_{ij} = H_j(\tau_1, \tau_2, \ldots) \exp(i\omega_j \tau_0) + \bar{H}_j(\tau_1, \tau_2, \ldots) \exp(-i\omega_j \tau_0) \]

\[ -i\bar{\bar{\Lambda}}_j \exp(ij\pi \xi \tau_0) - \exp(-ij\pi \xi \tau_0)), j = 1, 2, 3, \ldots \]

where \( i = \sqrt{-1} \) and \( \bar{H}_j \) is the complex conjugate of \( H_j \). Here, \( H_j = (1/2)a_j \exp(i\phi_j), j = 1, 2, \ldots \), with \( \phi_j \) being the phase of the \( j \)th mode.

To seek the solution of \( A_{ij} \) defined by Eq. (8), one substitutes Eq. (9) into Eq. (8) and obtains:

\[ D_0^2 A_{ij} + \omega_j^2 A_{ij} = 2\omega_j [(D_i a_j) \sin \beta_j + a_j (D_i \phi_j) \cos \beta_j] + \pi\bar{M}(V_{\xi}^2)^2 v_0 \sin(j\pi \xi \tau_0) \]

\[ - \frac{1}{2} \pi\bar{M}(V_{\xi}^2)^2 v_0 [(\sin(j+2)\pi \xi \tau_0 + \sin(j-2)\pi \xi \tau_0)] \]

\[ + \frac{1}{2} \pi\mu \bar{M}g V_{\xi}^2 v_0 \cos(j-2)\pi \xi \tau_0 - \cos(j+2)\pi \xi \tau_0 \]

\[ + \frac{1}{2} \mu \bar{M}g \sum_{n=1}^{\infty} [(n\pi) a_n [(\sin \beta_{uj}^+ - \sin \beta_{uj}^-) - (\sin \beta_{uj}^+ - \sin \beta_{uj}^-)] \]

\[ + \frac{1}{2} \frac{\mu \bar{M}g}{M^2} \sum_{n=1}^{\infty} [(n\pi) \omega_n a_n [(\cos \beta_{uj}^+ + \cos \beta_{uj}^-) - (\cos \beta_{uj}^+ + \cos \beta_{uj}^-)] \]

\[ + \frac{1}{2} \frac{\bar{M}g}{M^2} \sum_{n=1}^{\infty} [(n\pi) \omega_n a_n [(\cos \beta_{uj}^+ - \cos \beta_{uj}^-) - (\cos \beta_{uj}^+ - \cos \beta_{uj}^-)] \]

\[ + \frac{1}{2} \frac{\bar{M}g}{M^2} \sum_{n=1}^{\infty} [(n\pi)^2 \omega_n a_n [(\cos \beta_{uj}^+ + \cos \beta_{uj}^-) - (\cos \beta_{uj}^+ + \cos \beta_{uj}^-)] \]

\[ + \mu (\bar{M}^2 g) \sum_{n=1}^{\infty} [(n\pi \Lambda) a_n [(\cos(2n-j)\pi \xi \tau_0 - \cos(2n+j)\pi \xi \tau_0] \]

\[ + 4\bar{M}^2 g \sum_{n=1}^{\infty} (n\pi \xi)^2 \Lambda_n [(\sin(2n-j)\pi \xi \tau_0 - \sin(2n+j)\pi \xi \tau_0] \]

where

\[ \beta_{uj}^+ = \beta_u + (n+j)\pi \xi \tau_0 = (\omega_u + (n+j)\pi \xi) \tau_0 + \phi_u, \]

\[ \beta_{uj}^- = \beta_u - (n+j)\pi \xi \tau_0 = (\omega_u - (n+j)\pi \xi) \tau_0 + \phi_u. \]
In order to study the effects produced by the initial imperfection of the beam to the dynamics due to the occurrence of parametric resonance under primary resonance, the dynamics of a system with finite degrees of freedom as defined by Eq. (11) is investigated. It is known that a multi-degree-of-freedom dynamic system with parametric excitation will experience parametric resonance when internal frequencies and the excitation frequency are commensurable or nearly commensurable. For a dynamic system defined by Eq. (11), parametric resonance under primary resonance may exist when 

\[ \omega_j \approx \Omega = (j+2)\pi V_j, \]

and \( \omega_n = \omega_j, \ n, j = 1, 2, \ldots \), where \( \omega_j \) is the dimensionless internal frequency of the \( j \)-th mode of vibration and \( \Omega \) is the frequency of excitation.

To express the commensurable relations of \( \omega_j \approx \Omega = (j+2)\pi V_j \), the detuning parameter \( \sigma_j \) are introduced:

\[ \Omega = (j+2)\pi V_j = \omega_j + \omega \sigma_j \]

Therefore,

\[ \Omega \tau_0 = (\omega_j \tau_0 + \phi_j) + (\sigma_j \tau_1 - \phi_j) = \beta_j + \delta_j \]

where \( \delta_j = \delta_j(\tau_1, \tau_2, ...) \) is a new phase angle. The differences of the arguments of the cosine and sine functions of unequal and equal arguments one has

\[ \beta_j^- = (\omega_j - (j - 1)\pi V_j)\tau_0 + \phi_j = \beta_j \]

\[ \beta_j^+ = (\omega_j + (j + 1)\pi V_j)\tau_0 + \phi_j = \beta_j \]

Returning to Eq. (11) the solvability conditions are the vanishing of the secular terms. These are respectively

\[ 4\omega_j \hat{D}_H \hat{H} + \hat{i}((\hat{M} / (2\omega_j))\alpha_j \hat{H}) = \{\hat{M}[(1/2)\pi V_j]^3 \nu_0 + \hat{M}gF_{km}] + \hat{i} \mu \hat{M}g[(1/2)\pi V_j^3 \nu_0 + \hat{M}gF_{km}] \exp(i \sigma_j \tau_1) \]

where

\[ \alpha_j = [\omega_j + (j\pi V_j)^2], \]

\[ F_{km} = 4(\pi V_j)^2[k^{L} \Lambda_k \delta_{(2k+j+1,0)} - m^{L} \Lambda_m \delta_{(2m-j+1,0)}], \ k = 1 \ and \ m = j + 1 \]

\[ \delta_{km} = \text{the Kronecker delta symbol} \]

The main purpose of Eq. (16) is to study the response of motion.

To determine the solutions and correspondingly the local stability of the parametric resonance one lets \(^{7}\)

\[ H_j = (1/2)(x_j \hat{i}z_j) \exp(i \theta_j \tau_1). \]

Here \( x_j \) and \( z_j \) are real and \( \theta_j = d\phi_j / dt \).

For the resonant case, one substitutes Eq. (17) into the resonant equations defined by Eq. (16) and separate the real and imaginary parts. The result yields

\[ x_j' + [\theta_j + (\hat{M} / (2\omega_j))\alpha_j]x_j = \hat{M} / (2\omega_j)[(1/2)\pi V_j]^{3} \nu_0' + \hat{M}gF_{km} \]

\[ z_j' - (\theta_j + (\hat{M} / (2\omega_j))\alpha_j)x_j = -\mu \hat{M}g / (2\omega_j)[(1/2)\pi V_j^3 \nu_0' + \hat{M}gF_{km}] \]
where \( (\cdot)' = d/d\tau_i \). The solutions of the parametric resonance with the excitation frequency being near the resonant frequency \( \omega_j \) then can be obtained by setting \( x_j' = z_j' = 0 \) and \( x_j^2 + z_j^2 = a_j^2 \). This gives

\[
a_j = -\frac{\omega_j}{2\omega_j}\sqrt{\left(\frac{1}{2}\pi(V_j)^2\nu_0 + \frac{\alpha_j}{2}\right)^2 + \left(\frac{\pi\nu_0 + \alpha_j}{2}\right)^2}
\]

In Eq. (20), \( \nu_0^* \) is the geometric imperfection of the beam; \( F_{lm} \) and \( \overline{F}_{lm} \) are the terms produced by the existence of parametric resonance. As seen from Eq. (20) that the interaction between the initial curvature of the beam and the existence of parametric resonance plays an important role to the response of the system. It is generally true that the amplitude of response increases with the increase of geometric imperfection of the beam and the values of \( F_{lm} \) and \( \overline{F}_{lm} \). In addition, it is also observed that under the condition when \( \omega_j = \Omega = (j + 2)\pi V_j \), and \( \omega_{n,j} = \omega_j \; \text{n}, \; j = 1, 2, ..., \) the amplitude of response of the system is always bounded; this is due to the fact that the denominator of Eq. (20) is always nonzero.

In the following, the local stability of a fixed point with respect to a small perturbation for the resonant case is determined by the eigenvalues \( \lambda \) which are given by the zero of the determinant of the perturbation equations. For this, a small perturbation is superimposed on \( x_j \) and \( z_j \) and one has

\[
x_j = x_j^0 + \tilde{x}_j \; \text{and} \; z_j = z_j^0 + \tilde{z}_j
\]

Here \( x_j^0, z_j^0, \tilde{x}_j, \) and \( \tilde{z}_j \) are the fixed points and the disturbances respectively. The determinant then can be obtained by substituting Eq. (21) into Eqs. (18) and (19). The result yields

\[
\begin{vmatrix}
\lambda & \theta_j + (\overline{M}/(2\omega_j))\alpha_j^* \\
-[\theta_j + (\overline{M}/(2\omega_j))\alpha_j^*] & \lambda
\end{vmatrix} = 0
\]

where \( \theta_j = D_j\phi_j = \sigma_j \). Thus, the characteristic equation of (22) has the form

\[
\lambda^2 + [\theta_j + (\overline{M}/(2\omega_j))\alpha_j^*]^2 = 0
\]

The root of Eq. (23) is

\[
\lambda = \pm i[\theta_j + (\overline{M}/(2\omega_j))\alpha_j^*] = \pm i[\sigma_j + (\overline{M}/(2\omega_j))\alpha_j^*]
\]

It is known that the stability of response determines the existence of steady state reaction. As shown in Eq. (24), the root of the characteristic equation is the same result as the denominator of Eq. (20). Hence, the response of the system is bounded.

3. Numerical results and discussions

Numerical results refer to an assumed model wherein an initially curved finite beam rests on a uniform elastic foundation and carries a mass traveling on it with constant speed. With the purpose of checking the accuracy of the model, Eq. (6), agreement with previous work was considered. As shown in Fig. 2, the accuracy of the model is verified by numerically integrating the system, Eq. (6), by the Runge-Kutta method with sixth order
accuracy and then compares its results with the results reported by Ting et al. (8). It is well known that the latter is in agreement with experimental observation. In this specific case the parameters used are the same as those used in (8); they are \( \dot{\xi} = f = 0 \), \( \dot{M} = 0.5 \), \( \dot{M} = \frac{M \dot{g}}{48} \), and \( \dot{\xi} = 0.5\pi \) and \( \dot{\xi} = 0.75\pi \). The existence and validity of perturbation solutions are proofed by numerically integrating the modulation equations, Eqs. (18) and (19), by the Runge-Kutta method.

**Fig. 2.** The comparison between the result obtained by this study and the result reported by Ting et al. (8). This figure shows the trajectory of mass vs the position of mass along the beam with the same parameters as used in (8) for \( \dot{\xi} = 0.5\pi \) and \( 0.75\pi \).

Without loss of generality and considering the commensurable relations among frequencies and the probability of occurrence in nature, the case when \( \omega_j \approx \Omega = (j + 2)\pi \xi \), and \( \omega_n = \omega_0 \) the fundamental mode of vibration, \( j = 1 \), was selected to determine the basic characteristics of the occurrence of parametric resonance under primary resonance and the effects produced by the geometric imperfection of the beam to the response of the system. It is recalled that \( \omega_j \) is the dimensionless natural frequency and is defined by \( \omega_j = \sqrt{(j\pi)^2 + k} \) where \( k \) is the foundation stiffness; \( V_0 \) is the dimensionless velocity of the attached mass moving along the beam; \( v_0 \) is the geometric imperfection of the beam; \( F_{km} \) and \( \bar{F}_{km} \), \( k = j \) and \( m = j + 1 \), are the terms produced by the existence of parametric resonance and hence \( F_{km} = \bar{F}_{km} = 4(\pi \xi V_0)^2(\Lambda_1 - 4\Lambda_2) = 4(\pi \xi V_0)^2\{[1/(\omega_0^2 - (\pi \xi V_0)^2)] - [4/(\omega_0^2 - (2\pi \xi V_0)^2)]\} \) and \( \bar{F}_{km} = \bar{F}_{km} = \pi(\Lambda_1 - 2\Lambda_2) = \pi\{1/(\omega_0^2 - (\pi \xi V_0)^2)] - [1/(\omega_0^2 - (2\pi \xi V_0)^2)]\} \).

It is mentioned here that in the following five figures, Figs. 3 to 7, there are three small plots in each figure. The bottom plot (subplot (a)) is obtained from Eq. (20) by setting \( j = 1 \); it shows the influence produced by the variation of crucial parameters to the amplitude of response \( a_i \). Subplots (b) and (c) illustrate the time history of the amplitude \( a_i \) for specified parameters and are acquired by numerically integrating the modulation equations, Eqs. (18) and (19).

In Fig. 3, subplot 3(a) shows the variation of geometric imperfection of the beam \( a_0^* \) to the amplitude of response \( a_i \) for \( \mu = \dot{k} = 0 \), \( \dot{M} = 0.25 \) and three different values of detuning parameters \( \epsilon \sigma_i \), \( \epsilon \sigma_i = -0.2 \), 0.0, and 0.2. Subplot 3(b) illustrates the long time behavior of the amplitude \( a_i \) for \( \epsilon \sigma_i = -0.2 \) and \( \epsilon \sigma_i = 0.0 \) (initially straight beam). Subplot 3(c) presents the same manner as does subplot 3(b), except \( \epsilon \sigma_i = 0.1 \). The result of this figure clearly indicates that as the detuning parameter \( \epsilon \sigma_i \),
increases from negative value, say $\varepsilon_1 = -0.2$, to positive value, say $\varepsilon_1 = 0.2$, the decrease of the magnitude of amplitude $a_1$ occurs. In other words, larger value of the amplitude $a_1$ is found if the resonant internal frequency is lower than the excitation frequency.

The effects of elastic foundation and dimensionless mass $\varepsilon M$ to the deflection of the beam are given in Figs 4 and 5, respectively. In subplot 4(a), the amplitude $a_1$ is plotted as a function of the geometric imperfection of the beam $\varepsilon \nu_0$ for $\mu = \hat{k} = 0$, $\hat{g} = 1$, and $\varepsilon M = 0.25$ with three different values of detuning parameters $\varepsilon \sigma_1$, $\varepsilon \sigma_1 = -0.2$, 0.0, and 0.2 and correspondingly the time history of the amplitude $a_1$.

From Figs. 3-5, it is found that the variation of the amplitude $a_1$ w.r.t. the initial imperfection of the beam $\varepsilon \nu_0$ is linear. This phenomenon, the growth rate of the magnitude of response due to the increase of geometric imperfection of a beam being nearly constant, can also be clearly found by Eq. (20) if the friction $\mu$ is set to be zero. In addition, from these figures, it can be concluded that the interaction between the geometric imperfection of beam and the parametric resonance plays an important role to the dynamics of the system. It shows that under certain conditions, for example: the excitation frequency is close to the fundamental internal frequency, the amplitude of response grows significantly even if the initial curvature of the beam is tiny.
Fig. 4. The amplitude $a_i$ vs the geometric imperfection of the beam $\epsilon v_0$ for $\sigma_i = 0.1$, $\dot{g} = 1$, $\mu = 0$, $\bar{M} = 0.25$ and three different values of foundation stiffness $\hat{k}$, $\hat{k} = 0.0$, 50 and 100, and correspondingly the time history of the amplitude $a_i$ for $\epsilon v_0 = 0.05$ with $\hat{k} = 0.0$ (4(b)) and $\hat{k} = 50$ (4(c)).

Fig. 5. The amplitude $a_i$ vs the geometric imperfection of the beam $\epsilon v_0$ for $\sigma_i = 0.0$, $\dot{g} = 1$, $\mu = \hat{k} = 0$ and three different values of the normalized mass $\bar{M}$, $\bar{M} = 0.1$, 0.25 and 0.5, and correspondingly the time history of the amplitude $a_i$ for $\epsilon v_0 = 0.0$ with $\bar{M} = 0.1$ (5(b)) and $\bar{M} = 0.1$ (5(c)).
The effects produced by the variation of other parameters, such as the coefficient of friction \( \mu \) and the normalized gravity \( \hat{g} \), to the magnitude of response are shown in Figs. 6 and 7. In subplot 6(a), the amplitude \( a_1 \) develops as a function of the coefficient of friction \( \mu \) for \( \varepsilon \sigma = 0.0 \), \( \hat{k} = 0 \), \( \hat{g} = 1 \), \( \varepsilon M = 0.25 \) and three different values of geometric imperfection of the beam \( \alpha v^*_0 \), \( \alpha v^*_0 = 0.0 \), \( 0.05 \), and \( 0.1 \). Subplots 7(b) and 7(c) illustrate, respectively, the long time behavior of the amplitude \( a_1 \) for \( \alpha v^*_0 = 0.1 \) with \( \mu = 0.0 \) and \( \alpha v^*_0 = 0.1 \) with \( \mu = 0.2 \). It is interesting to find that the influence produced by the variation of the coefficient of friction to the amplitude of response is not significant. This can be evident from Eq. (20). It is recalled that the amplitude \( a_1 \) is given by

\[
a_1 = \frac{\varepsilon M \sqrt{\frac{1}{4} \pi (V_\xi)^2 v_0' + \bar{M} F_{12}^2} + \left[ \mu \hat{g} \left( \frac{1}{2} \pi v_0' + \bar{M} \hat{g} F_{12} \right) \right]^2}{2 \alpha_1 \epsilon \sigma \varepsilon + \frac{\epsilon M}{2 \alpha_1} \left( \alpha_1^2 + (\pi \xi)^2 \right)}
\]

where \( F_{12} = 4(\pi \xi)^2 (\Lambda_1 - 4 \Lambda_2) \) and \( \bar{F}_{12} = \pi (\Lambda_1 - 2 \Lambda_2) \) with \( \Lambda_1 = 1/\lambda \left( \alpha_1^2 - (\pi \xi)^2 \right) \), \( \Lambda_2 = 4/\lambda \left( \lambda_2^2 - (2 \pi \xi)^2 \right) \) and \( V_\xi = \left[ (\pi)^4 + \kappa \varepsilon \sigma \right] / (3 \pi) \). Hence, by comparison, it is found that the values produced by the term \( \mu \hat{g} \left( \frac{1}{2} \pi v_0' + \bar{M} \hat{g} F_{12} \right) \) are much smaller than those generated by the term \( (1/2) \pi (V_\xi)^2 v_0' + \bar{M} \hat{g} F_{12} \) and thus \( a_1 \approx \varepsilon M (1/2) \pi (V_\xi)^2 v_0' + \bar{M} \hat{g} F_{12} \) and correspondingly the time history of the amplitude \( a_1 \) for \( \alpha v^*_0 = 0.1 \) with \( \mu = 0.0 \) (7(b)) and \( \mu = 0.2 \) (7(c)).

The change of the amplitude of response due to the variation of normalized gravity \( \hat{g} \) is shown in Fig. 7 for \( \mu = \hat{k} = 0 \), \( \varepsilon \sigma = 0.0 \) and three different values of geometric imperfection of the beam \( \alpha v^*_0 \), \( \alpha v^*_0 = 0.0 \), \( 0.05 \), and \( 0.1 \). In subplots 7(a) and 7(b), the
dimensionless mass $\tilde{M}$ is set to be 0.5 and 0.25, respectively. The result indicates that geometric imperfection of the beam $\varepsilon_0^*$, the normalized gravity $\tilde{g}$ and the mass $\tilde{M}$ play crucial roles to the response of the system. As shown, the amplitude $a_1$ is tiny if an initially straight beam and small values of $\tilde{g}$ and $\tilde{M}$ are considered, for example: $\varepsilon_0^* = 0.0$, $\tilde{g} = 0.1$ and $\tilde{M} = 0.25$. However, the magnitude of response increases tremendously if either the effect created by the geometric imperfection of the beam or large values of $\tilde{g}$ and $\tilde{M}$ is taken into account.

![Graph](image.png)

Fig. 7. The amplitude $a_1$ vs the normalized gravity $\tilde{g}$ for $\mu = \tilde{k} = 0$, $\varepsilon_0^* = 0.0$ and three different values of geometric imperfection of the beam $\varepsilon_0^* = 0.0, 0.05, \text{and } 0.1$. The normalized mass $\tilde{M}$ used in subplots (a) and (b) is set to be 0.5 and 0.25, respectively.

4. Conclusions

In the modeling, based on the Euler-Bernoulli beam theory and Newtonian, the coupled equations of motion of a beam-mass system with an initially curved beam are derived. By employing Galerkin’s procedure, the coupled partial differential equations of motion are reduced to an initial value problem with finite dimensions. In the analysis, a mass traveling with constant speed is assumed and the method of multiple time scales is applied to investigate the phenomena generated by the existence of geometric imperfection of the beam to the dynamics of the system due to the occurrence of parametric resonance under the condition when the excitation is near the fundamental mode of vibration.

Results of the study shows that the excitation created by the geometric imperfection of the beam can cause significant effects to the response of the system if parametric resonance of the system occurs simultaneously. It is generally true that the interaction between the geometric imperfection of beam and the parametric resonance plays an important role to the dynamics of the system. It shows that under certain conditions, for example: the excitation frequency is close to the fundamental internal frequency, the amplitude of response grows significantly even if the initial curvature of the beam is tiny. The result also indicates that the magnification in response increases with the normalized mass of the attached mass as well as the normalized gravity, but decreases with the increase...
of foundation stiffness. However, the influence caused by the variation of coefficient of friction to the amplitude of response is negligible.

References


