Solution to Hertzian Contact Problem between Wheel and Rail for Small Radius of Curvature*

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Abstract
There have been several attempts to solve Hertz equation for curved surface contact problem. One application of Hertzian contact problem is to determine the contact properties between wheel and rail. It is important to understand the contact between wheel and rail so that excessive wear can be avoided and train accidents can be minimized. In this work, an attempt has been made to solve Hertzian contact for small radius of curvature using a simple newly invented formula. A finite element modeling was also performed to observe the variations of maximum contact stress in rail with respect to the change rail radius of curvature. The invented formula was to modify the formula of Fischer et.al especially for the major and minor axis so that a more accurate result was obtained in determining the contact dimensions and maximum contact pressure.

Key words: Hertz Contact, Wheel-rail, Elliptical Integral, Contact Dimensions, Contact Pressure, Finite Element Analysis, Approximation Method, Radius of Curvatures

1. Introduction
Accuracy of contact dimension is very important in the modeling of wear due to contact between wheel and rail. A practical and accurate formulation was so help to speed up the process of calculation and accurate the model. Fischer et.al(1) developed an approach to solve Hertz equation for contact dimension, contact pressure, and penetration. The approach is simple and can be manually performed. At wheel-rail contact, Hertz equation can only be applied at the contact between wheel and rail head, however, not at the contact between wheel flange and rail gage corner since the radius of the contact body is smaller than the radius of contact area(2). Besides Hertz equation uses the assumption that major curvature of the contacting bodies remains the same during contact. This paper is aiming for solving the Hertzian contact problems between wheel and rail with small radius of curvature by using finite element analysis and approximate solution to the elliptic integral.

Several researchers have attempted to simplify numerically the solution of the elliptic integral by approaching the ratio of elliptical axis, Brewe et.al(3) determined that approximate equation are used to obtain value for elliptical axis ratio r and elliptic integral ε as below, where a is the major elliptical semi axis and b is the minor elliptical semi axis. A
and B are positive constants relating to the radius of curvature.

\[ r = \frac{a}{b} \approx 1.0339 \left( \frac{B}{A} \right)^{0.636} \quad (1) \]

\[ \varepsilon \approx 1.003 + 0.5968 \left( \frac{A}{B} \right) \quad (2) \]

The approximation shows that the axis-ratio \( r \) is 3.4\% higher and elliptic integral \( \varepsilon \) is 1.7\% higher than the theoretical results. The work was then followed up by Hamrock et.al\(^{(4)}\) with a new approximate equation for elliptical axis ratio and elliptic integral,

\[ r = \frac{a}{b} \approx \left( \frac{B}{A} \right)^{\frac{1}{2}} \quad (3) \]

\[ \varepsilon \approx 1 + \left( \frac{\pi}{2} - 1 \right) \left( \frac{A}{B} \right) \quad (4) \]

This approximation gives correct value for the circular contact. Greenwood\(^{(5)}\) introduced effective radius method. This method provides the best value for the Hertz pressure, particularly for 1\( \leq B/A \leq 5 \). Later on Tanaka\(^{(6)}\) established a new method to calculate elliptical Hertz contact pressure in which the calculation elliptic integral is not necessary. The result shows that the Hertz contact pressure difference is about 0.0005 percent to the theoretical contact pressure. Subsequently, Antonie et.al\(^{(7)}\) obtained an approximate of Hertzian contact model by replacing the elliptical integral with polynomial approximation. The result shows that application of the model to Hertzian theory will cause an error of less than 0.003\% for \( 10^{-8} \leq r \leq 10^{8} \). Fischer et.al\(^{(1)}\) introduced a new method to simplify the elliptical integral to become simpler and the result shows a good correlation with the theoretical result.

Hanson et.al\(^{(8)}\) extended Hertz theory to another application of material and load. They discovered the elastic field expression for the transversely isotropic body with ellipsoidal variation of normal traction and shear traction loading on the surface. Liu et.al\(^{(9)}\) found the expression to calculate Hertzian contact elliptic dimensions and contact pressure for the coated bodies and give accurate predictions of contact characteristics. Beside using analytical-numerical approximation, Hertzian contact problem solution in wheel rail contact can be done using finite element method as Telliskviv et.al\(^{(10)}\) did in 2000, they compared contact pressure calculation using three method i.e. Hertz theory, finite element using Ansys code and program Contact from Kalker. The calculation was done in two cases of wheel rail contact. The first case was performed in gauge corner and the second in rail head. In both cases Hertz and program Contact provided similar results for the maximum contact pressure. For the first case: there was a significant difference of the results among finite element method and Hertz method and program Contact. The results of Hertz and Contact method are three times larger than finite element analysis. It was due to the half space and the elastic-plastic material model assumption. For the second case: there was about 25-30\% difference between finite element solution and Hertz as well as Contact, in this case half space assumption is valid so that the difference is due to employing plastic material model in finite element model. Then, Sladkowski et.al\(^{(11)}\) simulated the attack angle and wheel and rail profile. Angles of attack were 0° and 2° whereas the profiles that were applied on wheel profile standard and new wheel profile were 33 and 30. The development of new wheel profile can reduce contact pressure in flange about 20 until 50\%. Subsequently, Wiest et.al\(^{(12)}\) compared four calculation methods to obtain contact pressure value, contact area and mutual approach, i.e. Hertz theory, Program Contact, the elastic finite element, the elastic-plastic finite element. The analysis was done in rail crossing. Afterward, Parwata
et al (13) observed the effect of the evolution of principle radius of curvature with respect to contact pressure and Von Misses stress. For this case, if the principle radius of curvature is decreased it will increase the contact pressure and Von Misses stress.

Finite element method is implemented to solve the Hertzian contact problem. The modeling of wheel-rail contact is crucial in obtaining the correct and accurate results of the solution, and in this paper special finite element modeling techniques have been applied to reach the best approach in solving the Hertzian contact problem.

2. Approximate Solution

If two bodies were pressed together opposite to another with normal force, an area contact will be treated as a point contact. The shape and size of contact area between two elastic bodies in static condition is given by Hertz. If the material of the wheel and the rail are different then according to Hertz theory the contact area is elliptical and the contact pressure is semi elliptical (14).

The main curvatures are needed for the calculation of surface contact dimension and pressure distribution. The equations for the calculation are:

\[
A + B = \frac{1}{2} \left( \frac{1}{R_{11}} + \frac{1}{R_{12}} + \frac{1}{R_{21}} + \frac{1}{R_{22}} \right)
\]

\[
|B - A| = \frac{1}{2} \left( \left( \frac{1}{R_{11}} - \frac{1}{R_{12}} \right)^2 + \left( \frac{1}{R_{21}} - \frac{1}{R_{22}} \right)^2 + 2 \left( \frac{1}{R_{11}} - \frac{1}{R_{12}} \right) \left( \frac{1}{R_{21}} - \frac{1}{R_{22}} \right) \cos 2\alpha \right)^{1/2}
\]

Where A and B are positive constants, \( R_{11}, R_{12}, R_{21}, \) and \( R_{22} \) are defined as the principal relative radii of curvature each bodies and \( \alpha \) is an inclined angle between axis of principal radii of curvature in a body and axis of principal radii of curvature in other body. The equations of A, B, mutual approach \( \delta \), and maximum contact pressure \( p_0 \) are
\[ A = \frac{M}{\pi E} \quad M = \frac{\pi p_f ab}{2} \int_0^{\pi} \frac{dw}{\left( a z + w \right)^2} \left( b z + w \right) w^{-3/2}, \tag{7} \]

\[ B = \frac{N}{\pi E} \quad N = \frac{\pi p_f ab}{2} \int_0^{\pi} \frac{dw}{\left( a z + w \right)^2} \left( b z + w \right)^3 w^{-1/2}; \tag{8} \]

\[ \delta = \frac{L}{\pi E} \quad L = \frac{\pi p_f ab}{2} \int_0^{\pi} \frac{dw}{\left( a z + w \right)^2} \left( b z + w \right) w^{-3/2}. \tag{9} \]

\[ \rho = \frac{3 P}{2 \pi a b}. \tag{10} \]

With the new method introduced by Tanaka(6) the above elliptic integral can be changed into simpler form as

\[ A = \frac{3 r P}{2 a^3} \left( 1 - \frac{v_1^2}{E_1} + 1 - \frac{v_2^2}{E_2} \right) x^2 \int_0^{\pi} \frac{d\theta}{\left( 1 + \tan^2 \theta \right)^{\frac{3}{2}} \left( \cos^2 \theta + r^2 \sin^2 \theta \right)^{\frac{1}{2}}} \cos \theta \tag{11} \]

\[ B = \frac{3 r P}{2 a^3} \left( 1 - \frac{v_1^2}{E_1} + 1 - \frac{v_2^2}{E_2} \right) x^2 \int_0^{\pi} \frac{d\theta}{\left( 1 + r^2 \tan^2 \theta \right)^{\frac{3}{2}}} \cos \theta \tag{12} \]

\[ \delta = \frac{3 P}{2 b} \left( 1 - \frac{v_1^2}{E_1} + 1 - \frac{v_2^2}{E_2} \right) x^2 \int_0^{\pi} \frac{d\theta}{\left( 1 + r^2 \tan^2 \theta \right)^{\frac{3}{2}}} \cos \theta \tag{13} \]

With the assumption that the same identical material for both of wheel and rail and using simple program in Matlab to solve the integral for range \( 1 \leq r \leq 50 \) then the value of A and B were obtained. By using regression method the relation between \( r \) and \( B/A \) was found as follow

\[ \frac{B}{A} = \alpha r^\beta \quad \text{where} \quad \alpha = 0.82 \quad \text{and} \quad \beta = 1.6665 \tag{14} \]

\[ B = \frac{3 P}{2 \pi} \left( \frac{2 \left( 1 - \frac{v_i^2}{E_i} \right)}{E_i} \right) \frac{1}{a} \pi \epsilon_i r^\epsilon_i \tag{15} \]

where: \( \epsilon_i = 0.9034 \) and \( \epsilon_2 = 2.0311 \), therefore the equations of contact dimension are as follows

\[ a = p^{\gamma_5} \left( 1 - \frac{v_1^2}{E_1} \right) \gamma_5 \left( \frac{3 \epsilon_1}{\pi} \right) \gamma_5 \alpha^{-x_5} \left( \frac{B}{A} \right)^{x_5} B^{-\gamma_5} \tag{16} \]

\[ b = p^{\gamma_5} \left( 1 - \frac{v_2^2}{E_2} \right) \gamma_5 \left( \frac{3 \epsilon_2}{\pi} \right) \gamma_5 \alpha^{-x_5} \left( \frac{B}{A} \right)^{x_5} B^{-\gamma_5} \tag{17} \]

Equation (17) is simplified as

\[ b = 0.9161 \cdot p^{\gamma_5} \left( 1 - \frac{v_2^2}{E_2} \right) \gamma_5 \left( \frac{B}{A} \right)^{0.1938} B^{-\gamma_5} \tag{18} \]

The mutual approach could be determined by solving the integral in equation (13) and performing the regression as a result of the relation between \( r \) and \( \delta \) as follows.
\[ \delta = 3P \left( 1 - \nu^2 \right) \frac{1}{b} \lambda^{-\gamma} \]  
\[ \lambda = 1.9265 \text{ and } \eta = 0.732 \]  
\[ \delta = 3P \left( 1 - \nu^2 \right) \frac{1}{b} \lambda^{-\gamma} \left( \frac{3\pi}{\lambda} \right)^{\frac{3}{2}} \]  
\[ \frac{3\pi}{\lambda} \left( \frac{B}{A} \right)^{\frac{\gamma}{\beta}} = \left( \frac{B}{A} \right)^{-\Gamma} \]  
\[ = \left( \frac{B}{A} \right)^{\frac{3\gamma - 2\beta - \gamma}{2}} \]  
\[ \text{Equation (20)} \]  
\[ \delta = 1.8406 \cdot P \left( 1 - \nu^2 \right) \left( \frac{B}{A} \right)^{0.2454} \cdot B^{-\gamma} \]  
\[ \text{Equation (21)} \]  
The dimension of the major axis (a) is calculated based on ratio axis 1 ≤ r ≤ 10 to obtain the elliptical integral in equation (11) and (12) and the results are a = 0.9572 and \( \beta = 1.5712 \). With the same way to determine the dimension of minor axis (b), where \( \epsilon_1 \) and \( \epsilon_2 \) are given in equation (16) and they are 0.8399 and 2.0804 respectively. Thus the major axis dimension is 
\[ a = 0.9472 \cdot P^{\frac{1}{3}} \left( \frac{1 - \nu^2}{E} \right)^{\frac{1}{3}} \left( \frac{P}{A} \right)^{0.4414} \cdot B^{-\frac{1}{3}} \]  
\[ \text{Maximum contact pressure is} \]  
\[ P_s = \frac{3P}{2 \pi ab} \]  
\[ \text{Equation (23)} \]  
If \( E = 210 \text{ GPa} \) and \( \nu = 0.3 \) 
\[ a = 0.0154 \cdot P^{\frac{1}{3}} \left( \frac{B}{A} \right)^{0.4414} \cdot B^{-\gamma} \]  
\[ b = 0.0149 \cdot P^{\frac{1}{3}} \left( \frac{B}{A} \right)^{-0.1938} \cdot B^{-\gamma} \]  
\[ \delta = 2.8813 \times 10^{-4} \cdot P^{\frac{1}{3}} \left( \frac{B}{A} \right)^{0.2454} \cdot B^{-\gamma} \]  
\[ \text{Equation (26)} \]  
3. Application 

The above equations are used to calculate the contact dimension and contact pressure between wheel and rail. Contact between wheel and rail will cause wear, plastic deformation, and surface crack. In this case, we assumed that principle radius of curvature is varying from 300, 290, 280, 270, and 260. The wheel used in this observation is a conical wheel with \( R_{11} = 460 \text{ mm} \) and radius \( R_{12} \) becomes infinite, whereas in rail \( R_{21} \) is infinite. Mechanical properties of the material are Young’s modulus 210 GPa and Poisson’s ratio 0.3. The contact load is 80 kN.
The calculation was done using approximate solution from Fisher et al\textsuperscript{(1)} and from this paper. Then, the result was compared with exact solution based on Hertz theory as in Vijay et al\textsuperscript{(15)}. The comparison result is shown in Table 1 and difference of result is shown in Table 2.

Table 1. Results from approximate equation\textsuperscript{(1)}, this paper, and exact value by Hertz

<table>
<thead>
<tr>
<th>Radius R22 (mm)</th>
<th>Fisher</th>
<th>This paper</th>
<th>Hertz Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>a (mm)</td>
<td>b (mm)</td>
<td>po (N/mm\textsuperscript{2})</td>
<td>H (mm)</td>
</tr>
<tr>
<td>260</td>
<td>6.6368</td>
<td>4.5908</td>
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<tr>
<td>270</td>
<td>6.6071</td>
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<td>280</td>
<td>6.5787</td>
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<td>290</td>
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<tr>
<td>300</td>
<td>6.5250</td>
<td>4.9506</td>
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</tr>
</tbody>
</table>

Table 2. Comparison of the difference of results from approximation\textsuperscript{(1)}, and this paper to Hertz theory

<table>
<thead>
<tr>
<th>Radius R22 (mm)</th>
<th>Fisher</th>
<th>This paper</th>
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<tbody>
<tr>
<td>a</td>
<td>b</td>
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<tr>
<td>260</td>
<td>2.36%</td>
<td>1.41%</td>
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<tr>
<td>270</td>
<td>2.20%</td>
<td>1.30%</td>
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<td>280</td>
<td>2.14%</td>
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<td>290</td>
<td>2.05%</td>
<td>1.21%</td>
</tr>
<tr>
<td>300</td>
<td>1.95%</td>
<td>1.13%</td>
</tr>
</tbody>
</table>

4. Using Finite Element Method to Determine Contact Pressure

In order to make finite element model, wheel-rail contact model was performed with displaced wheel on the rail and make a point contact in rail head. Length of rail used in the model was L = 500 mm with incline angle of 1:40, as shown in Fig. 2.
Wheel receives load from wheelset in centre of gravity of the wheel. The load coincides with the point contact between wheel and rail. Loading model is shown in Fig.3.

![Fig. 3 Mechanical model of wheel](image)

The mechanical model of the wheel contacting with a rail is shown in Fig.3. Longitudinal direction, i.e. the wheel rolling direction, is indicated by X axis, the lateral direction and the vertical direction are indicated by Y axis and Z axis, respectively. The bottom of the rail is assumed as a fixed support. Furthermore, wheel profile is assumed to be constant and radius of curvature varies from 300, 290, 280, 270 and 260 mm.

Analysis was performed by Ansys Multiphysic software. In 3 D finite element model, wheel and rail were modeled as an elastic material. Finite element model was shown in Fig.4. Model element for wheel and rail was 20 node brick element and contact element was CONTA175 and TARGE170.

![Fig.4 Finite element model of wheel](image)

In order to model the wheel-rail contact, it is assumed that wheel as a target surface and rail as a contact surface. Contact model on wheel is surface to surface model contact, whereas on rail is node to surface model contact. Meshing process resulted in 10,716 elements and 35,845 nodes in the simulation. For faster execution, the meshing process is using different element configuration. Fine meshing in the neighborhood of the contact point, and coarse meshing in the other area.
Table 3. Contact pressure calculation from approximate equation\(^{(1)}\), this paper, finite element method, and exact value from Hertz

<table>
<thead>
<tr>
<th>Radius (R_{22})</th>
<th>Fisher</th>
<th>This paper</th>
<th>FEM</th>
<th>Hertz</th>
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Table 4. Comparison of the difference of contact pressure calculation from approximation\(^{(1)}\), this paper, and finite element method to Hertz theory

<table>
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<tr>
<th>Radius (R_{22})</th>
<th>Fisher</th>
<th>This paper</th>
<th>FEM</th>
</tr>
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<tbody>
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<tr>
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<td>3.61%</td>
<td>0.55%</td>
<td>3.23%</td>
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<tr>
<td>280</td>
<td>3.50%</td>
<td>0.69%</td>
<td>0.95%</td>
</tr>
<tr>
<td>290</td>
<td>3.34%</td>
<td>0.88%</td>
<td>2.69%</td>
</tr>
<tr>
<td>300</td>
<td>3.16%</td>
<td>1.09%</td>
<td>1.25%</td>
</tr>
</tbody>
</table>

5. Discussion

In this investigation, the obtained major elliptical radius of contact area (a), minor elliptical radius of contact area (b), and contact pressure (po) are compared with the results obtained by other researcher\(^{(1)}\) and also with the theory of Hertzian contact. Comparison is done by varying the rail radius of curvature while keeping the wheel radius of curvature a constant. The results of the new formula of this paper show a much closer agreement with the Hertzian theory, as shown in Table 2. For smaller value of \(R_{22}\), rail radius of curvature, the new formula shows a much better agreement with Hertzian theory while the Fischer and Wiest formula shows a greater difference. For the finite element analysis, the calculation of po shows a better agreement with Hertzian theory compared with Fischer and Wiest, however, it is worse than the new formula as shown in Table 4. There is an inconsistent trend of the FEM result as the value of \(R_{22}\) is decreasing. This is due to the difficulty of contact modeling where the intact compatibility between \(R_{11}\), wheel radius of curvature, and \(R_{22}\) has to be secured during the finite element modeling.

6. Conclusion

Further research is needed to formulate the appropriate contact problem in wheel-rail interaction due to the complicated shapes of both the wheel and rail, especially if the contact occurs at the rail track curve. The current approach using finite element method has provided a good solution to the problem and it needs further improvement for getting the best modeling of the contact area/region.

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