Development of High-Order Infinite Element Method for Stress Analysis of Elastic Bodies with Singularities*

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Abstract

Structural analysis problems are traditionally solved using the Finite Element Method (FEM). However, when the structure of interest contains discontinuities such as cracks or re-entrant corners, a large number of elements are required to accurately reproduce the stress characteristics in the region of the discontinuities. As a result, the FEM method requires a large storage space and has a slow convergence speed. It has been shown that the Infinite Element Method (IEM) overcomes these limitations and provides a feasible means of solving various types of elasticity and singularity problems. Previous studies have generally focused on the use of IEM formulations based on low-order elements (2×2). By contrast, this study develops a high-order (3×3) IEM formulation. The solutions obtained using the proposed IEM method for various 2D elasto-static problems are compared with the results obtained using the traditional low-order IEM method and the analytical solutions presented in the literature. It is shown that the results obtained using the proposed method are more accurate than those obtained using the low-order IEM method and are in excellent agreement with the analytical solutions.

Key words: Stress Singularity, Infinite Element Method, Stress Intensity Factor, Finite Element Method

1. Introduction

The stress analysis of elastic bodies with imbedded geometric singularities (e.g. cracks, dissimilar material junctions, re-entrant corners, and so on) is a topic of great practical importance in a variety of engineering applications. Due to the limited capability of analytical techniques, such problems are generally analyzed using the Finite Element Method (FEM) ⁽¹⁾. However, to accurately model the stress response in the vicinity of local singularities, the FEM method requires the use of a local mesh refinement scheme, which leads in turn to a significant number of elements, the need for a large memory storage, and a slow convergence speed. The Boundary Element Method (BEM), based on a boundary integral equation formulation, provides an alternative approach for modeling the regions of singularity in the body of interest ⁽²⁾. The BEM method requires the discretization of the problem boundary only and has been successfully applied to a variety of elastic crack problems. However, despite its many advantages, the BEM method has a number of serious drawbacks. For example, in deriving the BEM integral equation formulation, a fundamental singular function is required which satisfies the governing differential equation in all regions of the domain. Whilst such a function is relatively easily found for bodies with

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heterogeneous properties, those with inhomogeneous behaviors pose a far greater challenge. Various researchers have suggested that the limitations of the FEM and BEM methods can be overcome by the Infinite Element Method (IEM)⁽³⁾. The basic principle of IEM involves discretizing the problem domain into a large number of elements using a suitable partitioning technique and nodal sequence such that each element layer is similar to the other layers. Ying⁽⁴⁾, Han and Ying⁽⁵⁾, and Ying and Pan⁽⁶⁾ developed the mathematical foundations of the IEM formulation for the Laplace equation, exterior Stokes problem, corner problem, and planar elasticity problem in fracture mechanics. Ying⁽⁷⁾ proved the existence of a transformation matrix relating the nodal displacement vectors of the inner and outer layers, respectively, and showed that the stiffness matrices associated with each layer could therefore be combined to form an overall stiffness matrix defined in terms of the boundary nodes and tractions only. In addition, it was shown that the combined stiffness matrix converges to a certain constant quantity as the number of element layers approaches infinity. Guo⁽⁸⁾ proved that the concept of similar elements could be extended to arbitrary isoparametric elements and also presented a transformation procedure for constructing the overall stiffness matrix of the body of interest by combining the local stiffness matrices of each layer. Liu and Chiou⁽⁹⁾⁻⁽¹²⁾ recently developed both 2D (2×2) and 3D ($2 \times 2 \times 2$) IEM formulations to solve a variety of elasticity and singularity problems. In addition, the authors proposed a convergence criterion for the IEM formulation which extended the imaginary “infinity layer” concept to practical applications and ensured the convergence of the overall IE stiffness matrix.

Previous researchers have generally focused on low-order element IEM formulations. However, whilst such schemes offer a number of advantages relative to traditional FE methods, their accuracy is inevitably limited by the use of low order elements to mesh the problem domain. Accordingly, the present study develops a high-order element IEM formulation and a complementary degeneration process. The IEM formulation is further combined with the conventional FEM technique in order to solve a variety of 2D elasto-static problems. In the proposed approach, the problem domain is partitioned into two sub-domains; one containing the point of singularity and modeled using the IEM formulation, and another containing the remainder of the body of interest and modeled using the conventional FE formulation. In the solution procedure, the two sub-domains are coupled via the boundary nodes at the extremity of the IE sub-domain. Within the IE sub-domain, a similarity-based partitioning concept is applied to decompose the domain into multiple geometrically-similar meshed layers. All of the degrees of freedom of the multiple layers other than those associated with the boundary nodes at the coupling interface are then condensed to form an equivalent combined element referred to as an “infinite element” (IE) with master node degrees of freedom only. The stiffness matrix constructed in the IE sub-domain is then assembled into a system stiffness matrix for the FE sub-domain. The resulting IE-FE formulation, based on a symmetrical stiffness matrix involving IE effects, provides an ideal platform for the solution of a variety of elasticity problems. Specifically, the IE-FE approach combines the advantage of the FE method in modeling complex shape domains with the advantage of the IE method in modeling the points of discontinuity within the structure of interest with a high degree of precision. Furthermore, the round-off errors, CPU time, and PC memory storage requirements of the IE-FE scheme are significantly reduced. In addition, no prior governing assumption is required and no special finite elements (e.g. singular, hybrid or boundary elements) need be incorporated. Finally, the inherent convergence characteristics of the IEM method avoid the need for further parametric refinement analysis.

The validity of the proposed IE-FE approach is demonstrated by comparing the results obtained using the high-order IEM formulation for various crack problems (i.e. center cracks, collinear cracks, and slanted edge cracks) with those obtained using a traditional

low-order IEM formulation and presented in the literature, respectively. In general, the results confirm that the solutions obtained using the proposed high-order IE-FE formulation are more accurate than those obtained using the low-order IEM method and are in good agreement with the results obtained using the boundary collocation method ^(13,14), the displacement discontinuity method ⁽¹⁵⁾, the boundary force method ⁽¹⁶⁾, the fractal two-level FEM method ⁽¹⁷⁾ and the Bueckner weight function / boundary method ⁽¹⁸⁾.

Nomenclature

$[K]$	: stiffness matrix of a element	$[B]$	: strain-displacement matrix
$[D]$	: elastic stiffness matrix	$[J]$	: Jacobian matrix of the transformation
ζ	: master element natural coordinate system denoted ζ -axis	η	: master element natural coordinate system denoted η -axis
$[K]^I$	: stiffness matrix of element I	$[K]^II$	: stiffness matrix of lower-level element II
m	: the number of nodes in the outmost layer	Γ	: region boundary
ξ	: proportionality constant	O	: partition center point
s	: layer number	δ	: displacement vector
u	: displacement in the x direction	f	: force vector
K_z	: combined element stiffness matrix	v	: displacement in the y direction
K_I	: stress intensity factor	X	: transfer matrix
D	: sub-domain of infinite element mesh	Ω	: sub-domain of finite element mesh
I	: moment of inertia	E	: Young's modulus
θ	: incident angle	ν	: Poisson's ratio
r	: the distance of the nodal point from the crack tip	U_y	: the y -directional displacement
σ	: Stress	a	: crack length
W	: plate width	c	: height
b	: Width	e	: crack-spacing
$\bar{K}_a, \bar{K}_b, \bar{K}, M$ and $\bar{A}$	: incident symbol	β	: crack angle
K_a, K_b, K_m, A, B and C	: sub-matrices of the assembled stiffness matrix	F_{IA}, F_{IB}	: normalized SIF

Subscripts

T : transpose

2. Principle and Formulation of High-Order IEM method

2.1 High-Order Infinite Element Method

As discussed above, most previous studies in the IEM field focus on the use of low-order element (2×2) formulations. By contrast, the present study aims to improve the precision of the solutions obtained for singularity-type elasticity problems by means of a high-order element (3×3) IEM formulation.

2.2 Similarity of Nine-Node Isoparametric Elements

The similarity-based partitioning concept proposed in ⁽⁸⁾ is the key to the IEM method and is illustrated schematically in Figure 1 for the case of a nine-node (3×3) quadrilateral element. In the present study, the similar isoparametric elements are formulated utilizing the conventional FEM method. In general, the stiffness $[K]$ of each 2D isoparametric element in

Fig. 1 Schematic diagram of similar 2D nine-node elements.

Fig. 1 can be calculated as

$$[K] = \int_{-1}^1 \int_{-1}^1 [B]^T [D] [B] \det [J] d\zeta d\eta, \quad (1)$$

where $[B]$, $[D]$ and $[J]$ are the strain-displacement matrix, elastic stiffness matrix, and Jacobian matrix of the transformation, ζ and η denoted master element natural coordinate system respectively.

Due to the similarity property of the isoparametric elements, the stiffness of the lower-level isoparametric element in Fig. 1 is equal to that of the element in the layer above⁽⁹⁾, i.e.

$$[K]^{II} = [K]^I. \quad (2)$$

Equation (2) indicates the stiffness dimensional independence of the similar 2-D nine-node isoparametric elements shown in Fig. 1 and therefore provides a suitable basis for the condensing operation in the proposed 3×3 IEM formulation. Note that while the present study focuses specifically on the case of 3×3 elements, the derivation process described in the remainder of this section is equally applicable to higher-order elements (e.g. 4×4, 5×5, and so on).

2.3 Basic Infinite Element Method Formulation

This section derives the proposed nine-node IEM formulation for the solution of general 2-D elasto-static problems. In general, the similarity-based partitioning concept⁽⁸⁾ can be applied to either perfect bodies or bodies containing an imbedded singularity (e.g. a crack). Referring to Fig. 2, the basic steps in the meshing procedure can be described as follows. First, the outer boundary, Γ_0 , is discretized using a total of nine-node type master nodes, ordered in the counterclockwise direction. Second, by choosing a similar partitioning center, O , and taking a proportionality constant in the interval $\xi \in (0,1)$, a series of similar boundaries $\Gamma_1, \Gamma_2, \dots, \Gamma_s$ of Γ_0 are constructed with centers O and proportionality constants $\xi^1, \xi^2, \dots, \xi^s$, respectively. The region bounded by Γ_{i-1} and Γ_i is referred to as the i -th element layer ($i = 1, 2, \dots, s$), where s is the total number of element layers. Every layer inserts one row of 9-node-element, therefore the IE region would have $2s$ rows of node as Fig. 2 shown. Third, each boundary row is regularly discretized using the same number of nodes as that used in row 0 (Γ_0) (see Fig. 2). Thus, the coordinates of each node on each individual boundary row can be determined directly from the master node coordinates in row 0 in accordance with the similar geometry property used in the partitioning process. That is, the node coordinates in rows 0, 2, 4, 6, $\dots$, $2s$ are related via proportionality constants $\xi^1, \xi^2, \dots, \xi^s$, respectively. Similarly, the node coordinates in rows 1, 3, 5, $\dots$, $2s-1$ are related via the same proportionality constants $\xi^1, \xi^2, \dots, \xi^s$, respectively. Fourth, every element layer is auto-meshed into multiple quadrilateral nine-node type elements which are geometrically similar to those in the layers above and below, respectively.

Fig. 2 Infinite element meshing scheme used in high-order IEM formulation.

The element stiffness matrix of each quadrilateral element within each layer can be calculated and assembled to create an element-layer stiffness matrix using the conventional FE formulation. The assembled stiffness matrix of the “first element layer” can be expressed as

$$\begin{bmatrix} K_a & -B^T & -A^T \\ -B & K_m & -C^T \\ -A & -C & K_b \end{bmatrix}_{6m \times 6m}, \quad (3)$$

where K_a , K_b , K_m , A , B and C are sub-matrices of the assembled stiffness matrix with the same dimension $2m \times 2m$ (where m is the number of nodes in the outmost layer), and A^T , B^T and C^T are the transpose matrices of A , B and C , respectively. Since the global layer stiffness matrix is symmetrical and banded matrices, sub-matrices K_a , K_b , K_m , A , B and C are also symmetrical and banded matrices. The nodal displacement vector δ_i of the nodes in the i -th row is defined as

$$\delta_i \equiv [u_1^i \quad v_1^i \quad u_2^i \quad v_2^i \quad \dots \quad u_m^i \quad v_m^i]^T. \quad (4)$$

Meanwhile, the force vector f_i acting on the nodes in the i -th row is defined as

$$f_i \equiv [f_{1x}^i \quad f_{1y}^i \quad f_{2x}^i \quad f_{2y}^i \quad \dots \quad f_{mx}^i \quad f_{my}^i]^T. \quad (5)$$

The i -th element-layer stiffness matrix represents the combined effects of the nodal force and displacement vectors associated with the $(i-1)$ -th, i -th and $(i+1)$ -th rows, respectively. Recalling Eq. (3), and taking the first element layer as an example, it follows that

$$\begin{bmatrix} K_a & -B^T & -A^T \\ -B & K_m & -C^T \\ -A & -C & K_b \end{bmatrix} \begin{Bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} f_0 \\ f_1 + (-f_1) \\ f_2 \end{Bmatrix}. \quad (6)$$

Eq. (6) yields the following algebraic equations:

$$K_a \delta_0 - B^T \delta_1 - A^T \delta_2 = f_0, \quad (7)$$

$$-B \delta_0 + K_m \delta_1 - C^T \delta_2 = 0, \quad (8)$$

$$-A \delta_0 - C \delta_1 + K_b \delta_2 = f_2. \quad (9)$$

Eq. (8) can be rewritten as

$$\delta_1 = K_m^{-1} (B \delta_0 + C^T \delta_2). \quad (10)$$

Substituting Eq. (10) into Eqs. (7) and (9) yields

$$(K_a - B^T K_m^{-1} B) \delta_0 - (A^T + B^T K_m^{-1} C^T) \delta_2 = f_0. \quad (11)$$

$$-(A + C K_m^{-1} B) \delta_0 - (K_b - C K_m^{-1} C^T) \delta_2 = f_2. \quad (12)$$

Eqs. (11) and (12) can be consolidated as follows:

$$\begin{bmatrix} (K_a - B^T K_m^{-1} B) & -(B^T K_m^{-1} C^T + A^T) \\ -(A + C K_m^{-1} B) & (K_b - C K_m^{-1} C^T) \end{bmatrix} \begin{Bmatrix} \delta_0 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} f_0 \\ f_2 \end{Bmatrix}. \quad (13)$$

Eq. (13) can be rewritten as

$$\begin{bmatrix} \bar{K}_a & -\bar{A}^T \\ -\bar{A} & \bar{K}_b \end{bmatrix} \begin{Bmatrix} \delta_0 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} f_0 \\ f_2 \end{Bmatrix}. \quad (14)$$

where $\bar{K}_a = (K_a - B^T K_m^{-1} B)$, $\bar{K}_b = (K_b - C K_m^{-1} C^T)$, $\bar{A} = (A + C K_m^{-1} B)$. In Eq. (13), it is easily shown that the transpose of $(A + C K_m^{-1} B)$ is $(A^T + B^T K_m^{-1} C^T)$.

According to the similarly principle and regular meshing process described above, it follows that the stiffness matrices of all the element layers are the same. Consequently, the stiffness matrix of the s element layers (i.e. from the 1st element layer to the s -th element layer) can be expressed as the following set of algebraic equations:

$$\begin{bmatrix} \bar{K}_a & -\bar{A}^T \\ -\bar{A} & \bar{K}_b \end{bmatrix} \begin{Bmatrix} \delta_0 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} f_0 \\ f_2 \end{Bmatrix}, \quad \text{for layer 1} \quad (15)$$

$$\begin{bmatrix} \bar{K}_a & -\bar{A}^T \\ -\bar{A} & \bar{K}_b \end{bmatrix} \begin{Bmatrix} \delta_2 \\ \delta_4 \end{Bmatrix} = \begin{Bmatrix} -f_2 \\ f_4 \end{Bmatrix}, \quad \text{for layer 2} \quad (16)$$

$\vdots$

$$\begin{bmatrix} \bar{K}_a & -\bar{A}^T \\ -\bar{A} & \bar{K}_b \end{bmatrix} \begin{Bmatrix} \delta_{2s-2} \\ \delta_{2s} \end{Bmatrix} = \begin{Bmatrix} -f_{2s-2} \\ f_{2s} \end{Bmatrix}. \quad \text{for layer (s)} \quad (17)$$

Adding the second equation for the i -th element layer to the first equation for the $(i+1)$ -th element layer, and letting $\bar{K} = \bar{K}_a + \bar{K}_b$, it can be shown that

$$\bar{K}_a \delta_0 - \bar{A}^T \delta_2 = f_0, \quad (18)$$

$$-\bar{A} \delta_0 + \bar{K} \delta_2 - \bar{A}^T \delta_4 = 0, \quad (19)$$

$\vdots$

$$-\bar{A} \delta_{2s-4} + \bar{K} \delta_{2s-2} - \bar{A}^T \delta_{2s} = 0, \quad (20)$$

$$-\bar{A} \delta_{2s-2} + \bar{K}_b \delta_{2s} = f_{2s} = 0. \quad (21)$$

Letting $M_{2s} = \bar{K}_b = (K_b - C K_m^{-1} C^T)$ in Eq. (21), it follows that

$$\delta_{2s} = M_{2s}^{-1} \bar{A} \delta_{2s-2}. \quad (22)$$

Substituting Eq. (22) into Eq. (20) gives

$$-\bar{A} \delta_{2s-4} + (\bar{K} - \bar{A}^T M_{2s}^{-1} \bar{A}) \delta_{2s-2} = 0. \quad (23)$$

Comparing Eqs. (22) and (23), the following iteration formulas can be inferred:

$$M_i = \bar{K} - \bar{A}^T M_{i+2}^{-1} \bar{A}, \quad (i=2, 4, 6, \dots, 2s-2) \quad (24)$$

$$\delta_i = (M_i^{-1} \bar{A}) \delta_{i-2} = X \delta_{i-2}, \quad (i=2, 4, 6, \dots, 2s) \quad (25)$$

Applying the condition given in Eq. (10), and the following iterative formula is obtained:

$$\delta_i = K_m (B \delta_{i-1} + C^T \delta_{i+1}), \quad (i=1, 3, 5, \dots, 2s-1) \quad (26)$$

where X is a transfer matrix representing the relationship between δ_i and δ_{i-2} . Since M_{2s} is known to equal $\bar{K}_b$, then M_{2s-2} , M_{2s-4} , ..., and M_2 can be iterated out using Eq. (24). From Eq. (25), it follows that $\delta_2 = M_2^{-1} \bar{A} \delta_0$. Substituting δ_2 into Eq. (18), yields the following fundamental IEM formula:

$$(\bar{K}_a - \bar{A}^T M_2^{-1} \bar{A}) \delta_0 = K_z \delta_0 = f_0. \quad (27)$$

Where $K_z = (\bar{K}_a - \bar{A}^T M_2^{-1} \bar{A})$ is the combined element stiffness matrix which

Fig. 3 Consolidation of infinite elements to boundary nodes only.

Fig. 4 Coupling of IE sub-domain and FE sub-domain.

preserves the inherent symmetry characteristic of the global stiffness matrix used in the conventional FE procedure. Once f_0 (the outer surface traction) is given, δ_0 can be obtained from Eq. (27), and δ_2 , δ_4 , ..., and δ_{2s} can then be found sequentially from Eq. (25). Using the procedure described in Eqs. (18) to (21), all of the inner layer elements can be condensed and transformed into a single super-element with master nodes at the outer boundary Γ_0 . The resulting nine-node "Infinite Element (IE)" is shown in Fig. 3. Note that while Fig. 3 shows a regular rectangular element, the IE element can in fact have any arbitrary configuration, as defined by the coordinates of the nine master nodes. From a physical point of view, the K_z term represents the equivalent stiffness matrix of the IE. The proposed IE approach yields a significant reduction in the number of DOFs, and the memory storage requirement, respectively. Importantly, even though the number of DOFs is significantly reduced, the IE model retains the ability to capture the effects of physical discontinuities and non-uniform material properties in the body of interest.

2.4 IE-FE Coupling Method

Using IEM to model the entire domain is impractical particularly when dealing with the singular problem, e.g. the structure with cracks. And the IE-FE coupling technique is proposed ⁽⁹⁾. As Fig. 4 illustrated, two sub-domains, Ω and D are separated with the coupling interface, Γ_0 , and are modeled using FEM and IEM schemes, respectively.

In this study, the coupled IE-FE formulation is used to solve various 2D elasto-static analysis problems for the structure containing single or multiple cracks. The crack tips are modeled using IEs within the sub-domain D , which are then assembled into the global stiffness matrix formed within the FE sub-domain Ω . Since the similar element concept is embodied within the IE approach, all of the crack tips with a similar IE sub-domain have the same stiffness matrix, i. e., a single IE stiffness matrix is sufficient to describe the stiffness properties of all the others of the sub-domain D . As a result, the effort required to construct the mesh and the computational cost of the subsequent solution procedure are both significantly reduced.

2.5 Evaluation of Stress Intensity Factor (SIF)

In Chan et al., work ⁽¹⁹⁾, the stress intensity factor in plane elasticity are introduced as follows

$$K_I = \frac{E}{8C} \lim_{r \rightarrow 0} \sqrt{\frac{2\pi}{r}} \left[U_y(r) \Big|_{\theta=\pi} - U_y(r) \Big|_{\theta=-\pi} \right], \quad (28)$$

where U_y is the y -directional displacement as Fig. 5 shown, r is the distance of the nodal point from the crack tip, where the stresses near the tip of the crack are modeled using a

Fig. 5 Modeling of stresses near tip of crack in elastic body.

polar coordinate framework (r, θ) mounted at the crack tip. $C = 1$ denotes the plane stress condition, $C = 1 - \nu^2$ denotes the plane strain condition, E and ν are the Young's modulus and the Poisson ratio of the material, respectively.

3. Numerical Examples

In order to evaluate the accuracy of the proposed IE-FE method, this section presents various numerical examples characterized by differences in the crack element size, the mesh density, the crack size, and the crack geometry. In every case, the IE-FE formulation is solved using MATLAB v7.3.0 simulation software ⁽²⁰⁾. The simulations commence by comparing the solutions obtained using the high-order IEM formulation for the nodal displacement of a rectangular plate subject to a tension force at two ends with the equivalent analytical results obtained using the commercial ABAQUS software. The simulations then compare the high-order IE solutions obtained for the SIFs in the center crack tension (CCT) problem with the low-order IE solutions presented by Liu and Chiou ⁽⁹⁾ and the numerical results presented by Isida ⁽¹³⁾. In the third example, the normalized SIFs obtained for the collinear crack problem using the high-order IE method are compared with the corresponding low-order IE solutions ⁽⁹⁾ and the solutions obtained by Wang ⁽¹⁶⁾ using the boundary force method. The fourth example compares the high-order IE solutions for the nodal displacement of a cantilever beam with a tip load with the equivalent results obtained using the low-order IE method ⁽⁹⁾ and the analytical results obtained using commercial ABAQUS software. The fifth example compares the low-order and high-order IE solutions for a single edge crack subject to pure bending, with the solutions obtained using the boundary collocation method proposed by Gross and Srawley ⁽¹⁴⁾. Finally, in the sixth example, the solutions obtained using the low-order and high-order IE methods for the SIF in a body containing a slanted edge crack and subjected to pure bending are compared with the results presented by Leung and Su ⁽¹⁷⁾ and Aliabadi et al. ⁽¹⁸⁾

Example 3.1 A plate with a central hole subjected to uniform tension stress at its two free ends

Consider a thin rectangular plate with panel dimensions $(400\text{mm} \times 200\text{mm})$, as shown in Fig. 6. The elastic modulus and Poisson's ratio are $E=70000 \text{ N/mm}^2$ and $\nu=0.3$, respectively. Assume that a uniform tensile stress $\sigma=100 \text{ N/mm}^2$ is applied to the two free ends of the plate, causing a displacement of the nodal points in both the x - and y axial directions.

As shown in Fig. 6, the IEM solution procedure assumes that the outer boundary Γ_0 comprises 26 uniformly distributed master nodes and the proportionality ratio is set to $\xi = 0.9$. Given the geometric symmetry and the load, only a quarter of the full strip, under the given load and boundary conditions, needs to be considered. For comparison, the theoretical nodal displacement is also determined using commercial ABAQUS software. The corresponding results in Figs. 7 and 8 agree very closely with the results obtained herein.

Fig.6 Schematic of central hole rectangular plate in tension.

Fig.7 Comparisons between results of proposed approach and analytical solution of displacement u along IEM boundary shown in Fig.6. (triangle symbols for FEM solution and square symbols for IEM solution)

Fig.8 Comparisons between results of proposed approach and analytical solution of displacement v along IEM boundary shown in Fig.6. (triangle symbols for FEM solution and square symbols for IEM solution)

Example 3.2 Center Crack Tension (CCT) Problem

Consider the central crack tension (CCT) problem, shown in Fig. 9, in which a rectangular plate that has a crack in its center is subjected to plane tension. Because of the geometric symmetry and the load, only half of the full strip, under the given load and boundary conditions, needs to be addressed. A total of 61 master nodes are uniformly distributed around the circumferential boundary. The proportionality ratio ξ is set to 0.9 to calculate the stiffness matrix of K_{IEM} . Figure 10 compares the low-order and high-order IEM solutions obtained for SIF (K_I) using Eq. (28). Notably, the ratio of the half-crack length (a) to the plate width (W) is set to $a/W=0.2$ and the ratio of plate length (H) to the plate width (W) is set to $(H/W=1)$. The higher-order element is preferred to solve the singular problem, as shown in Fig. 10, in which the K_I curve close to the crack tip position converges.

The CCT problem was further analyzed for crack-length-to-plate-width ratios of $a/W=0.2$ and 0.3 and rectangular plates with crack position ratios of $H/W=0.4, 0.5, 0.6, 0.7, 0.8, 0.9$, and 1.0. Table 1 shows the corresponding values of the normalized SIFs ($K_I/\sigma\sqrt{\pi a}$), computed using the low-order and high-order IEM formulations, respectively. For comparison, Table 2 shows the equivalent numerical results of Isida⁽¹³⁾. Figure 11 plots the three sets of results obtained for $a/W=0.2$. The results in Fig. 11 and Tables 1 and 2 reveal that excellent agreement between the high-order IEM solutions and the numerical solutions

presented by Isida ⁽¹³⁾ for all a/W and H/W ratios. Restated, the ability of the high-order IEM formulation to approximate the SIFs in the CCT problem is confirmed.

Fig. 9 CCT problem: (a) original crack problem; (b) symmetry of analytical domain; (c) schematic high-order node arrangement.

Table 1
Normalized SIFs in CCT problem computed using low-order IEM method ⁽⁹⁾ and proposed high-order IEM method.

a/W	Method	H/W						
		0.4	0.5	0.6	0.7	0.8	0.9	1
0.2	Low-Order ⁽⁹⁾	1.2417	1.1702	1.1312	1.1065	1.0887	1.0750	1.0644
	High[present]	1.2551	1.1746	1.1309	1.1036	1.0840	1.0688	1.0569
0.3	Low-Order ⁽⁹⁾	1.4986	1.3617	1.2820	1.2273	1.1853	1.1519	1.1257
	High[present]	1.5191	1.3706	1.2854	1.2279	1.1843	1.1500	1.1233

Table 2
Normalized SIFs in CCT problem presented by Isida ⁽¹³⁾.

a/W	H/W						
	0.4	0.5	0.6	0.7	0.8	0.9	1
0.2	1.256	1.175	1.130	1.103	1.083	1.067	1.055
0.3	1.520	1.371	1.285	1.228	1.184	1.150	1.123

Fig. 10 Comparison of low-order and high-order IEM solutions for SIF in CCT problem.

Fig. 11 Comparison of 2×2 and 3×3 IEM solutions for SIF in CCT problem with those of Isida ⁽¹³⁾ for $a/W=0.2$.

Example 3.3 Collinear Internal Crack Problem

Consider the collinear internal crack problem shown in Fig. 12 (reproduced from ⁽⁹⁾), in which a rectangular plate containing two collinear cracks is subject to a uniform tensile stress. Adopting the same geometric proportionality definition and material properties as those considered by Wang ⁽¹⁶⁾, the SIFs within the rectangular plate were solved using the high-order IEM formulation for a crack-spacing-to-width ratio of $e/b = 0.5$, height-to-width ratios of $c/b = 0.5$, and crack-length-to-width ratios of $a/e = 0.4, 0.5$ and 0.6 . Figure 13 compare the low- and high-order IEM solutions for the normalized SIFs at crack tips A and B with the solutions presented by Wang ⁽¹⁶⁾ for a/e ratios of 0.4, 0.5 and 0.6 and c/b ratios of 0.5, respectively. Table 3 present the corresponding results in a tabular format. In general, the results show that the SIF increases with an increasing crack-length-to-width ratio or a decreasing height-to-width ratio. From inspection, the low-order solutions are found to deviate by up to 3.2% from the boundary force solutions presented by Wang ⁽¹⁶⁾. By contrast, the solutions computed using the high-order IEM formulation deviate from those of Wang ⁽¹⁶⁾ by no more than 1%. Thus, the results confirm that the proposed high-order IE-FE method yields highly accurate solutions for the collinear internal crack problem.

Fig.12 IE-FE coupled computational model of two collinear crack problem ⁽⁹⁾.

Fig. 13 Comparison of 2×2 and 3×3 IEM solutions for SIF in two collinear crack problem with those of Wang ⁽¹⁶⁾ for $c/b=0.5$ and normalized F_{IA} .

Table 3

Comparison of normalized SIFs in two collinear crack problem with $c/b = 0.5$ and $a/e = 0.4, 0.5$ and 0.6 computed using low-order IEM method ⁽⁹⁾, numerical method ⁽¹⁶⁾ and proposed high-order IEM method.

$c/b=0.5$						
a/e	F_I	Ref. ⁽¹⁶⁾	Lower ⁽⁹⁾	R.D.(%)	Present	R.D.(%)
0.4	F_{IA}	1.1678	1.1872	1.66%	1.1687	0.08%
	F_{IB}	1.1714	1.1896	1.55%	1.1725	0.09%
0.5	F_{IA}	1.2562	1.2850	2.29%	1.2570	0.06%
	F_{IB}	1.2584	1.2879	2.34%	1.2593	0.07%
0.6	F_{IA}	1.3697	1.4060	2.65%	1.3702	0.04%
	F_{IB}	1.3629	1.4066	3.20%	1.3629	0.00%

$$F_{IA} = \frac{K_{IA}}{\sigma\sqrt{\pi a}}; \quad F_{IB} = \frac{K_{IB}}{\sigma\sqrt{\pi a}}$$

Example 3.4 Cantilever Beam with Tip Load Problem

Consider the cantilever beam shown in Fig. 14 with dimensions of 4 in x 1 in, a Young’s modulus of $E = 1000000\text{ psi}$ and a Poisson’s ratio of $\nu = 0.3$. Assume that a uniform stress of $P=1000\text{ psi}$ is applied to the free end of the beam causing the nodal points along segment $\overline{AB}$ to be displaced in the y -axis direction. The analytical solution for the displacement of the nodal points in the y -direction is given by

$$v = \frac{Px^2}{6EI}(3L - x),$$

(29)

where P, E, I, L and x denote the load, the Young’s modulus, the moment of inertia, beam length and the nodal coordinate position, respectively.

As shown in Fig. 14(b), the IEM solution procedure considers the outer boundary Γ_0 to comprise 24 uniformly distributed master nodes. For the node arrangements, the high-order IE formulation is used to obtain the y -axis displacement of node B using proportionality ratios of $\xi = 0.9$ and $\xi = 0.5$, respectively. For comparison purposes, the theoretical value of the node B displacement is also computed using commercial ABAQUS software. Tables 4 and 5 compare the solutions obtained from the low-order and high-order IEM methods for the node B displacement with that predicted by ABAQUS software for the case where the outer boundary comprises 24 nodes and the proportionality ratio is assigned values of $\xi = 0.9$ and $\xi = 0.5$, respectively. For the meshing schemes, it can be seen that the node displacement obtained using the high-order IEM method is in better agreement with the theoretical solution than that obtained using the low-order method. To clarify the effect of the proportionality ratio ξ on the solutions obtained using the IEM formulation, the nodal displacement field along segment $\overline{AB}$ of the beam (see Fig. 14) was computed using both the low-order IEM scheme and the high-order IEM scheme for values of ξ varying from $\xi = 0.5$ to $\xi = 0.9$. The corresponding results are presented in Figs. 15 and 16, respectively. Figure 15 shows that the results obtained using the low-order IEM method are highly sensitive to the value assigned to the proportionality ratio. By contrast, Fig. 16 shows that the high-order IEM solutions are insensitive to the proportionality ratio, i.e. the same value of the nodal displacement is obtained irrespective of the proportionality ratio used to decompose the IE sub-domain.

Fig. 14 (a) Original cantilever beam subject to tensile load; (b) arrangement of 24 nodes in IEM formulation

Table 4
Comparison of B nodal displacement computed by low-order IEM method, proposed high-order IEM method and ABAQUS for $\xi = 0.9$ and 24 nodes.

Theoretical=−0.2560in			
Number of node	Method	Present	R.D. (%)
24	Low-Order(IEM)	−0.2070	19.1287%
24	High-Order(IEM)	−0.2667	4.1691%
24	ABAQUS(Lower)	−0.2106	17.7460%
24	ABAQUS(Higher)	−0.2674	4.4460%

Table 5

Comparison of B nodal displacement computed by low-order IEM method, proposed high-order IEM method and ABAQUS for $\xi = 0.5$ and 24 nodes.

Theoretical = -0.2560 in			
Number of node	Method	Present	R.D. (%)
24	Low-Order(IEM)	-0.1438	43.8259%
24	High-Order(IEM)	-0.2656	3.7583%
24	ABAQUS(Lower)	-0.1813	29.1820%
24	ABAQUS(Higher)	-0.2646	3.3670%

Fig. 15 Effect of proportionality ratio on $\overline{AB}$ displacement in low-order IEM scheme.

Fig. 16 Effect of proportionality ratio on $\overline{AB}$ displacement in high-order IEM scheme.

Example 3.5 Single Edge Crack Problem in Body Subject to Pure Bending or Three-point Bending

Consider a rectangular plate containing a single edge crack of length a and subject to either pure bending or three-point bending, respectively. The corresponding results for the variation of the SIF with the crack-length-to-width ratio are shown in Fig. 17. In this case, it can be seen that the result obtained using the high-order IEM method are in closer agreement with those presented by Gross and Srawley⁽¹⁴⁾ than the solutions obtained using the low-order IEM method.

Example 3.6 Slanted Edge Crack Problem in Body Subject to Pure Bending

Figures 18 and 19 compare the low-order and high-order IEM solutions for the normalized SIF in a body containing a single slanted edge crack and subjected to pure bending with those presented by Leung and Su⁽¹⁷⁾ and Aliabadi et al.⁽¹⁸⁾, respectively. In both figures, results are presented for three crack angles in the range $\beta < 90^\circ$. Note that the solutions presented for Leung and Su⁽¹⁷⁾ and Aliabadi et al.⁽¹⁸⁾ are based on the fractal two-level FE method and the Bueckner weight function / boundary analysis method, respectively. In both figures, it can be seen that the solutions obtained using the high-order IEM method are in better agreement with the results presented in the literature than those calculated using the low-order IEM method.

To evaluate the sensitivity of the low-order and high-order IEM formulations to the number of layers used to mesh the IE sub-domain, the normalized SIF close to the point of singularity in Fig. 18 was computed for cracks with slant angles of $\beta = 45^\circ$ and a constant crack-length-to-plate-width ratio of $a/b = 0.5$ using various number of layers in the range 20 ~ 500. For comparison purposes, the SIF was also calculated using the fractal two-level FE

method used by Leung and Su⁽¹⁷⁾. The corresponding results are presented in Fig. 20. It can be seen that for all three methods, a convergent solution for the SIF is obtained given a minimum of approximately 100 layers. Moreover, it is observed that the results obtained using the high-order IEM method are consistently higher than those obtained using the low-order IEM method and are in excellent agreement with the FE solutions provided by Leung and Su⁽¹⁷⁾.

Fig.17 Comparison of 2×2 and 3×3 IEM solutions for SIF in single edge crack problem with pure bending with those of Gross and Srawley⁽¹⁴⁾.

Fig. 18 Comparison of 2×2 and 3×3 IEM solutions for mode I and mode II SIF in slanted edge crack problem with pure bending with those of Leung and Su⁽¹⁷⁾.

Fig. 19 Comparison of 2×2 and 3×3 IEM solutions for mode I and mode II SIF in slanted edge crack problem with pure bending with those of Aliabadi et al.⁽¹⁸⁾.

Fig. 20 Convergence of SIF solutions computed using low-order IEM method, high-order IEM method and method proposed by Leung and Su⁽¹⁷⁾.

4. Conclusion

This study has presented a high-order element IEM formulation for the stress analysis of bodies containing singularities such as cracks. In the proposed method, the problem domain is partitioned into two sub-domains; one domain containing the tip of the singularity

of interest and modeled using the IE method, and the other containing the remainder of the problem domain and modeled using the conventional FE method. Within the first sub-domain, a similarity-based partitioning scheme is used to construct an infinitesimal mesh in the region of the crack tip. Due to the fractal geometry condition imposed in the partitioning process, the stiffness matrix associated with one layer of the decomposed mesh is identical to that of the layers immediately above and below it. Consequently, a single IE stiffness matrix is sufficient to describe the stiffness properties of all the meshed layers. In the solution procedure, the IE and FE sub-domains are coupled via the master nodes located at the interface between the two domains. The resulting IE-FE scheme combines the inherent advantage of the FE method in modeling complex shape domains with the advantage of the IE method in modeling the points of discontinuity within the structure of interest with a high level of precision and computational efficiency. Meanwhile, the use of a proportionality ratio at the interface boundary enables the problem of interest to be analyzed using either low-order or high-order IEM schemes with no change in the nodal arrangement. Compared to the traditional FEM method, the proposed IE-FE formulation results in lower rounding errors, a reduced memory storage requirement and a faster convergence time. The validity of the proposed approach has been confirmed by solving a variety of common 2D elasto-static problems. It has been shown that the solutions obtained using the high-order IEM method are more accurate than those obtained using the low-order IEM method and are in excellent agreement with the analytical results presented in the literature. Finally, it has been shown that in contrast to the solutions obtained using the low-order IEM method, those obtained using the high-order IEM method are insensitive to the value of the proportionality ratio used in the IE meshing procedure. Overall, the results presented in this study confirm that the high-order IEM method provides an accurate and computationally efficient means of solving a variety of 2D elasto-static problems.

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