Abstract
The propagation of transverse surface waves in a functionally graded material carrying a piezoelectric layer is investigated analytically. The material properties in the substrate change gradually with the depth coordinate. We here assume that all material properties of the substrate have the same exponential function distribution along the depth direction. The dispersion equations relating phase velocity to the material gradient in the substrate for the existence of the waves are obtained in a simple mathematic form for class 6mm piezoelectric materials. It is demonstrated that the material gradient in the elastic substrate significantly affects the phase velocity and cut-off frequency of long waves but has only negligible effects on short waves. The effects of the material gradient on the penetration depth and electromechanical coupling factor, which are two parameters of practical interest, are also calculated and plotted. The significant influence of the material gradient on the wave propagation behavior provides a potential factor for designing acoustic wave devices.

Key words: Transverse Surface Waves, Functionally Graded Materials, Piezoelectric Layer, Dispersion Relation

1. Introduction
Wave propagation and vibration in a pure piezoelectric plate have received considerable attention previously (1). Nowadays, the study of piezoelectric devices over the last three decades spans from a simple plate model to piezoelectric coupled structure model. Transverse surface waves in piezoelectric coupled materials and structures are attractive for designing signal-processing devices due to their high performance and simple particle motion and have obtained extensive study (2-8). More recently, a study on the propagation of transverse surface waves in a homogeneous elastic substrate carrying a finite-thickness piezoelectric layer has been investigated analytically (9). One permitted wave by virtue of piezoelectricity in such structure when the bulk-shear-wave velocity in the layer is greater than that in the substrate has only one mode consisting of partly normal dispersion and partly anomalous dispersion, and hence has potential importance in practical applications. However, in such layered laminate the development of high local stress fields arises inevitably due to mismatch of thermal strain, which leads to cracking and delaminating at the interfaces of such layered structures; a concept that may be used to reduce the magnitude of residual and thermal stresses would be the introduction of functionally graded
materials (FGMs). The development of FGMs has demonstrated that they have the potential to reduce the stress concentration near the ends and increase the fracture toughness \(^{(10)}\).

Recently, to improve the efficiency and natural life of the surface acoustic wave (SAW) devices, the potential application of FGMs to surface wave propagation has been explored \(^{(11-16)}\). However, the research work on how material gradient affects the surface wave propagation is still limited up to now. In this paper, we will study the transverse surface waves in a functionally graded elastic substrate coated with a piezoelectric layer of finite thickness. The dispersion equations relating phase velocity to the material gradient in the substrate for the existence of the waves are given in a simple mathematic form. The effects of the material gradient on the dispersion behavior, penetration depth and electromechanical coupling factor of the transverse surface waves are discussed in detail. To the best of our knowledge, such work has not been done yet.

The manuscript is organized as follows: the mathematical formulation is given in Section 2; a surface wave solution is obtained in Section 3, followed by discussions and numerical results in Section 4; some conclusions are drawn in Section 5.

2. Mathematical formulation

Consider an FGM elastic substrate occupying the half-space \(x > 0\), in which the properties change gradually along the \(x\)-axis direction, as shown in Fig. 1. Let a piezoelectric layer of uniform thickness \(h\) be deposited perfectly on the substrate, which results in a surface at \(x = -h\) free of external forces. Here the piezoelectric material is taken to be of class 6 \(mm\) (or \(∞\) \(m\)), with its polar axis oriented along the \(z\) direction of Cartesian coordinates \((x, y, z)\). There is a perfect electrode at the interface \(x = 0\), so the substrate can be either metal or dielectric. Here, transverse surface waves (i.e., pure shear-horizontal mode) propagating in such layered piezoelectric structure will be taken into account. It is assumed that the waves propagate in the positive direction of the \(y\)-axis, such that the nonzero field quantities representing the motion are only functions of the coordinates \((x, y)\) and time \(t\).

Let \(w\) and \(\phi\) denote separately the mechanical displacement and electric potential function of the piezoelectric layer. Following Bleustein \(^{(2)}\), the coupled field equations are given by

\[
\begin{align*}
\nabla^2 w - \left(\frac{1}{c_p^2}\right) \ddot{w} &= 0 \\
\nabla^2 \left[\phi - \left(\frac{e_{15}}{\varepsilon_{11}}\right) w\right] &= 0
\end{align*}
\]

where \(\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2\), and \(c_p = \left[(c_{44} + e_{15}^2/\varepsilon_{11})/\rho\right]^{1/2}\) is the bulk-shear-wave velocity in the piezoelectric material, with \(c_{44}, e_{15}, \varepsilon_{11}\) and \(\rho\) representing the elastic, piezoelectric,
dielectric constants and mass density, respectively.

For the FGM elastic substrate, occupying the region \( x > 0 \), let \( w' \) denote the mechanical displacement and assume that the shear modulus \( \mu \) and the mass density \( \rho \) have the same exponential function variation, i.e., \( \mu(x) = \mu_0 e^{\alpha x} \) and \( \rho(x) = \rho_0 e^{\alpha x} \), with \( \mu_0 = \mu(0) \) and \( \rho_0 = \rho(0) \). The assumption is often encountered in the literatures \(^{15,16} \). It presents the advantages of mathematical simplicity and familiarity, but the inconvenience of describing a somewhat unrealistic inhomogeneity, because it either blows up or vanishes as \( x \to \infty \). These problems can be overcome by considering that it occurs sufficiently far away from the interface, and by focusing on the near-the-surface localization of the wave \(^{16} \). Then from basic equilibrium equations and constitutive equations in elasticity, we have

\[
\nabla^2 w' + \frac{\alpha}{c} \frac{\partial w'}{\partial x} = \frac{\ddot{w}'}{c_m^2}
\]

where \( c_m = (\mu_0/\rho_0)^{1/2} \) is the bulk-shear-wave velocity in the FGM elastic substrate.

The wave propagation problem specified by (1) and (2) should satisfy the following boundary and continuity conditions:

1) \( \sigma_{zx} = 0 \) at \( x = -h \);
2) \( w = w', \sigma_{zx} = \sigma_{zx}', \varphi = 0 \) at \( x = 0 \);
3) \( w' \to 0 \) as \( x \to +\infty \).

The electric conditions at the free surface can be classified into two categories, i.e.

4) electrically open circuit: \( D_x = 0 \) at \( x = -h \);
5) electrically short circuit (or metalized surface): \( \varphi = 0 \) at \( x = -h \),

which is based on the fact that the space above the piezoelectric layer is vacuum or air and its permittivity is much less than that of the piezoelectric layer.

Then the propagation problem of the transverse surface waves in the layered half-space under question becomes the solution of Eqs. (1) and (2) under conditions 1)-3) and 4) or 5).

3. Solution of the problem

Built upon some earlier work \(^{2,3} \), we consider the following transverse surface waves satisfying attenuation condition 3):

\[
w(x, y, t) = \left( A_1 e^{-i2\pi Hx} + A_2 e^{i2\pi Hx} \right) \exp \left[ ik \left( y - ct \right) \right]
\]

\[
\varphi(x, y, t) = \left[ A_3 e^{-i2\pi Hx} + A_4 e^{i2\pi Hx} \right] \exp \left[ ik \left( y - ct \right) \right], \quad -h \leq x \leq 0,
\]

\[
w' = A_5 e^{-i2\pi Hx} \exp \left[ ik \left( y - ct \right) \right], \quad x \geq 0,
\]

where \( A_1, A_2, A_3, A_4 \) and \( A_5 \) are arbitrary constants, \( x = x/h \), and \( H = h/\lambda \) is the dimensionless wavelength with \( k = 2\pi/\lambda \) being the wave number, \( i = \sqrt{-1} \), and \( c \) is the phase velocity.

(3) and (4) satisfy separately Eqs. (1) and (2) when

\[
s^2 = 1 - c^2/c_p^2, \quad r = \frac{m}{4\pi H} + \frac{m^2}{(4\pi H)^2 + r^2},
\]

where the dimensionless quantity \( m = \alpha h \) denotes the degree of the material gradient inside the FGM elastic substrate, and \( r^2 = 1 - c^2/c_m^2 \).

Substitution of (3), (4) and the corresponding stress components into the remaining boundary and continuity conditions 1), 2) and 4) yields the following homogeneous linear algebraic equations with respect to \( A_1 \) through \( A_5 \)
\[ A_1 + A_2 = A_3 \]
\[ A_3 + A_4 + (A_1 + A_2) \varepsilon_{15} / \varepsilon_{11} = 0 \]
\[ s \varepsilon_{44} (A_2 - A_4) + \varepsilon_{15} (A_4 - A_1) = -\mu_0 A_1 r \]
\[ s \varepsilon_{44} (A_2 e^{-2\pi i} - A_4 e^{2\pi i}) + \varepsilon_{15} (A_4 e^{-2\pi i} - A_1 e^{2\pi i}) = 0 \]
\[ \varepsilon_{11} (A_4 e^{-2\pi i} - A_1 e^{2\pi i}) = 0 \]

where \( \varepsilon_{44} = c_{44} + \varepsilon_{15} / \varepsilon_{11} \) is the piezoelectrically stiffened elastic constant of the piezoelectric material layer (2).

The non-trivial solution of Eq. (6) exists if and only if the determinant of the coefficient matrix equals zero, which leads to the following dispersion relation

\[ k_p^2 \tanh(2\pi H) - s \tanh(s2\pi H) - \mu_p r / \varepsilon_{44} = 0 \]  

for the case of electrically open circuit. In (7), \( k_p^2 = \varepsilon_{15} / \varepsilon_{11} e_{44} \) is the piezoelectric coupling factor (2).

Through the similar procedure, we have the dispersion relation

\[ \left( k_p^2 + s^2 \right) \tanh(2\pi H) \tanh(s2\pi H) - \frac{\mu_0}{\varepsilon_{44}} k_p^2 r \tanh(s2\pi H) + \]
\[ \frac{\mu_0}{\varepsilon_{44}} sr \tanh(2\pi H) + 2k_p^2 s \left[ \frac{1}{\cosh(2\pi H) \cosh(s2\pi H)} - 1 \right] = 0 \]  

for the case of electrically short circuit.

Equations (7) and (8) are the dispersion relations determining the propagation behavior of the transverse surface waves studied in the paper. Obviously, the waves are dispersive.

We examine a special case below.

When \( \alpha = 0 \), i.e., the elastic substrate is non-graded, we note that \( r = r' = \sqrt{1 - c_p^2 / c_m^2} \). Then Eqs. (7) and (8) degenerate exactly into the results of Ref. (9). This agreement confirms to some extent the validity of the formulation in this paper.

4. Numerical examples and discussion

4.1 Dispersion characteristics

Based on some previous work (2, 9), two physical situations for the transverse surface waves are possible for non-graded elastic substrate, i.e., \( m = 0 \): 1) Type 1: \( c_m > c_p > c_{BG} \), \( s \) real or imaginary, 2) Type 2: \( c_p > c_m > c_{BG} \), \( s \) always real. Here, \( c_{BG} = c_p \left( 1 - k_p^2 \right)^{1/2} \) is the phase velocity of the B-G waves on the surface of a piezoelectric substrate coated with an infinitely thin layer of conducting material. The Type 1 wave is a Love-type wave, which has been well studied. We thus focus on the effect of the material gradient in the elastic substrate on the Type 2 wave.

Table 1 Material properties

<table>
<thead>
<tr>
<th>Material</th>
<th>Shear Modulus ( c_{44} )/ GPa</th>
<th>Piezoelectric constant ( e_{15} )/ C m(^{-2})</th>
<th>Dielectric constant ( \varepsilon_{11} )/ F m(^{-1})</th>
<th>Mass density ( \rho )/ kg m(^{-3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>PZT-4</td>
<td>25.6</td>
<td>12.7</td>
<td>730 ( \varepsilon_0 )</td>
<td>7500</td>
</tr>
<tr>
<td>Zinc</td>
<td>41.2</td>
<td></td>
<td>730 ( \varepsilon_0 )</td>
<td>6920</td>
</tr>
</tbody>
</table>

* \( \varepsilon_0 = 8.854 \times 10^{-12} \) F m\(^{-1}\) is the permittivity in vacuum.

The dispersion curves plotted for selected values of the material gradient coefficient...
are illustrated in Fig. 2, corresponding to a graded zinc substrate carrying a PZT-4 ceramic layer for the cases of both electrically open circuit and electrically short circuit, respectively. The material parameters for both the metal and the piezoelectric material are listed in Table 1 (2). It can be seen from Fig. 2 that positive $m$ not only increases the starting value of the phase velocity but also the cut-off frequency of the mode, while negative $m$ decreases only the starting value of the phase velocity and has no influence on the cut-off frequency (the cut-off frequency is defined as the product of the starting phase velocity of a mode and the corresponding wave number), regardless of the electrically boundary conditions. In other words, graded metal substrates with material properties being from “soft” to “hard” in the positive direction of $x$-axis provide higher starting values of the phase velocity and cut-off frequency for the wave propagation, while those with material properties being from “hard” to “soft” in the positive direction of $x$-axis give lower starting values of the phase velocity and unchanged cut-off frequency. However, the influence of the material gradient on the mode is only significant at the low frequency range and negligible at the high frequency range. With the increase in the dimensionless wavelength, dispersion curves in the case of graded metal substrates trend to the limit case of homogeneous substrate (i.e., $m = 0$).

![Fig. 2 Phase velocity $c$ of the transverse surface waves in a zinc substrate carrying a PZT-4 layer plotted as a function of $H = h/\lambda$ for selected values of the material gradient coefficients $m$; $c_m = 2464$ m/s, $c_p = 2597$ m/s, $c_{BG} = 2258$ m/s.]

4.2 Penetration depth

From Eq. (6), we can obtain that

$$A_2 = e^{4\pi H} A_1, \quad A_3 = -\frac{e_{15}^2 + 1 + e^{4\pi H}}{\varepsilon_{11}} e^{4\pi H} A_1,$$

$$A_4 = -\frac{e_{15}^2 + 1 + e^{4\pi H}}{\varepsilon_{11}} e^{4\pi H} A_1, \quad A_5 = (1 + e^{4\pi H}) A_1. \quad (9)$$

Then the mechanical displacement can be rewritten as

$$w = \left(e^{-2\pi H h} + e^{2\pi H h} e^{2\pi H h} \right) A_4 \exp \left[i k \left(y - ct \right)^{i}ight] \quad -h < x < 0,$$
\[ w' = \left(1 + e^{4\pi iH}\right) A e^{-r^2 \pi H} \exp\left[ ik\left(y - ct\right)\right], \quad x > 0. \] (11)

Similarly, the displacement expressions for the case of electrically short circuit can also be obtained, which is suppressed for brevity. It is found from our calculation that the material gradient has the similar effects on the penetration depth in the electrically open case to that in the electrically short case. Here, only the numerical results in the case of electrically open circuit are shown.

The variations of mechanical displacement \( w \) vs. \( x/h \) at \( y = 0 \) for the waves are shown in Fig. 3 for selected values of \( m \). It is seen from Fig. 3 that the mechanical displacement decays away from the interface in both directions. The mechanical displacement in the substrate attenuates to zero within several wavelengths for the case of \( m = 0 \) (i.e., nongraded elastic substrate), which is altered by the presence of the material gradient in the substrate. The material gradient has no effect on the displacement distribution inside the piezoelectric layer, while in the substrate positive \( m \) decreases the penetration depth and negative \( m \) increases the penetration depth. That means that positive material gradient coefficient can confine the transverse surface waves within the vicinity of the interface more efficiently, which is in favor of designing SAW devices.

![Figure 3 Variation of \( w \) with \( x/h \) for selected values of \( m \)](image)

4.3 Electromechanical coupling factor

For the transverse surface waves, the electromechanical coupling factor defined as follows plays an important role in their characteristic analysis \(^{16}\)

\[ K^2 = \frac{c_{\text{open}} - c_{\text{short}}}{c_{\text{open}}} \] (12)

Figure 4 shows the plot of the electromechanical coupling factor as a function of \( H=h/\lambda \) for selected values of the material gradient coefficient \( m \). It can be seen that positive \( m \) increases the maximum value of the electromechanical coupling factor a little bit while negative \( m \) decreases it a little bit. In general, the influence of the material gradient on the electromechanical coupling factor is not so significant. Considering the significant influence of the material gradient on the mode shown in Fig. 2, we expect to manipulate the wave
propagation behavior through proper design of the FGM substrate materials without loss of energy transformation efficiency for SAW devices.

Fig. 4 Electromechanical coupling factor for selected values of $m$

5. Conclusion

The propagation behavior of the transverse surface waves in a layered structure concerning a piezoelectric layer and an FGM elastic substrate has been investigated analytically. Some conclusions are drawn through the numerical example and discussion:

1) The presence of the material gradient in the elastic substrate significantly affects the phase velocity and cut-off frequency of long waves but has only negligible effects on short waves.

2) The material gradient has no effect on the displacement distribution inside the piezoelectric layer. Positive material gradient coefficient can confine the wave propagation within the vicinity of the interface more efficiently.

3) The influence of the material gradient on the electromechanical coupling factor is not so significant. Positive material gradient coefficient can increase it a little bit while negative material gradient coefficient decreases it.

These results are meaningful to both theoretical research and engineering application of the transverse surface waves. Although this analysis has been confined to the elastic substrates with properties of exponential gradient function, it might be possible to draw from it some general information concerning the physical characteristics of the transverse surface waves in FGM substrates with properties of other variation functions.

References


(3) Wang, Q., Quek, S.T. and Varadan, V.K., Love waves in piezoelectric coupled solid


