Interaction between Euler Buckling and Brazier Instability*

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Abstract
The interaction between Euler buckling and Brazier instability in an orthotropic cylindrical tube was investigated. A simple closed-form interaction curve was obtained by using the principle of minimum potential energy including the 3D extended Brazier’s 2D type cross-section ovalization and the potential energy of axial compression. The interaction curve consists of an Euler buckling load and a bending instability load, which slightly differs from Brazier’s bending instability load. Hence, the differences between these bending instability loads were also investigated. The interaction curve was validated by comparing it with the finite element analysis results from the open literature, and it was confirmed that it gives good estimates.

Key words: Shell, Buckling, Bending

1. Introduction

Long cylindrical tubes are important structural parts for fuselages of airplanes, main spars of human powered airplanes, and members of truss and rahmen (rigid frame) structures. In particular, truss and rahmen structures constructed from carbon-fiber-reinforced-plastic (CFRP) struts are often used in space structures because of the material’s structural efficiency and hygrothermal stability. Often, long cylindrical tubes are subjected to both axial compression and a bending moment at the ends and are at a risk of collapse through the interaction between Brazier instability and Euler buckling. Brazier [1] studied the instability phenomenon of cylindrical tubes subjected to a bending moment. As the applied moment increases, the cross section becomes an oval, which induces a limit moment instability. Turkin [2] and Resse and Bert [3] studied buckling of short orthotropic sandwich cylinders under axial compression and bending by using linearized bifurcation theory. Huyan et al. [4] studied imperfect metallic and laminated short cylinders under axial compression and bending by using ANSYS nonlinear finite element computer code. Tatting and Gürdal [5] investigated the interaction between Brazier instability and shell-type local axial buckling of a cylindrical shell. Houliara and Karamanos [6] investigated local shell-type bifurcation buckling on the prebuckling ovalization of long cylindrical shells of transversely-isotropic elastic material subjected to bending. None of these studies discusses the interaction between Brazier instability and Euler buckling. Recently, however, Fam et al. [7] studied the interactions between Brazier instability, Euler buckling, and material failure in composite cylindrical tubes. They also used ANSYS nonlinear finite element computer code and analyzed seven numerical cases of Brazier instability together with Euler buckling and without material failure. In addition, they deduced an empirical linear interaction curve for Brazier instability and Euler buckling (Fig. 1). However, because this empirical linear interaction curve is based on only seven numerical results, the interaction...
curves may be of another shape when the cylindricical tube has different properties ("candidates" in Fig. 1).

![Graph showing normalized moment and axial load](image)

**Fig. 1** Candidates, Fam’s results, theoretical interaction curves.

Thus, the aim of this paper is to obtain a theoretical interaction curve for Brazier instability and Euler buckling. Brazier’s 2D analysis was extended to a 3D one by using the principle of minimum potential energy including the energy of axial compression, and a simple interaction curve was obtained. The obtained interaction solution consists of a column buckling load and a bending instability load. The column buckling load coincides with the well-known Euler buckling load, but the bending instability load is slightly different from Brazier’s instability load. Hence, the obtained bending instability load was compared with Brazier’s 2D solution and differences between them were analyzed. The obtained interaction curve was in good agreement with Fam’s numerical results.

**2. Deriving the Equation**

Consider a cylinder subject to a moment $M$ and axial compression $P$ on both ends, as shown in Fig. 2. The cross section of the cylinder deforms into an ellipse, as shown in Fig. 3.

![Diagram of a cylinder under moment and axial compression](image)

**Fig. 2** A cylinder subject to moment $M$ and axial compression $P.$
Fig. 3 The cross section ovalization.

The radial and circumference deformation of the cross section are \( w \) and \( \nu \), respectively. The potential energy of the cylinder \( \Pi \) can be written as follows:

\[
\Pi = U + W, \tag{1}
\]

where \( U \) is strain energy of the orthotropic shell that consists of the energy of axial elongation \( U_1 \) and the energy of cross section ovalization \( U_2 \):

\[
U = U_1 + U_2 \tag{2}
\]

\[
U_1 = \frac{1}{2} \int_0^l \int_0^{2\pi} E_t r^2 \kappa_1^2 r d\theta d\tau \tag{3}
\]

\[
U_2 = \frac{1}{2} \int_0^l \int_0^{2\pi} D_{tt} \kappa_2^2 r d\theta d\tau \tag{4}
\]

Here, \( d \) is the distance from an arbitrary point on the middle surface of the deformed shell to the neutral surface and \( \kappa_2 \) is the curvature of the deformed cross section.

The relationship between the curvature of the neutral surface of the cylindrical tube \( \kappa_1 \) and the displacement of the neutral surface \( w_0 \) can be written as

\[
\kappa_1 = -\frac{d^2 w_0}{dx^2}. \tag{5}
\]

The potential energy of the external load \( W \) consists of the energy of axial compression \( W_C \) and the energy of the external moment applied to both ends \( W_B \):

\[
W = W_C + W_B \tag{6}
\]

\[
W_C = -\frac{1}{2} \int_0^l \int_0^{2\pi} P \left( \frac{dw_0}{dx} \right)^2 d\tau d\theta \tag{7}
\]

\[
W_B = -2M\theta_0 \tag{8}
\]

From the kinematics relationship shown in Fig. 3, \( d \) is

\[
d = (r + w)\sin \theta + \nu \cos \theta. \tag{9}
\]

For simplicity, we shall introduce Brazier’s assumption that the deformation of the cross section contour is inextensional, that is, there is no stretching in the circumferential direction. This assumption is valid in the case of a thin cylinder, and it leads us to the following relationship.

\[
w = -r \frac{d\nu}{rd\theta}. \tag{10}
\]

Substituting Eq (10) into Eq (9), one gets

\[
d = (1 - \frac{d\nu}{rd\theta})r \sin \theta + \nu \cos \theta. \tag{11}
\]

On the other hand, the curvature of the deformed cross section \( \kappa_2 \) is obtained as follows:
\[ \kappa_s = \frac{d^2 w}{r^3 d\theta^2} - \frac{1}{r} \frac{dv}{rd\theta} = -r \frac{d^2 v}{r^3 d\theta^2} - \frac{1}{r} \frac{dv}{rd\theta}. \]  

Brazier’s 2D instability analysis was expanded to a 3D one by assuming the following deformations \( w_0 \) and \( v \).

\[ w_0 = \sum_{n=0}^{N} w_{0n} \sin((2n-1)\pi x/L), \quad \text{for} \ 0 \leq x \leq L, \]  

\[ v = v_1 \sin(2\theta) w_0 / (L \theta_0 / \pi). \]  

Brazier considered only a cross section deformation and neglected the effect of boundary conditions by assuming an infinite-length tube. We can extend Brazier’s 2D analysis to a 3D one by accounting for the deformation of the neutral surface of the tube \( w_0 \) in Eqs. (13) and (14). To obtain a simple closed-form interaction curve, only \( N=1 \) was used in Eqs. (13) and (14), i.e.,

\[ w_0 = \frac{L}{\pi} \theta_0 \sin(\pi x / L), \quad \text{for} \ 0 \leq x \leq L, \]  

\[ v = v_1 \sin(2\theta) \frac{w_0}{(L \theta_0 / \pi)} = v_1 \sin(2\theta) \sin(\pi x / L). \]  

Note that the deformation defined by Eqs. (10), (15), and (16) represents both a rigid body displacement and oval deformation of the cross section. In addition, although polynomial trial functions for \( w_0 \) are also the allowable function, they do not give good estimates. Substituting Eqs. (15) and (16) into Eqs. (1)–(8), (11) and (12) and after some manipulation, we get the following potential energy \( \Pi \).

\[ \Pi = E_x r^2 \frac{v_1^2}{r^3} (\pi + 8v_1) \theta_0^2 + \frac{9D_{32} L \pi v_1^2}{r^3} - 2M \theta_0 - P \frac{L \theta_0^2}{4}. \]  

In calculating Eq (17), we neglected the third and higher orders of \( v_1 \) and sixth and higher orders of \( \theta_0 \).

The condition of minimum potential energy is

\[ \frac{\partial \Pi}{\partial v_1} = \frac{2 \pi^2 E_x r^3}{L} \theta_0^2 + \frac{18 \pi D_{32} L v_1}{r^3} = 0. \]  

Equation (18) gives the following relationship between \( v_1 \) and \( \theta_0 \):

\[ v_1 = \frac{\pi E_x r^3 \theta_0^2}{9D_{32} L^2}. \]  

Substituting Eq. (19) into Eq. (17), the condition of minimum potential energy for \( \theta_0 \) becomes

\[ \frac{\partial \Pi}{\partial \theta_0} = -2M - P \frac{L \theta_0}{4} + \frac{\pi^2 E_x r^3}{2L} \theta_0 - \frac{2 \pi^3 (E_x r^3)^2 r^7 (\theta_0^2)}{9D_{32} L^2} = 0. \]  

The relationship between \( M \) and \( \theta_0 \) is obtained from Eq. (20) as follows:

\[ M = -P \frac{L \theta_0}{4} + \frac{\pi^2 E_x r^3}{4L} \theta_0 - \frac{2 \pi^3 (E_x r^3)^2 r^7}{9D_{32} L^2} \theta_0^3. \]  

The maximum moment is determined by following condition:

\[ \frac{\partial M}{\partial \theta_0} = \frac{P L}{4} - \frac{\pi^2 E_x r^3}{4L} - \frac{2 \pi^3 (E_x r^3)^2 r^7}{3D_{32} L^2} \theta_0^2 = 0. \]  

Solving Eq. (22) obtains the rotation angle of the neutral surface that gives the maximum (limit) moment.

\[ \theta_{0,\text{lim}} = \frac{L}{r^2} \sqrt{\frac{3D_{32}}{8E_x r} \left( \frac{P}{P_r} - 1 \right)}, \]  

where
\[ P_{cr} = \frac{\pi^3 E_r r^3 t}{L^2} = \frac{\pi^2 E_r L}{L}, \]  
(24)

\( P_{cr} \) is the Euler buckling load at both ends of a simply supported cylindrical tube. Substituting Eq. (23) into Eq. (21), the following the maximum (limit) moment \( M_{\text{lim}} \) is obtained.

\[ M = \frac{\pi^3 \sqrt{3 E_s t D_{22}} \left(1 - \frac{P}{P_{cr}}\right)^{3/2}}{12 \sqrt{2}} \]
\[ r = M_{\text{lim}} \left(1 - \frac{P}{P_{cr}}\right)^{3/2} \]  
(25)

where
\[ M_{\text{lim}} = \frac{\pi^3 \sqrt{3 E_s t D_{22}}}{12 \sqrt{2}} r \]  
(26)

Note that for a homogeneous-through thickness shell, i.e., the layup has enough layers, \( D_{22} \) can be written as
\[ D_{22} = \frac{E_r t^3}{12(1 - v_x v_y)} \]  
(27)

Substituting Eqs. (27) into Eq. (26) gives
\[ M_{\text{lim}} = \frac{E_r t^3}{24 \sqrt{2(1 - v_x v_y)}} = 0.914 \sqrt{\frac{E_r}{1 - v_x v_y}} \]  
(28)

In addition, when \( P = 0 \), the amplitude of \( w \) (amplitude of ovalization) and curvature of the neutral surface \( \kappa_1 \) are
\[ w_{\text{lim}} \bigg|_{\theta = 0, \phi = L/2} = \frac{dv}{d\phi} \bigg|_{\theta = 0, \phi = L/2} = \frac{\pi}{12} r \]
\[ w_{\text{lim}} \bigg|_{\theta = 0, \phi = L/2} = 0.262r \]  
(29)

\[ \kappa_{\text{lim},1} = \frac{d^2 w_0}{dx^2} \bigg|_{\theta = 0, \phi = L/2} = \frac{\pi \sqrt{3}}{2 \sqrt{2} r^2} \sqrt{\frac{D_{22}}{E_s t}} = \frac{\pi}{4 \sqrt{2} r^2} \sqrt{\frac{E_r}{1 - v_x v_y}} \]  
(30)

\[ \kappa_{\text{lim},1} = 0.555 \frac{t}{r^2} \sqrt{\frac{E_r}{1 - v_x v_y}} \]

3. Discussion

Equation (28) corresponds to the following extended Brazier’s limit moment for an orthotropic cylindrical tube as determined by Hayashi [8] and Baruch [9] et al.

\[ M_{\text{lim,br}} = \frac{4 \sqrt{6} \sqrt{E_s t D_{22}}}{9} r = \frac{2 \sqrt{2} \sqrt{E_r t}}{9 \sqrt{(1 - v_x v_y)}} = 0.987 \sqrt{\frac{E_r}{1 - v_x v_y}} \]  
(31)

The constant 0.914 in Eq. (28) is 7.4% lower than Brazier’s 0.987.

Equations (29) and (30) correspond to the following Brazier’s maximum ovalization and extended Brazier’s limit curvature for an orthotropic cylindrical tube obtained by Hayashi [8] and Baruch [9] et al.

\[ w_{0,\text{lim,br}} \bigg|_{\theta = 0, \phi = L/2} = \frac{2}{9} r = 0.222r \]  
(32)

\[ \kappa_{1,\text{lim,br}} = \frac{8}{3} \sqrt{\frac{D_{22}}{E_s t r^2}} = \frac{2}{9} \sqrt{\frac{E_r}{1 - v_x v_y}} \]  
(33)

The ovalization on the limit moment (Eq. (29)) obtained from the 3D theory is 18% larger than Brazier’s ovalization (Eq. (32)). In addition, the limit curvature (Eq. (30)) obtained from the 3D theory is 15% smaller than Brazier’s limit curvature (Eq. (33)). These differences are caused by the differences between the 3D and 2D theories. This assertion is easily confirmed by noting that if the following trial function (constant ovalization and
quadratic deformation of the neutral surface, which is essentially Brazier’s 2D assumption

\[ w_0 = L \theta_0 \left( 1 - \frac{x}{L} \right) \frac{x}{L}, \quad \text{for } 0 \leq x \leq L \]  

(34)

\[ \nu = \nu_0 \sin(2\theta) \]  

(35)

were used, the same values as Brazier’s limit moment, limit curvature, and maximum ovalization would be obtained.

Note that if a quadratic polynomial trial function is used in \( w_0 \) (Eq. (15)), similar values for the limit moment, limit curvature, and maximum ovalization are obtained. In addition, a three-halves power relationship between \( P_{cr} \) and \( M_{lim} \) which is the same form as Eq. (25), is obtained. However, a quadratic polynomial trial function in \( w_0 \) gives an inaccurate Euler buckling load; hence, we used a sinusoidal trial function for \( w_0 \) instead.

Figure 4 compares the 3D solution for the moment-curvature plot for an isotropic cylindrical tube (Eq. (21) with \( P=0 \)) with Brazier’s solution.

The moment is normalized with \( \{E_x/E_y/(1-\nu_x \nu_y)\}^{1/2}r \) and \( \kappa_t \) is normalized with \( \{E_x/E_y (1-\nu_x \nu_y)\}^{1/2}/r^2 \) for the sake of comparison. Both curves have similar nonlinear relationships but the limit moments and limit curvature of the 3D solution are slightly different than those of Brazier’s solution.

Note that the length of the tube (i.e., the boundary condition) has no influence on the limit moment (Eq. (28)), the maximum ovalization (Eq. (29)), or limit curvature (Eq. (30)) in spite of the 3D analysis using Eqs. (15) and (16). This problem should be investigated in the future. The effect of boundary condition for Brazier instability is only important for short cylindrical tubes. But Euler buckling is not important for such a short cylindrical tubes. Hence the effect of boundary condition for Brazier instability is not important because the major aim of this paper is the investigation of interaction between Euler buckling and Brazier instability.

Figure 5 shows the effect of axial compression \( P \) on the moment-curvature relationship. In this figure, \( M \) is normalized with \( M_{lim} \) (Eq. (28)) and \( \kappa_t \) is normalized with \( \kappa_{t,lim} \) (Eq. (30)). The limit moment decreases as \( P \) increases. In addition, the limit curvature is not a constant of \( P \); it decreases as \( P \) increases.

Equation (25) can be rewritten as

\[ \frac{M}{M_{lim}} = \left( 1 - \frac{P}{P_{cr}} \right)^{(3/2)} \]  

(36)
Equation (36) is a simple closed-form solution, and it clearly shows that a nonlinear interaction exists in a cylindrical tube subjected to both axial compression and bending. The interaction curve in Fig. 1 shows that the interaction results in a slightly lower estimate than Fam’s linear interaction.

4. Comparison with FE Analyses

Figures 6–9 compare the theoretical curves given by Eq. (36) and Fam’s FE analyses [7]. The specimens were GFRP cylindrical tubes. Table 1 lists their material properties, and Table 2 shows the layup sequence and dimensions. Good agreement between the theoretical curves and the FE analyses is evident in all figures.

Equation (36) gives a slightly lower estimate than Fam’s FE analysis for the maximum moment. This means that Eq. (36) gives a conservative estimate suitable for design purposes. Note that the analytical results from Eq. (36) on the bending instability moment loads for D10 (in Fig. 8) and E10 (in Fig. 9) are significantly lower than those of Fam’s analysis. Because those layups have only +10- and –10-degree layers and those of the circumferential stiffnesses are significantly smaller, the reinforcement of the end caps in Fam’s FE analysis may increase the bending instability loads and the differences.
Fig. 7 Interaction curves for specimen B31.

Fig. 8 Interaction curves for specimens D10 and D45.

Fig. 9 Interaction curves for specimen E10.
Table 1 Material properties

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<tr>
<td>$E_{11}$</td>
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<tr>
<td>$E_{22}$</td>
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<tr>
<td>$v_{12}$</td>
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</tr>
</tbody>
</table>

Table 2 Layup sequences and dimensions of specimens.

<table>
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<tr>
<th>Specimen</th>
<th>Layup sequences</th>
<th>$D$ [mm]</th>
<th>$t$ [mm]</th>
<th>$L$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A31</td>
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<td>300</td>
<td>2.4</td>
<td>6</td>
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<tr>
<td>A11</td>
<td>[90/0/90/0]</td>
<td>300</td>
<td>2.4</td>
<td>6</td>
</tr>
<tr>
<td>A13</td>
<td>[90/90/90/0]</td>
<td>300</td>
<td>2.4</td>
<td>6</td>
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<tr>
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<td>300</td>
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<td>[10/–10/10/–10]</td>
<td>300</td>
<td>7.5</td>
<td>6</td>
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</tbody>
</table>

In addition, it should be noted that Eq. (36) is independent of the material properties and dimensions of the cylindrical tube. It is affected by only $M_{\text{lim}}$ and $P_{\text{cr}}$.

5. Conclusion

A simple closed-form interaction curve for Brazier instability and Euler buckling was obtained by using a 3D version of Brazier’s analysis and the principle of minimum potential energy including the energy of axial compression. A slightly lower (–7.4%) limit moment than Brazier’s was obtained, but the equation did not include the effect of length in spite of using a 3D deformation function. The obtained theoretical interaction curve is in good agreement with Fam’s FE-analysis and slightly lower in moment. The interaction curve is closed form, can be easily calculated, and gives a conservative estimate. Hence, it is useful for design purposes.

Nomenclature

- $D_{22}$ = circumferential bending stiffness
- $E_x, E_y$ = Young’s moduli for each direction
- $d$ = distance from the middle surface of the deformed shell to a neutral surface
- $I$ = moment of inertia of cylindrical tube ($\pi r^4 t$)
- $L$ = length of cylindrical tube
- $M$ = moment
- $M_{\text{lim}}$ = limit moment
- $N, n$ = integers
- $P$ = axial compression
- $P_{\text{cr}}$ = Euler buckling load
- $r$ = radius of cylindrical tube
- $U, U_1, U_2$ = strain energies
- $W, W_{\text{th}}, W_{\text{C}}$ = potential energies of the external load
- $v, w$ = deformation of cross section
- $v_1, w_{\text{lin}}$ = amplitudes
- $w_0$ = displacement of neutral surface
- $x$ = axial coordinate
- $\kappa_1$ = curvature of neutral surface
- $\kappa_2$ = curvature of deformed cross section
ν_x, ν_y = Poisson’s ratios for each direction
Π = total potential energy
θ = circumferential coordinate
θ_0 = rotation angle of neutral surface

References


