Analysis of In-Plane Problems for an Isotropic Elastic Medium with Many Circular Elastic Inclusions*

Mutsumi MIYAGAWA **, Takuo SUZUKI **, Takanobu TAMIYA ** and Jyo SHIMURA ***

** Dept. of Creative Manufacturing Tokyo Metropolitan College of Industrial Technology, Arakawa Campus
8-17-1, Minami-senju, Arakawa-ku, Tokyo 116-0003, Japan

E-mail: miyagawa@acp.metro-cit.ac.jp

*** Tokyo National College of Technology
1220-2, Kunugida-machi, Hachioji-shi, Tokyo 193-0997, Japan

Abstract

This paper presents general solutions for problems involving many circular elastic inclusions that are perfectly bonded to an elastic medium(matrix) of infinite extent under in-plane deformation. These many elastic inclusions may have different radii, central points and possess different elastic properties. The matrix is assumed to be subjected to arbitrary loading, for example, by uniform stresses at infinity. The solutions were obtained through iterations of the Möbius transformation as series with an explicit general term involving the complex potential functions of the corresponding homogeneous problem. This procedure is referred to as heterogenization. Using these solutions, several numerical examples are presented graphically.

Key words : Elasticity, In-Plane Problem, Many Circular Elastic Inclusions, Uniform Stresses

1. Introduction

A number of studies have examined the problems associated with disturbances around a single circular inclusion under in-plane loading, such as loading due to uniform stresses or a concentrated force at an arbitrary point. Therefore, these problems have many applications in engineering fields. These inclusion problems have proved to be very useful for mechanical analysis.

These problems have been developed further in order to observe the interacting disturbances for multiple circular inclusions. However, these techniques have been applied using different numerical analysis methods such as the finite element method (FEM) or the boundary element method (BEM). So, for example, if one engineer is an expert in FEM analysis of a model, while another engineer is not, their results will not be the same. General solutions of multi-inclusion problems for the in-plane situation using the theory of elasticity have not yet been produced. The governing equation for the anti-plane problem is a harmonic equation, but for the in-plane problem, it is a biharmonic equation. Thus, it is difficult to satisfy the continuity of stresses and displacement on the boundary.

The purpose of the present study is to apply the reflection principle of Moriguchi⁽¹⁾, Dunders⁽²⁾ and Sendekyj⁽³⁾ who investigated a single hole or inclusion in the in-plane problems, and the techniques of Honein⁽⁴⁾ and Hirashima⁽⁵⁾⁻⁽⁷⁾ to consider anti-plane multi-inclusion problems. We obtained general solutions⁽⁸⁾⁽⁹⁾ for up to two circular inclusions. Using these techniques, we expanded these problems to cases involving many circular inclusions. To analyze the problem of rigid inclusions or holes⁽¹⁰⁾, we need only change the elastic modulus of the inclusion to ∞ (Rigid inclusions) or to 0 (Holes). In the present study, these inclusions have arbitrary arrangements, elastic modulus and sizes inside the matrix. Using this explicit

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general solution, we present several numerical examples under uniform stresses at infinity.

2. Fundamental Equation and General Solution

2.1. Formulation of elastostatics for in-plane problems

In this section, we review the fundamental formulation of in-plane elastostatics and present the notation used herein. We consider the complex region $z = x + iy$, where i is the imaginary unit ($i = \sqrt{-1}$), to be infinite. Under in-plane deformations, there exist displacements u_x and u_y and stresses σ_x , σ_y , and τ_{xy} , which are obtained in Cartesian coordinates only.

The formulation used to find the stresses and displacements is satisfied by the complex potential functions $\varphi(z)$ and $\psi(z)$, which are also used in the techniques of Moriguchi⁽¹⁾.

$$u_x - iu_y = \frac{1}{2G_M} \left[\kappa_M \overline{\varphi(z)} - \{\bar{z}\varphi'(z) + \psi'(z)\} \right]. \quad (1)$$

$$-P_y - iP_x = \overline{\varphi'(z)} + \{\bar{z}\varphi''(z) + \psi''(z)\}. \quad (2)$$

where a prime indicates differentiation with respect to the complex variables z and κ_M as follows:

$$\kappa_M = \begin{cases} (3 - \nu_M)/(1 + \nu_M) & \text{Plane Stress} \\ 3 - 4\nu_M & \text{Plane Strain} \end{cases} \quad (3)$$

where G_M and ν_M are the shear modulus and Poisson's ratio for the matrix, respectively. P_x and P_y indicate the resultant forces that act from right to left along an arbitrary course from point A to point B in the matrix.

$$P_x = \int_A^B (\sigma_x dy - \tau_{xy} dx), \quad P_y = \int_A^B (\tau_{xy} dy - \sigma_y dx). \quad (4)$$

Hence, the stresses are obtained as follows:

$$\left. \begin{aligned} \sigma_x &= 2\text{Re}[\varphi'(z)] - \text{Re}[\bar{z}\varphi''(z) + \psi''(z)], \\ \sigma_y &= 2\text{Re}[\varphi'(z)] + \text{Re}[\bar{z}\varphi''(z) + \psi''(z)], \\ \tau_{xy} &= \text{Im}[\bar{z}\varphi''(z) + \psi''(z)], \end{aligned} \right\} \quad (5)$$

where $\text{Re}[\]$ and $\text{Im}[\]$ are the real and imaginary parts, respectively, of the complex function in parentheses, and the overbar indicates complex conjugation.

2.2. General solution in the presence of a single circular hole

We first investigated the problem in the presence of a single circular elastic inclusion disturbing the in-plane loading, which was given by the complex potential functions $\phi(z)$ and $\psi(z)$ in the matrix. We considered the heterogeneous problem of the j^{th} elastic circular inclusion perfectly bonded to an elastic matrix of infinite extent. The matrix and the boundary produce an in-plane deformation, as shown in Fig. 1. We set the general boundary conditions of the tractions and displacements on the boundary L_j (i.e., $z = z_j + a_j e^{i\theta}$) between the j^{th} inclusion and the matrix, where a_j and $z_j (= (0, 0))$ are the radius and origin, respectively, of the j^{th} inclusion.

We will call the required continuity of the tractions and displacements the "boundary conditions" along the circular interface.

Boundary conditions at L_j :

$$\left. \begin{aligned} P_x^{(M)} &= P_x^{(j)}, & P_y^{(M)} &= P_y^{(j)}, \\ u_x^{(M)} &= u_x^{(j)}, & u_y^{(M)} &= u_y^{(j)}. \end{aligned} \right\} \quad (6)$$

where, the upper subscripts (M) and (j) mean the matrix and j^{th} inclusion, respectively. In this subsection, we use $j = 1$ because there is only one inclusion. The most general complexes for each region can be written as follows:

Fig. 1 Geometry of an infinite elastic medium with a single hole or rigid inclusion.

Matrix:

$$\varphi_M(z) = \varphi(z) + \hat{f}(z). \quad (7)$$

$$\chi_M(z) = \chi_m(z) + \hat{g}(z). \quad (8)$$

j^{th} inclusion:

$$\varphi_{I_j}(z) = \varphi(z) + h(z). \quad (9)$$

$$\chi_{I_j}(z) = \chi_j(z) + k(z). \quad (10)$$

Based on Eqs.(1) and (2), we set $\chi_{I_j}(z)$ in Eq.(10) as a first assistant function of the j^{th} inclusion.

$$\chi_{I_j}(z) = \bar{z}\varphi'_{I_j}(z) + \psi'_{I_j}(z). \quad (11)$$

When the matrix is an isotropic material, using the mirror projection of a point that is produced by Moriguchi⁽¹⁾ on the boundary $|z| = a_j$, we set $\chi_M(z)$ in Eq.(8) as a first assistant function of the matrix.

$$\chi_M(z) = \frac{a_j^2}{z}\varphi'_M(z) + \psi'_M(z). \quad (12)$$

Also, we set $\chi_j(z)$ in Eq.(10) as a second assistant function of the j^{th} inclusion.

$$\chi_j(z) = \gamma_j \bar{z}\varphi'(z) + \psi'(z). \quad (13)$$

Eq.(11) is continuous with Eq.(12) on the boundary L_j . Therefore, we set $\chi_M(z)$ in Eq.(8) as a second assistant function for the matrix in the following equation.

$$\chi_m(z) = \gamma_j \frac{a_j^2}{z}\varphi'(z) + \psi'(z). \quad (14)$$

where, γ_j is the unknown constant. Note that f, g, h , and k in Eqs. (7) ~ (10) are determinate functions, after we establish $\hat{f}(z)$ and $\hat{g}(z)$ in the following equations using the principle of mirror projection:

$$\hat{f}(z) = \overline{f(a_j^2/\bar{z})}, \quad \hat{g}(z) = \overline{g(a_j^2/\bar{z})}. \quad (15)$$

To satisfy the boundary conditions of Eq.(6) on the interface L_j , we continue the analysis of the complex potential functions $\varphi(z)$, $\chi(z)$ in the following equations. There is a detailed derivation in reference⁽⁹⁾. Finally, we obtain the relations.

Matrix:

$$\varphi_M(z) = \varphi(z) + \alpha_j \overline{\chi_m(a_j^2/\bar{z})}. \quad (16)$$

$$\chi_M(z) = \chi_m(z) + \beta_j \overline{\varphi(a_j^2/\bar{z})}. \quad (17)$$

j^{th} inclusion:

$$\varphi_{I_j}(z) = \varphi(z) + \beta_j \varphi(z). \quad (18)$$

$$\chi_{I_j}(z) = \chi_j(z) + \alpha_j \chi_j(z). \quad (19)$$

Eqs. (16)~(19) are obtained when the center of the j^{th} inclusion coincides with the origin. This problem is reduced to finding f, g, h , and k such that the continuities given by Eq. (6) are satisfied.

Matrix:

$$\varphi_M(z) = \varphi(z) + \alpha_j \overline{\chi_m(A_j z)}. \quad (20)$$

$$\chi_M(z) = \chi_m(z) + \beta_j \overline{\varphi(A_j z)}. \quad (21)$$

j^{th} inclusion:

$$\varphi_{I_j}(z) = \varphi(z) + \beta_j \varphi(z). \quad (22)$$

$$\chi_{I_j}(z) = \chi_j(z) + \alpha_j \chi_j(z). \quad (23)$$

Where, α_j, β_j and γ_j are given as

$$\alpha_j = \frac{G_j - G_M}{\kappa_M G_j + G_M}, \quad \beta_j = \frac{\kappa_M G_j - \kappa_j G_M}{G_j + \kappa_j G_M}, \quad \gamma_j = \frac{1 + \beta_j}{1 + \alpha_j}. \quad (24)$$

$$\kappa_M = \begin{cases} (3 - \nu_M)/(1 + \nu_M) & \text{Plane Stress} \\ 3 - 4\nu_M & \text{Plane Strain} \end{cases} \quad (25)$$

G_j and ν_j are the shear modulus and Poisson's ratio for the j^{th} inclusion, respectively. When the j^{th} inclusion is a rigid inclusion or hole, we only have to set G_j to a special value in the following.

$$\begin{cases} \alpha_j = 1/\kappa_M, \beta_j = \kappa_M. & \text{Rigid inclusion } (G_j \rightarrow \infty) \\ \alpha_j = -1, \beta_j = -1. & \text{Hole } (G_j \rightarrow 0) \end{cases} \quad (26)$$

In Eqs.(20) ~ (23), we consider the problem in the single circular inclusion ($j = 1$) in the matrix. We have a strong conviction that these solutions coincide with the solution of Dunders⁽²⁾ and Sendeckyj⁽³⁾. The fundamental complex potential functions $\phi(z)$ and $\psi(z)$ are given by the arbitrary loading described in a later section. To this end, we define

$$A_j z = \frac{a_j^2}{\bar{z} - \bar{z}_j} + z_j, \quad (j = 1, 2). \quad (27)$$

Moreover, A_j specifies the operator with respect to the complex variable z . Normally, the left-hand side of Eq. (27) would be written as $A_j(z)$. However, for convergence, in the present paper we denote $A_j z$ as in Eq. (27). Thus, $A_i A_j z$, for example, is expressed as follows:

$$\begin{aligned} A_i A_j z &= A_i \left[A_j(z) \right] \\ &= \frac{a_i^2}{A_j(z) - \bar{z}_i} + z_i = \frac{a_i^2}{\left(\frac{a_j^2}{z - z_j} + \bar{z}_j \right) - \bar{z}_i} + z_i. \end{aligned} \quad (28)$$

We treat the above operation similarly in the following section.

Fig. 2 Infinite elastic matrix with a single inclusion under uniform stresses.

2.3. Problem in the presence of a single circular inclusion under uniform stresses

In this section, we consider the problem of disturbing the uniform stresses σ_x^∞ , σ_y^∞ , and τ_{xy}^∞ at infinity in Fig. 2. The fundamental complex potential functions $\phi(z)$, $\psi'(z)$ are given at infinity $|z| = \infty$ as

$$\varphi(z) = \tau^* z, \quad \psi'(z) = 2\tau^{**} z. \quad (29)$$

where

$$\tau^* = \frac{\sigma_x^\infty + \sigma_y^\infty}{4(1 + \alpha_1 \gamma_1)}, \quad \tau^{**} = \frac{\sigma_y^\infty - \sigma_x^\infty}{4} + i \frac{\tau_{xy}^\infty}{2}. \quad (30)$$

These functions do not have a singularity inside the region $a_j < |z| < \infty$ ($j = 1$). We next considered the functions obtained by substituting Eq. (29) into Eqs. (20) and (23), that coincide in the initial conditions σ_x^∞ , σ_y^∞ , and τ_{xy}^∞ at infinity. The second assistant function $\chi_1(z)$, defined in Eq. (13) at infinity $|z| = \infty$, reduces to

$$\chi_1(z) = \gamma_1 \bar{z} \phi'(z) + \psi'(z). \quad (31)$$

From Eq.(31), a second assistant function $\chi_m(z)$ for the matrix is given as

$$\chi_m(z) = \gamma_1 \frac{a_1^2}{z} \phi'(z) + \psi'(z). \quad (32)$$

The functions obtained by substituting Eq. (31) into Eqs. (20) and (23) are general solutions in this subsection, and we obtain the following:

Matrix:

$$\varphi_M(z) = (\tau^* z + \alpha_1 \gamma_1 \bar{\tau}^*) z + 2\alpha_1 \overline{\tau^{**} A_1(z)}. \quad (33)$$

$$\chi_M(z) = 2\tau^{**} z + (\gamma_1 \tau^* + \beta_1 \bar{\tau}^*) \overline{A_1(z)}. \quad (34)$$

1st inclusion:

$$\varphi_{I_1}(z) = (1 + \beta_1) \tau^* z. \quad (35)$$

$$\chi_{I_1}(z) = (1 + \beta_1) \tau^* \bar{z} + 2(1 + \alpha_1) \tau^{**} z. \quad (36)$$

Fig. 3 Geometry of an infinite elastic matrix with many circular elastic inclusions.

2.4. General solution in the presence of many circular inclusions

In this section, we use the solution presented in the previous section as a starting point for obtaining the solution for many circular inclusions that have radii a_j and origins z_j ($j = 1, 2, \dots, Q$), as shown in Fig. 3, where G_j and ν_j are the shear modulus and Poisson's ratio for the j^{th} inclusion, respectively. To analyze the problem of rigid inclusions or holes, we need only change the coefficient $G_j \rightarrow \infty$ (Rigid inclusions) or to $G_j \rightarrow 0$ (Holes).

This matrix is given by the complex potential functions $\phi(z)$ and $\psi(z)$. We set the general boundary conditions given by Eq.(6), of the tractions and displacements. For the purpose of the multi-inclusion problem, we first considered the problem with three holes (i.e. $Q = 3$) as a simple case. After that, we produced the general solution for many holes (Q is arbitrary number). Using the same technique, we could satisfy the continuities given by Eqs. (1) and (2) for the boundary L_j . We first set the general functions $\hat{f}_1(z)$, $\hat{g}_1(z)$, $h_1(z)$ and $k_1(z)$ on L_1 as follows:

Matrix:

$$\varphi_M(z) = \varphi(z) + \hat{f}_1(z), \quad (37)$$

$$\chi_M(z) = \chi_m(z) + \hat{g}_1(z). \quad (38)$$

1st Inclusion:

$$\varphi_{I_1}(z) = \varphi(z) + h_1(z), \quad (39)$$

$$\chi_{I_1}(z) = \chi_1(z) + k_1(z). \quad (40)$$

2nd Inclusion:

$$\varphi_{I_2}(z) = \varphi(z) + \hat{f}_1(z), \quad (41)$$

$$\chi_{I_2}(z) = \chi_2(z) + \hat{g}_1(z). \quad (42)$$

3rd Inclusion:

$$\varphi_{I_3}(z) = \varphi(z) + \hat{f}_1(z), \quad (43)$$

$$\chi_{I_3}(z) = \chi_3(z) + \hat{g}_1(z). \quad (44)$$

These functions reduce to finding $\hat{f}_1(z)$, $\hat{g}_1(z)$, $h_1(z)$ and $k_1(z)$ such that the continuities on L_1 are satisfied. The following equations are obtained using sets $\hat{f}_2(z)$, $\hat{g}_2(z)$, $h_2(z)$ and $k_2(z)$ on L_2 :

Matrix:

$$\varphi_M(z) = \varphi(z) + \alpha_1 \overline{\chi_m(A_1 z)} + \hat{f}_2(z), \quad (45)$$

$$\chi_M(z) = \chi_m(z) + \beta_1 \overline{\varphi(A_1 z)} + \hat{g}_2(z). \quad (46)$$

1st Inclusion:

$$\varphi_{I_1}(z) = \varphi(z) + \beta_1 \overline{\varphi(A_1 z)} + \hat{f}_2(z), \quad (47)$$

$$\chi_{I_1}(z) = \chi_1(z) + \alpha_1 \chi_1(A_1 z) + \hat{g}_2(z). \quad (48)$$

2nd Inclusion:

$$\varphi_{I_2}(z) = \varphi(z) + \alpha_1 \overline{\chi_2(A_1 z)} + h_2(z), \quad (49)$$

$$\chi_{I_2}(z) = \chi_2(z) + \beta_1 \overline{\varphi(A_1 z)} + k_2(z). \quad (50)$$

3rd Inclusion:

$$\varphi_{I_3}(z) = \varphi(z) + \alpha_1 \overline{\chi_3(A_1 z)} + \hat{f}_2(z), \quad (51)$$

$$\chi_{I_3}(z) = \chi_3(z) + \beta_1 \overline{\varphi(A_1 z)} + \hat{g}_2(z). \quad (52)$$

These functions reduce to finding $\hat{f}_2(z)$, $\hat{g}_2(z)$, $h_2(z)$ and $k_2(z)$ such that the continuities on L_2 are satisfied. The following equations are obtained using sets $\hat{f}_3(z)$, $\hat{g}_3(z)$, $h_3(z)$ and $k_3(z)$ on L_3 :

Matrix:

$$\varphi_M(z) = \varphi(z) + \alpha_1 \overline{\chi_m(A_1 z)} + \alpha_2 \overline{\chi_m(A_2 z)} + \beta_1 \alpha_2 \overline{\varphi(A_1 A_2 z)} + \hat{f}_3(z), \quad (53)$$

$$\chi_M(z) = \chi_m(z) + \beta_1 \overline{\varphi(A_1 z)} + \beta_2 \overline{\varphi(A_2 z)} + \alpha_1 \beta_2 \overline{\chi_m(A_1 A_2 z)} + \hat{g}_3(z). \quad (54)$$

1st Inclusion:

$$\varphi_{I_1}(z) = \varphi(z) + \beta_1 \overline{\varphi(A_1 z)} + \alpha_2 \overline{\chi_1(A_2 z)} + \beta_1 \alpha_2 \overline{\varphi(A_1 A_2 z)} + \hat{f}_3(z), \quad (55)$$

$$\chi_{I_1}(z) = \chi_1(z) + \alpha_1 \chi_1(A_1 z) + \beta_2 \overline{\varphi(A_2 z)} + \alpha_1 \beta_2 \overline{\chi_1(A_1 A_2 z)} + \hat{g}_3(z). \quad (56)$$

2nd Inclusion:

$$\varphi_{I_2}(z) = \varphi(z) + \alpha_1 \overline{\chi_2(A_1 z)} + \beta_2 \overline{\varphi(z)} + \alpha_1 \beta_2 \overline{\chi_2(A_1 z)} + \hat{f}_3(z), \quad (57)$$

$$\chi_{I_2}(z) = \chi_2(z) + \beta_1 \overline{\varphi(A_1 z)} + \alpha_2 \overline{\chi_2(A_2 z)} + \beta_1 \alpha_2 \overline{\varphi(A_1 z)} + \hat{g}_3(z). \quad (58)$$

3rd Inclusion:

$$\varphi_{I_3}(z) = \varphi(z) + \alpha_1 \overline{\chi_3(A_1 z)} + \alpha_2 \overline{\chi_3(A_2 z)} + \beta_1 \alpha_2 \overline{\varphi(A_1 A_2 z)} + h_3(z), \quad (59)$$

$$\chi_{I_3}(z) = \chi_3(z) + \beta_1 \overline{\varphi(A_1 z)} + \beta_2 \overline{\varphi(A_2 z)} + \alpha_1 \beta_2 \overline{\chi_3(A_1 A_2 z)} + k_3(z). \quad (60)$$

Applying the continuity on L_3 , we found $\hat{f}_3(z)$, $\hat{g}_3(z)$, $h_3(z)$ and $k_3(z)$. Note that the boundary condition on L_1 is not satisfied for L_2 and L_3 by the previous steps. For this reason, we may set $\hat{f}_4(z)$, $\hat{g}_4(z)$, $h_4(z)$ and $k_4(z)$ to satisfy the continuity on L_1 .

Matrix:

$$\begin{aligned} \varphi_M(z) = & \varphi(z) + \alpha_1 \overline{\chi_m(A_1 z)} + \alpha_2 \overline{\chi_m(A_2 z)} + \beta_1 \alpha_2 \overline{\varphi(A_1 A_2 z)} \\ & + \alpha_3 \overline{\chi_m(A_3 z)} + \beta_1 \alpha_3 \overline{\varphi(A_1 A_3 z)} + \beta_2 \alpha_3 \overline{\varphi(A_2 A_3 z)} \\ & + \alpha_1 \beta_2 \alpha_3 \overline{\chi_m(A_1 A_2 A_3 z)} + \hat{f}_4(z), \end{aligned} \quad (61)$$

$$\begin{aligned} \chi_M(z) = & \chi_m(z) + \beta_1 \overline{\varphi(A_1 z)} + \beta_2 \overline{\varphi(A_2 z)} + \alpha_1 \beta_2 \overline{\chi_m(A_1 A_2 z)} \\ & + \beta_3 \overline{\varphi(A_3 z)} + \alpha_1 \beta_3 \overline{\chi_m(A_1 A_3 z)} + \alpha_2 \beta_3 \overline{\chi_m(A_2 A_3 z)} \\ & + \beta_1 \alpha_2 \beta_3 \overline{\varphi(A_1 A_2 A_3 z)} + \hat{g}_4(z). \end{aligned} \quad (62)$$

1st Inclusion:

$$\begin{aligned}\varphi_{I_1}(z) = & \varphi(z) + \beta_1 \overline{\varphi(A_1 z)} + \alpha_2 \overline{\chi_1(A_2 z)} + \beta_1 \alpha_2 \overline{\varphi(A_1 A_2 z)} \\ & + \alpha_3 \overline{\chi_1(A_3 z)} + \beta_1 \alpha_3 \overline{\varphi(A_1 A_3 z)} + \beta_2 \alpha_3 \overline{\varphi(A_2 A_3 z)} \\ & + \alpha_1 \beta_2 \alpha_3 \overline{\varphi(A_1 A_2 A_3 z)} + h_4(z),\end{aligned}\quad (63)$$

$$\begin{aligned}\chi_{I_1}(z) = & \chi_1(z) + \alpha_1 \chi_1(A_1 z) + \beta_2 \overline{\varphi(A_2 z)} + \alpha_1 \beta_2 \overline{\chi_1(A_1 A_2 z)} \\ & + \beta_3 \overline{\varphi(A_3 z)} + \alpha_1 \beta_3 \overline{\chi_1(A_1 A_3 z)} + \alpha_2 \beta_3 \overline{\chi_1(A_2 A_3 z)} \\ & + \beta_1 \alpha_2 \beta_3 \overline{\varphi(A_1 A_2 A_3 z)} + k_4(z).\end{aligned}\quad (64)$$

2st Inclusion:

$$\begin{aligned}\varphi_{I_2}(z) = & \varphi(z) + \alpha_1 \overline{\chi_2(A_1 z)} + \beta_2 \overline{\varphi(z)} + \alpha_1 \beta_2 \overline{\chi_2(A_1 z)} \\ & + \alpha_3 \overline{\chi_2(A_3 z)} + \beta_1 \alpha_3 \overline{\varphi(A_1 A_3 z)} + \beta_2 \alpha_3 \overline{\varphi(A_2 A_3 z)} \\ & + \alpha_1 \beta_2 \alpha_3 \overline{\varphi(A_1 A_2 A_3 z)} + \hat{f}_4(z),\end{aligned}\quad (65)$$

$$\begin{aligned}\chi_{I_2}(z) = & \chi_2(z) + \beta_1 \overline{\varphi(A_1 z)} + \alpha_2 \chi_2(A_2 z) + \beta_1 \alpha_2 \overline{\varphi(A_1 z)} \\ & + \beta_3 \overline{\varphi(A_3 z)} + \alpha_1 \beta_3 \overline{\chi_2(A_1 A_3 z)} + \alpha_2 \beta_3 \overline{\chi_2(A_2 A_3 z)} \\ & + \beta_1 \alpha_2 \beta_3 \overline{\varphi(A_1 A_2 A_3 z)} + \hat{g}_4(z).\end{aligned}\quad (66)$$

3st Inclusion:

$$\begin{aligned}\varphi_{I_3}(z) = & \varphi(z) + \alpha_1 \overline{\chi_3(A_1 z)} + \alpha_2 \overline{\chi_3(A_2 z)} + \beta_1 \alpha_2 \overline{\varphi(A_1 A_2 z)} \\ & + \beta_3 \overline{\varphi(z)} + \alpha_1 \beta_3 \overline{\chi_3(A_1 z)} + \alpha_2 \beta_3 \overline{\chi_3(A_2 z)} \\ & + \beta_1 \alpha_2 \beta_3 \overline{\varphi(A_1 A_2 z)} + \hat{f}_4(z),\end{aligned}\quad (67)$$

$$\begin{aligned}\chi_{I_3}(z) = & \chi_3(z) + \beta_1 \overline{\varphi(A_1 z)} + \beta_2 \overline{\varphi(A_2 z)} + \alpha_1 \beta_2 \overline{\chi_3(A_1 A_2 z)} \\ & + \alpha_3 \overline{\chi_3(z)} + \beta_1 \alpha_3 \overline{\varphi(A_1 z)} + \beta_2 \alpha_3 \overline{\varphi(A_2 z)} \\ & + \alpha_1 \beta_2 \alpha_3 \overline{\varphi(A_1 A_2 z)} + \hat{g}_4(z).\end{aligned}\quad (68)$$

We applied the continuity on L_1 , repeating the previous steps and obtaining these additional terms each time. In this way, we could find the following explicit solution of the in-plane problem in the presence of many circular elastic inclusions. To this end, we used

Matrix:

$$\varphi_M(z) = \varphi(z) + \sum_{n=1}^{+\infty} \omega_{p^{(n)}q^{(n)}}^{M*} \varphi(M_{p^{(n)}q^{(n)}}^M z) + \sum_{n=0}^{+\infty} \alpha_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{M*} \overline{\chi_m(A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^M z)}, \quad (69)$$

$$\chi_M(z) = \chi_m(z) + \sum_{n=1}^{+\infty} \omega_{p^{(n)}q^{(n)}}^{M**} \chi_m(M_{p^{(n)}q^{(n)}}^M z) + \sum_{n=0}^{+\infty} \beta_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{M**} \overline{\varphi(A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^M z)}. \quad (70)$$

jth Inclusion:

$$\varphi_{I_j}(z) = (1 + \beta_j) \left\{ \varphi(z) + \sum_{n=1}^{+\infty} \omega_{p^{(n)}q^{(n)}}^{j*} \varphi(M_{p^{(n)}q^{(n)}}^j z) + \sum_{n=0}^{+\infty} \alpha_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{j*} \overline{\chi_j(A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^j z)} \right\}, \quad (71)$$

$$\chi_{I_j}(z) = (1 + \alpha_j) \left\{ \chi_j(z) + \sum_{n=1}^{+\infty} \omega_{p^{(n)}q^{(n)}}^{j**} \chi_j(M_{p^{(n)}q^{(n)}}^j z) + \sum_{n=0}^{+\infty} \beta_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{j**} \overline{\varphi(A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^j z)} \right\}. \quad (72)$$

The coefficients in the above expressions are given as follows. The arguments $p^{(i)}, q^{(i)}$ indicate different arguments from the index i and $p^{(i)}, q^{(i)}$ have values from 1 to Q . In addition, $\delta_{q^{(i)}}^{p^{(i)}}$ is Kronecker's delta. Now, the right-side functions of Eqs.(69) and (70) are shown by

Matrix:

$$\omega_{p^{(n)}q^{(n)}}^{M*} \varphi(M_{p^{(n)}q^{(n)}}^M z) = \prod_{i=1}^n \sum_{p^{(i)}=1}^Q \sum_{q^{(i)}=1}^Q \{1 - (1 - \delta_1^i) \delta_{p^{(i)}}^{q^{(i-1)}}\} (1 - \delta_{q^{(i)}}^{p^{(i)}}) \beta_{p^{(i)}} \alpha_{q^{(i)}} \varphi(A_{p^{(i)}} A_{q^{(i)}} z). \quad (73)$$

$$\alpha_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{M*} \overline{\chi_m(A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^M z)} = \delta_0^n \sum_{q^{(0)}=1}^Q \alpha_{q^{(0)}} \overline{\chi_m(A_{q^{(0)}} z)} + (1 - \delta_0^n) \prod_{i=1}^n \sum_{q^{(0)}=1}^Q \sum_{p^{(i)}=1}^Q \sum_{q^{(i)}=1}^Q (1 - \delta_{p^{(i)}}^{q^{(i-1)}}) (1 - \delta_{q^{(i)}}^{p^{(i)}}) \alpha_{q^{(0)}} \beta_{p^{(i)}} \alpha_{q^{(i)}} \overline{\chi_m(A_{q^{(0)}} A_{p^{(i)}} A_{q^{(i)}} z)}. \quad (74)$$

$$\omega_{p^{(n)}q^{(n)}}^{M**} \chi_m(M_{p^{(n)}q^{(n)}}^M z) = \prod_{i=1}^n \sum_{p^{(i)}=1}^Q \sum_{q^{(i)}=1}^Q \{1 - (1 - \delta_1^i) \delta_{p^{(i)}}^{q^{(i-1)}}\} (1 - \delta_{q^{(i)}}^{p^{(i)}}) \alpha_{p^{(i)}} \beta_{q^{(i)}} \chi_m(A_{p^{(i)}} A_{q^{(i)}} z). \quad (75)$$

$$\beta_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{M**} \overline{\varphi(A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^M z)} = \delta_0^n \sum_{q^{(0)}=1}^Q \beta_{q^{(0)}} \overline{\varphi(A_{q^{(0)}} z)} + (1 - \delta_0^n) \prod_{i=1}^n \sum_{q^{(0)}=1}^Q \sum_{p^{(i)}=1}^Q \sum_{q^{(i)}=1}^Q (1 - \delta_{p^{(i)}}^{q^{(i-1)}}) (1 - \delta_{q^{(i)}}^{p^{(i)}}) \beta_{q^{(0)}} \alpha_{p^{(i)}} \beta_{q^{(i)}} \overline{\varphi(A_{q^{(0)}} A_{p^{(i)}} A_{q^{(i)}} z)}. \quad (76)$$

j^{th} Inclusion:

$$\omega_{p^{(n)}q^{(n)}}^{j*} \varphi(M_{p^{(n)}q^{(n)}}^j z) = \prod_{i=1}^n \sum_{p^{(i)}=1}^Q \sum_{q^{(i)}=1}^Q \{1 - (1 - \delta_1^i) \delta_{p^{(i)}}^{q^{(i-1)}}\} (1 - \delta_{q^{(i)}}^{p^{(i)}}) (1 - \delta_j^{q^{(i)}}) \beta_{p^{(i)}} \alpha_{q^{(i)}} \varphi(A_{p^{(i)}} A_{q^{(i)}} z). \quad (77)$$

$$\alpha_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{j*} \overline{\chi_j(A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^j z)} = \delta_0^n \sum_{q^{(0)}=1}^Q (1 - \delta_j^{q^{(0)}}) \alpha_{q^{(0)}} \overline{\chi_j(A_{q^{(0)}} z)} + (1 - \delta_0^n) \prod_{i=1}^n \sum_{q^{(0)}=1}^Q \sum_{p^{(i)}=1}^Q \sum_{q^{(i)}=1}^Q (1 - \delta_{p^{(i)}}^{q^{(i-1)}}) (1 - \delta_{q^{(i)}}^{p^{(i)}}) (1 - \delta_j^{q^{(i)}}) \times \alpha_{q^{(0)}} \beta_{p^{(i)}} \alpha_{q^{(i)}} \overline{\chi_j(A_{q^{(0)}} A_{p^{(i)}} A_{q^{(i)}} z)}. \quad (78)$$

$$\omega_{p^{(n)}q^{(n)}}^{j**} \chi_j(M_{p^{(n)}q^{(n)}}^j z) = \prod_{i=1}^n \sum_{p^{(i)}=1}^Q \sum_{q^{(i)}=1}^Q \{1 - (1 - \delta_1^i) \delta_{p^{(i)}}^{q^{(i-1)}}\} (1 - \delta_{q^{(i)}}^{p^{(i)}}) (1 - \delta_j^{q^{(i)}}) \alpha_{p^{(i)}} \beta_{q^{(i)}} \chi_j(A_{p^{(i)}} A_{q^{(i)}} z). \quad (79)$$

$$\beta_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{j**} \overline{\varphi(A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^j z)} = \delta_0^n \sum_{q^{(0)}=1}^Q (1 - \delta_j^{q^{(0)}}) \beta_{q^{(0)}} \overline{\varphi(A_{q^{(0)}} z)} + (1 - \delta_0^n) \prod_{i=1}^n \sum_{q^{(0)}=1}^Q \sum_{p^{(i)}=1}^Q \sum_{q^{(i)}=1}^Q (1 - \delta_{p^{(i)}}^{q^{(i-1)}}) (1 - \delta_{q^{(i)}}^{p^{(i)}}) (1 - \delta_j^{q^{(i)}}) \times \beta_{q^{(0)}} \alpha_{p^{(i)}} \beta_{q^{(i)}} \overline{\varphi(A_{q^{(0)}} A_{p^{(i)}} A_{q^{(i)}} z)}. \quad (80)$$

when we omit upper subscripts M and j from the above equation. Note that these functions have the following relations:

$$\varphi(M_{p^{(n)}q^{(n)}} z) = \sum_{p^{(n)}=1}^Q \sum_{q^{(n)}=1}^Q (1 - \delta_{p^{(n)}}^{q^{(n-1)}}) (1 - \delta_{q^{(n)}}^{p^{(n)}}) \varphi(M_{p^{(n-1)}q^{(n-1)}} A_{p^{(n)}} A_{q^{(n)}} z). \quad (81)$$

$$\overline{\chi_m(A_{q^{(0)}} M_{p^{(n)}q^{(n)}} z)} = \sum_{p^{(n)}=1}^Q \sum_{q^{(n)}=1}^Q (1 - \delta_{p^{(n)}}^{q^{(n-1)}}) (1 - \delta_{q^{(n)}}^{p^{(n)}}) \overline{\chi_m(A_{q^{(0)}} M_{p^{(n-1)}q^{(n-1)}} A_{p^{(n)}} A_{q^{(n)}} z)}. \quad (82)$$

The above coefficients are given by Eqs.(91) ~ (99) in subsection 2.5. From the above results,

we obtained the external theoretical solutions. After that, we show the analytic solutions in a concrete example.

2.5. General solution in the presence of many circular elastic inclusions under uniform stresses

Fig. 4 Infinite elastic medium with many elastic inclusions under uniform stresses.

In this section, we consider the problem in the presence of many circular elastic inclusions ($j = 1, 2, \dots, Q$) disturbing the uniform stresses σ_x^∞ , σ_y^∞ and τ_{xy}^∞ at infinity. The j^{th} inclusions have the boundary L_j , origin z_j , radius a_j , shear modulus G_j , and Poisson's ratio ν_j , respectively, in Fig. 4.

where, τ^* and τ^{**} are known constants under the initial conditions that the uniform stresses σ_x^∞ , σ_y^∞ , and τ_{xy}^∞ load at infinity $|z| = \infty$. These constants are given as

$$\tau^* = \frac{\sigma_x^\infty + \sigma_y^\infty}{4 \left(Q + \sum_{j=1}^Q \alpha_j \gamma_j \right)} Q, \quad \tau^{**} = \frac{\sigma_y^\infty - \sigma_x^\infty}{4} + i \frac{\tau_{xy}^\infty}{2}. \quad (83)$$

After that, we consider the second assistant functions $\chi_m(z), \chi_j(z)$. At first, an arbitrary point z has total Q mirror points as $A_j z$ ($j = 1, 2, \dots, Q$) on the circular boundaries L_j respectively. Then, the point z needs to coincide with the mirror points on L_j . From this, we set the mirror point of the matrix as $\sum_{j=1}^Q \Gamma_j A_j z$ using the unknown value Γ_j . Therefore, the second assistant function $\chi_m(z)$ of the matrix is given as

$$\chi_m(z) = \sum_{j=1}^Q \Gamma_j \gamma_j \overline{A_j z} \phi'(z) + \psi'(z) \quad (84)$$

Next, we consider the second assistant function inside the j^{th} inclusion. When, the arbitrary point z is inside the j^{th} inclusion, the second assistant function of the j^{th} inclusion is given as

$$\chi_j(z) = \gamma_j \bar{z} \phi'(z) + \psi'(z) \quad (85)$$

These assistant functions are continuous on the boundary L_j . We obtained the Γ_j with the following equation.

$$\Gamma_j = \frac{1}{d_j \Delta_{d-1}}, \quad d_j = |z - z_j| - a_j, \quad \Delta_{d-1} = \sum_{j=1}^Q \frac{1}{d_j}. \quad (86)$$

In this problem, these functions do not have a singularity inside the region $a_j < |z| < \infty$ ($j = 1, 2, \dots, Q$). Therefore, the fundamental complex potential functions $\phi(z)$ and $\psi'(z)$ are given at infinity $|z| = \infty$, as shown in Eq. (29). The functions obtained by substituting the above equation into Eqs. (69) and (70) are general solutions to this problem, and are obtained as follows:

Matrix:

$$\varphi_M(z) = \tau^* z + \sum_{n=1}^{+\infty} \omega_{p^{(n)}q^{(n)}}^{M*} \tau^* M_{p^{(n)}q^{(n)}}^M z + \sum_{n=0}^{+\infty} \alpha_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{M*} \left(\tau^* \sum_{j=1}^Q \Gamma_j \gamma_j A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^M A_j z + 2\tau^{**} \overline{A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^M} \right), \quad (87)$$

$$\chi_M(z) = \tau^* \sum_{j=1}^Q \Gamma_j \gamma_j \overline{A_j z} + 2\tau^{**} z + \sum_{n=1}^{+\infty} \omega_{p^{(n)}q^{(n)}}^{M**} \left(\tau^{**} \sum_{j=1}^Q \Gamma_j \gamma_j \overline{M_{p^{(n)}q^{(n)}}^M A_j z} + 2\tau^{**} M_{p^{(n)}q^{(n)}}^M z \right) + \sum_{n=0}^{+\infty} \beta_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{M**} \overline{\tau^* A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^M} z. \quad (88)$$

j^{th} Inclusion:

$$\varphi_{I_j}(z) = (1 + \beta_j) \left\{ \tau^* z + \sum_{n=1}^{+\infty} \omega_{p^{(n)}q^{(n)}}^{j*} \tau^* M_{p^{(n)}q^{(n)}}^j z + \sum_{n=0}^{+\infty} \alpha_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{j*} \left(\gamma_j \tau^* \overline{A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^j z} + 2\tau^{**} \overline{A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^j} z \right) \right\}, \quad (89)$$

$$\chi_{I_j}(z) = (1 + \alpha_j) \left\{ \gamma_j \tau^* \overline{z} + 2\tau^{**} z + \sum_{n=1}^{+\infty} \omega_{p^{(n)}q^{(n)}}^{j**} \left(\gamma_j \tau^* \overline{M_{p^{(n)}q^{(n)}}^j z} + 2\tau^{**} M_{p^{(n)}q^{(n)}}^j z \right) + \sum_{n=0}^{+\infty} \beta_{q^{(0)}} \omega_{p^{(n)}q^{(n)}}^{j**} \overline{\tau^* A_{q^{(0)}} M_{p^{(n)}q^{(n)}}^j} z \right\}. \quad (90)$$

These solutions coincide with the solution of Moriguchi⁽¹⁾ and Hirashima⁽⁵⁾⁽⁶⁾ for reduction to the single-hole problem. The above coefficients are given by

$$\left. \begin{aligned} A_{p^{(n)}} A_{q^{(n)}} z &\stackrel{\text{set}}{=} \frac{A_n^I z + B_n^I}{C_n^I z + D_n^I}, \quad (n \geq 1) \\ A_n^I &= a_{q^{(n)}}^2 - |z_{q^{(n)}}|^2 + z_{p^{(n)}} \overline{z_{q^{(n)}}}, \\ B_n^I &= (a_{q^{(n)}}^2 - |z_{q^{(n)}}|^2) z_{p^{(n)}} - (a_{p^{(n)}}^2 - |z_{p^{(n)}}|^2) z_{q^{(n)}}, \\ C_n^I &= \overline{z_{q^{(n)}}} - \overline{z_{p^{(n)}}}, \quad D_n^I = a_{q^{(n)}}^2 - |z_{q^{(n)}}|^2 + \overline{z_{p^{(n)}}} z_{q^{(n)}}. \end{aligned} \right\} \quad (91)$$

where we set

$$M_{p^{(n-1)}q^{(n-1)}} z \stackrel{\text{set}}{=} \frac{A_{n-1}^{II} z + B_{n-1}^{II}}{C_{n-1}^{II} z + D_{n-1}^{II}}. \quad (92)$$

We obtained the following relations from the above recursions.

$$M_{p^{(n)}q^{(n)}} z = M_{p^{(n-1)}q^{(n-1)}} A_{p^{(n)}} A_{q^{(n)}} z \stackrel{\text{set}}{=} \frac{A_n^{II} z + B_n^{II}}{C_n^{II} z + D_n^{II}}. \quad (93)$$

where, A_n^{II} , B_n^{II} , C_n^{II} and D_n^{II} are expressed as the following recursions.

$$\left. \begin{aligned} A_0^{II} &= D_0^{II} = 1, \quad B_0^{II} = C_0^{II} = 0. \\ A_n^{II} &= (A_{n-1}^{II} A_n^I + B_{n-1}^{II} C_n^I), \\ B_n^{II} &= (A_{n-1}^{II} B_n^I + B_{n-1}^{II} D_n^I), \\ C_n^{II} &= (C_{n-1}^{II} A_n^I + D_{n-1}^{II} C_n^I), \\ D_n^{II} &= (C_{n-1}^{II} B_n^I + D_{n-1}^{II} D_n^I). \quad (n \geq 1) \end{aligned} \right\} \quad (94)$$

and

$$\left. \begin{aligned} \overline{A_{q^{(0)}}z} &\stackrel{\text{set}}{=} \frac{A_0^{\text{III}}z + B_0^{\text{III}}}{C_0^{\text{III}}z + D_0^{\text{III}}}, \\ A_0^{\text{III}} &= \overline{z_{q^{(0)}}}, \quad B_0^{\text{III}} = a_{q^{(0)}}^2 - |z_{q^{(0)}}|^2, \\ C_0^{\text{III}} &= 1, \quad D_0^{\text{III}} = -z_{q^{(0)}}. \end{aligned} \right\} \quad (95)$$

From the above relations, we obtained

$$\overline{A_{q^{(0)}}M_{p^{(n)}q^{(n)}}z} = \overline{A_{q^{(0)}}M_{p^{(n-1)}q^{(n-1)}}A_{p^{(n)}}A_{q^{(n)}}z} \stackrel{\text{set}}{=} \frac{A_n^{\text{IV}}z + B_n^{\text{IV}}}{C_n^{\text{IV}}z + D_n^{\text{IV}}}. \quad (96)$$

where

$$\left. \begin{aligned} A_n^{\text{IV}} &= A_0^{\text{III}}A_n^{\text{III}} + B_0^{\text{III}}C_n^{\text{III}}, \\ B_n^{\text{IV}} &= A_0^{\text{III}}B_n^{\text{III}} + B_0^{\text{III}}D_n^{\text{III}}, \\ C_n^{\text{IV}} &= C_0^{\text{III}}A_n^{\text{III}} + D_0^{\text{III}}C_n^{\text{III}}, \\ D_n^{\text{IV}} &= C_0^{\text{III}}B_n^{\text{III}} + D_0^{\text{III}}D_n^{\text{III}}. \quad (n \geq 1) \end{aligned} \right\} \quad (97)$$

Subsequently,

$$\left. \begin{aligned} \overline{A_jz} &\stackrel{\text{set}}{=} \frac{A_0^{\text{V}}z + B_0^{\text{V}}}{C_0^{\text{V}}z + D_0^{\text{V}}}, \\ A_0^{\text{V}} &= \overline{z_j}, \quad B_0^{\text{V}} = a_j^2 - |z_j|^2, \quad C_0^{\text{V}} = 1, \quad D_0^{\text{V}} = -z_j. \end{aligned} \right\} \quad (98)$$

and

$$\left. \begin{aligned} \overline{A_{q^{(0)}}M_{p^{(n)}q^{(n)}}A_jz} &\stackrel{\text{set}}{=} \frac{A_n^{\text{VI}}z + B_n^{\text{VI}}}{C_n^{\text{VI}}z + D_n^{\text{VI}}}, \\ A_n^{\text{VI}} &= A_0^{\text{V}}A_n^{\text{V}} + C_0^{\text{V}}B_n^{\text{V}}, \quad B_n^{\text{VI}} = B_0^{\text{V}}A_n^{\text{V}} + D_0^{\text{V}}B_n^{\text{V}}, \\ C_n^{\text{VI}} &= A_0^{\text{V}}C_n^{\text{V}} + C_0^{\text{V}}D_n^{\text{V}}, \quad D_n^{\text{VI}} = B_0^{\text{V}}C_n^{\text{V}} + D_0^{\text{V}}D_n^{\text{V}}. \quad (n \geq 1) \end{aligned} \right\} \quad (99)$$

In addition, $A_n^{\text{I}} \sim A_n^{\text{VI}}$, $B_n^{\text{I}} \sim B_n^{\text{VI}}$, $C_n^{\text{I}} \sim C_n^{\text{VI}}$, $D_n^{\text{I}} \sim D_n^{\text{VI}}$ are complex constants using the index n , which means the calculation of n -count, and they are known constants because they satisfy the above recursions. The sign $\stackrel{\text{set}}{=}$ means that the sign makes a connection with the complex variables and complex coefficients on the right-hand side of the equation.

3. Numerical Examples

The fundamental series solutions derived in the preceding section are used here to analyze the following examples. We define the ratio of the shear modulus of the j^{th} inclusion and matrix $e_j = G_j/G_M$. If e_j is greater than 1, the elasticity of the j^{th} inclusion is harder than that of the matrix; if e_j is smaller than 1, it is softer than the matrix. We call e_j the elastic constant ratio in this paper.

Then, we must account for the convergence of Eqs. (69) ~ (72). Generally, when adjoining inclusions are near or tangential to each other, the relative error may be large; that is to say, the convergences of the series tend to be large. We use n with a tolerance of relative error within 1% under the n and $n - 1$ counts.

As an example, we produce the convergence properties of numerical examples in Fig.7. We consider a geometry having three circular inclusions with $e_j = 0.1$ ($j = 1 \sim 3$) and distance $D/a_1 = 0.1$. In the plane stress problem, we denote the convergence properties of the problem under the uniform stress σ_y^∞ . Table1 shows the relative error [RE] of stresses and displacements on the boundary as the value of n changes, and Fig.5 is given by the Table1. From this figure, we can obtain a very precise value when we set n at a higher level. In this example, we set n that [RE] is lower than 1% using $n = 9$. Specifically, the tolerable value is confirmed to be sufficiently satisfied when we perform a general analysis using $n = 10$.

Table 1 Table of the relative errors $[RE]$ u_x and σ_θ at $\theta = 0$ in the calculation of the n -count.

n	$u_x/a_1 \times 10^6$	RE [%]	$\sigma_\theta/\sigma_y^\infty$	RE [%]
1	-3.1267	—	2.495	—
2	-1.1195	179.299	4.302	41.992
3	-1.7639	36.535	5.415	20.550
4	-1.3375	31.881	6.014	9.969
5	-1.4778	9.494	6.318	4.805
6	-1.3859	6.634	6.465	2.281
7	-1.4161	2.135	6.535	1.070
8	-1.3964	1.409	6.568	0.499
9	-1.4029	0.462	6.583	0.232
10	-1.3987	0.300	6.590	0.107

Fig. 5 Graph of the relative errors $[RE]$ in the calculation of the n -count.

3.1. Problems under uniform shear stresses

In this section, we show the stresses and displacements under a uniform stress σ_y^∞ in the plane stress state using Eqs. (87) and (90) given in subsections 2.3 and 2.5.

Fig. 6 Graph of σ_θ around L_2 under σ_y^∞ , when L_1 , L_3 approaches.

Fig. 7 Distribution of τ_{max} for the case of 3 soft inclusions under σ_y^∞ .

Three soft inclusions that are the same shape ($a_1 = a_2 = a_3$, $e = e_j$ ($j = 1 \sim 3$)) and elastic constant ratio $e = 0.1$, $e = 0$ (void) are arranged on the x -axis in the matrix. We observed the disturbances of the 2nd inclusion, when the 1st and 3rd inclusions approach the 2nd inclusion from a great distance $D/a_2 = 10$ to a close distance $D/a_2 = 0.1$. Fig. 6 shows the stress σ_θ on the boundary L_2 inside the matrix. When $e = 0$ (void) and $D/a_1 = 10$, these results were in complete agreement with the results $\sigma_\theta = 3\sigma_y^\infty$ at $\theta = 0^\circ, 180^\circ$ and $\sigma_\theta = -\sigma_y^\infty$ at $\theta = 90^\circ, 270^\circ$ reported by Moriguchi⁽¹⁾. We can thus find the interacting disturbances of the inclusions on each other. Fig. 7 shows the distribution of τ_{max} for the case of a single inclusion (top figure) and three inclusions (bottom figure) under σ_y^∞ , when $D/a_1 = 0.1$.

Fig. 8 Graph of σ_θ , σ_r , $\tau_{r\theta}$ around L_2 under σ_y^∞ , when L_1 , L_3 approaches.

Fig. 9 Distribution of τ_{max} for the case of 3 hard inclusions under σ_y^∞ .

Three hard inclusions that are the same shape and elastic constant ratio $e = 10, e = \infty$ (rigid inclusions) are arranged on the x -axis in the matrix.

We observed the disturbances of the 2nd inclusion, when the 1st and 3rd inclusions approach the 2nd inclusion from a great distance $D/a_2 = 10$ to a close distance $D/a_2 = 0.1$. Fig. 8 shows the stresses σ_θ , σ_r , $\tau_{r\theta}$ on the boundary L_2 . Fig. 9 shows the distribution of τ_{max} for the case of a single rigid inclusion (top figure) and three rigid inclusions (bottom figure) under σ_y^∞ , when $D/a_1 = 0.1$.

Fig. 10 Distribution of τ_{max} for the case of 4 and 5 soft inclusions under σ_y^∞ .

Fig. 11 Distribution of τ_{max} for the case of 4 and 5 hard inclusions under σ_y^∞ .

Fig. 10 shows the distribution of τ_{max} for the case of four soft inclusions (left figure) and five soft inclusions (right figure) under σ_y^∞ . In the case of five inclusions, the radii are all the same and the interference D/a_1 with each other is 0.1. The geometry in the case of the four inclusions is the same as in the case of five inclusions. Fig. 11 shows the case of four hard inclusions (left figure) and five hard inclusions (right figure) when $D/a_1 = 10$.

Fig. 12 Distribution of τ_{max} for the case of 7 inclusions under σ_y^∞ .

Fig. 13 Distribution of τ_{max} for the case of a RC with an air hole under σ_y^∞ .

Fig. 12 shows the distribution of τ_{max} for the case of seven soft inclusions $e = 0.1$ (left figure) and seven hard inclusions $e = 10$ (right figure) under σ_y^∞ .

Finally, we show a realistic example so that we can consider the reinforced concrete problem under σ_y^∞ in Fig. 13. In this problem, we replace the matrix with concrete and the inclusion with an iron reinforcing bar. The bottom figure shows that a dead air space having a radius $a_1/2$ comes in contact with the two bars. When the distance of two bars is close, the air space occurs easily. We may find high maximum shear stress τ_{max} around the concrete near the air space and bar.

4. Concluding Remarks

In the present paper, we examined the in-plane problem of a two-dimensional isotropic matrix containing many circular elastic inclusions subjected to arbitrary loading and produced general solutions to find the stresses and displacements. The purpose of this characteristic study was to apply Moriguchi's reflection principle. Using these solutions, several numerical examples were presented graphically.

These problems have previously been solved using different numerical analysis methods such as the finite element method (FEM) and the boundary element method (BEM). However, our studies were developed in order to observe the interacting disturbances for many circular inclusions with high precision.

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