NOTES AND CORRESPONDENCE

Wave Energy at Critical Level

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Abstract

The behaviour of wave energy (projected onto a wave vector) at the lowest critical level (corresponding to the wave vector) is investigated analytically in a linear 3-dimensional framework by the ray tracing method. Both the magnitude and direction of the environmental flow are arbitrary functions of the vertical coordinate, with a slowly varying assumption. The Coriolis parameter is constant.

The obtained results show the following. The wave energy infinitely increases only at such a critical level that the magnitude of environmental wind vanishes as a linear function of vertical coordinate, and that the direction of environmental wind does not become perpendicular to the wave vector, and only when the Coriolis effect is ignored. The wave energy remains finite at all other types of critical level.

The well-known 2-dimensional non-rotating result that the wave energy infinitely increases at the critical level can not be realized in the 3-dimensional rotating system.

1. Introduction

A mountain in an environmental basic flow generates waves. Assuming horizontal uniformness of the basic flow, the linearized wave equation can be projected onto a horizontal wave number vector (hereafter Wave Vector for brevity) space. The projected wave equation, which is a differential equation in the vertical direction, has singular points. These singular points are called critical levels. The waves propagate upward and encounter the lowest critical level.

In a 2-dimensional non-rotating system, the basic flow is unidirectional. The wave vector is also unidirectional and parallel to the basic flow. The lowest critical level is the lowest vanishing point of the basic flow, which is the same for all wave vectors. The waves are almost absorbed at the critical level if the Richardson number is greater than $1/4$ (Booker and Bretherton 1967), and can

scarcely pass through there. It is well-known that the group velocity becomes zero and the wave energy accumulates to become infinite at the critical level in the 2-dimensional non-rotating system.

In a 3-dimensional non-rotating system, the basic flow may veer and/or back. The wave vector is also no more unidirectional. The lowest critical level is the lowest vanishing point of the inner product of the basic flow and wave vector, which is dependent on the direction of wave vector and is distributed continuously in the vertical direction (Broad 1995; Shutts 1995). Shutts (1998) examined the wave structure in the case of a basic flow of uniform shear. Shutts and Gadian (1999) examined the case of uniform shear and uniform rotation in height. While, Broad (1999) considered a basic flow, whose direction and magnitude vary linearly with height. In almost all these particular cases, the wave energy (projected onto the wave vector) decays to zero as approaching the critical level. The wave energy become infinite only at such a critical level that the magnitude of the basic flow vanishes, and that the direction of the basic flow does not become perpendicular to the wave vector.

For a rotating system, several authors considered

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the case of unidirectional basic flow, and the wave structure is examined in detail (e.g., Jones 1967; Grimshaw 1975; Wurtele et al. 1996; Shen and Lin 1999). The case of directionally sheared basic flow has also already been considered (e.g., Broad 1995). (Although the directionally sheared flow can not be a steady free solution in a rotating system, it may be regarded as steady in gravity wave time scale, see Broad 1995.) However, the behaviour of wave energy at the critical level has not yet been clarified.

In this note, in a linear 3-dimensional rotating system, the case of a slowly varying basic flow with arbitrary shears both of magnitude and direction is considered, and the behaviour of wave energy at the lowest critical level by ray tracing method is examined.

The organization of this note is as follows. In section 2, critical levels are introduced. In section 3, the ray trajectory equation is derived. In section 4, an equation for wave energy to be estimated is presented. In sections 5, 6, 7 and 8, the cases of 2-dimensional non-rotating, 2-dimensional rotating, 3-dimensional non-rotating and 3-dimensional rotating systems are considered respectively. In section 9, concluding remarks are given. In the Appendix, the conservation theorem of wave action is presented.

2. Critical level

We consider a horizontal basic flow, dependent only on the vertical coordinate. Both the magnitude and direction vary with height. Linearizing about the basic flow, equations of a non-hydrostatic Boussinesq system become

$$Du_i + \Lambda_i w + \partial_i p - f \varepsilon_{ij} u_j = 0, \quad (2-1)$$

$$Dw + \partial p / \partial z - b = 0, \quad (2-2)$$

$$Db + N^2 w - f \varepsilon_{ij} \Lambda_i u_j = 0, \quad (2-3)$$

$$\partial_i u_i + \partial w / \partial z = 0. \quad (2-4)$$

The subscripts $i, j, \dots$, which run from 1 to 2, denote the horizontal directions, with the convention of summation, i.e., $A_i B_i = \sum_{i=1}^2 A_i B_i$. $D = \partial_t + U_i \partial_i$, $\partial_t = \partial / \partial t$ and t is the time. $\partial_i = \partial / \partial x_i$ and x_i are the horizontal coordinates. $U_i = U_i(z)$ is the basic horizontal velocity, and $\Lambda_i = dU_i / dz$. ε_{ij} is defined as $\varepsilon_{12} = -\varepsilon_{21} = 1$ and $\varepsilon_{11} = \varepsilon_{22} = 0$. z is the vertical coordinate. f is the Coriolis parameter. N^2 is the buoyancy frequency of the basic flow. u_i and w are, respectively, the horizontal and vertical component of velocity disturbance. p is the

pressure disturbance (divided by a mean density). b , which is proportional to the potential temperature disturbance, represents buoyancy.

Eliminating the other variables than w in (2-1, 2, 3, 4), a differential equation for w is obtained as follows.

$$\begin{aligned} D(D^2 + f^2)\partial^2 w / \partial z^2 + D(N^2 + D^2)\partial_i \partial_i w \\ - 2f^2 \Lambda_i \partial_i \partial w / \partial z - (d\Lambda_i / dz)\partial_i D^2 w \\ + 2f \varepsilon_{ij} \Lambda_i \partial_j \{D \partial w / \partial z\} + f \varepsilon_{ij} (d\Lambda_i / dz)\partial_j D w \\ - 2f \Lambda_i \Lambda_j \varepsilon_{jk} \partial_i \partial_k w = 0. \end{aligned} \quad (2-5)$$

Since the coefficients U_i , N^2 , Λ_i and $d\Lambda_i / dz$ are dependent only on the vertical coordinate z , it is possible to project (2-5) onto a Fourier base $(2\pi)^{-3/2} \text{Exp}[i(\mathbf{K} \cdot \mathbf{x} - \omega t)]$. Here and hereafter, $\mathbf{A} \cdot \mathbf{B}$ stands for the inner product $A_i B_i$, and K_i is the wave vector and ω the frequency.

$$\begin{aligned} \Omega(\Omega^2 - f^2)d^2 W / dz^2 + \Omega(N^2 - \Omega^2)K^2 W \\ + 2f^2(\mathbf{K} \cdot \boldsymbol{\Lambda})dW / dz - (\mathbf{K} \cdot d\boldsymbol{\Lambda} / dz)\Omega^2 W \\ + i2f(\mathbf{K} \times \boldsymbol{\Lambda})\Omega dW / dz + if(\mathbf{K} \times d\boldsymbol{\Lambda} / dz)\Omega W \\ - i2f(\mathbf{K} \cdot \boldsymbol{\Lambda})(\mathbf{K} \times \boldsymbol{\Lambda})W = 0. \end{aligned} \quad (2-6)$$

$\Omega = \mathbf{K} \cdot \mathbf{U} - \omega$ and $\mathbf{A} \times \mathbf{B}$ stands for the outer product $\varepsilon_{ij} A_i B_j$. $W = W(z)$ is the projection of w onto the Fourier base $(2\pi)^{-3/2} \text{Exp}[i(\mathbf{K} \cdot \mathbf{x} - \omega t)]$, and $K = (\mathbf{K} \cdot \mathbf{K})^{1/2}$. For the steady state $\omega = 0$, the levels, where $\{\mathbf{K} \cdot \mathbf{U}(z)\}^2 - f^2$ or $\mathbf{K} \cdot \mathbf{U}(z)$ vanishes, are singular points of Eq.(2-6). These levels are called the critical levels. In this note, we consider meso-scale phenomena, with the Rossby

Fig. 1. The critical level $z_c(\mathbf{K})$ for a wave vector $\mathbf{K}$ and a basic flow $\mathbf{U}(z)$ of linear shear, in the non-rotating system $f = 0$

number greater than unity. So, we may assume that $\mathbf{K} \cdot \mathbf{U}(0) > f > 0$ (if $\mathbf{K} \cdot \mathbf{U}(0)$ is negative, then by replacing K_i with $-K_i$, we can make $\mathbf{K} \cdot \mathbf{U}(0)$ positive). Then, the lowest critical level $z = z_C(\mathbf{K})$ is where $\mathbf{K} \cdot \mathbf{U}(z) - f$ vanishes.

$$\mathbf{K} \cdot \mathbf{U}(z) = f \quad \text{at} \quad z = z_C(\mathbf{K}). \quad (2-7)$$

In the non-rotating case $f = 0$, the critical level $z = z_C(\mathbf{K})$ is where the magnitude of the basic flow velocity $U_i(z)$ vanishes and/or the direction of $U_i(z)$ becomes perpendicular to the wave vector K_i .

$$U_i(z) \perp K_i$$

$$\text{and/or} \quad U(z) = \{\mathbf{U}(z) \cdot \mathbf{U}(z)\}^{1/2} = 0 \\ \text{at} \quad z = z_C(\mathbf{K}). \quad (2-8)$$

An example of critical level for a basic flow of linear shear in the non-rotating system is shown in Fig. 1.

3. Ray trajectory

Although U_i and N_2 in (2-6) are functions of z , they are assumed to be sufficiently slowly varying. Then, wave-like disturbances are locally possible.

$$W \propto \exp[imz]. \quad (3-1)$$

Substituting (3-1) into (2-6), neglecting the vertical derivatives of U_i , the following dispersion relation is obtained.

$$\Omega^2(K^2 + m^2) = K^2 N^2 + f^2 m^2. \quad (3-2)$$

With a condition that the vertical group velocity $C_z^G = \partial\omega/\partial m$ must be positive, and with the assumption $\mathbf{K} \cdot \mathbf{U}(0) > f > 0$, the vertical wave number m and group velocity $\{C_i^G, C_z^G\}$ for steady state $\omega = 0$ are determined from (3-2) as

$$m = K(N^2 - \Omega^2)^{1/2}/(\Omega^2 - f^2)^{1/2}, \quad (3-3)$$

$$C_z^G = \partial\omega/\partial m \\ = (N^2 - \Omega^2)^{1/2}(\Omega^2 - f^2)^{3/2}/\{K\Omega(N^2 - f^2)\}, \quad (3-4)$$

$$C_i^G = \partial\omega/\partial K_i \\ = U_i - (N^2 - \Omega^2)(\Omega^2 - f^2)K_i/\{K^2\Omega(N^2 - f^2)\}, \quad (3-5)$$

where $\Omega = \mathbf{K} \cdot \mathbf{U}(z)$. From (3-4) and (3-5), the ray trajectory is determined by the following differential equation.

$$dx_i/dz = C_i^G/C_z^G \\ = K\Omega(N^2 - f^2)U_i/\{(N^2 - \Omega^2)^{1/2}(\Omega^2 - f^2)^{3/2}\} \\ - (N^2 - \Omega^2)^{1/2}K_i/\{K(\Omega^2 - f^2)^{1/2}\}. \quad (3-6)$$

Integrating (3-6), the ray trajectory equation is obtained as

$$x_i \\ = \int_0^z dz K\Omega(N^2 - f^2)U_i/\{(N^2 - \Omega^2)^{1/2}(\Omega^2 - f^2)^{3/2}\} \\ - \int_0^z dz (N^2 - \Omega^2)^{1/2}K_i/\{K(\Omega^2 - f^2)^{1/2}\}, \quad (3-7)$$

where $x_i(z=0)$ is set to the origin without loss of generality.

4. Disturbance energy equation

In order to obtain an equation for disturbance energy E , the steady version of wave-action equation (A-15) is integrated in a segment of ray tube between two horizontal cross sections at $z = z_1$ and $z = z_2$ to give

$$0 = \iiint d^3x \partial_\lambda (C_\lambda^G E/\Omega) \\ = \left[\iint d^2x C_z^G E/\Omega \right] (z = z_2) \\ - \left[\iint d^2x C_z^G E/\Omega \right] (z = z_1). \quad (4-1)$$

Eq.(4-1) implies that

$$\iint d^2x C_z^G E/\Omega = \text{constant} \\ (\text{independent of } z). \quad (4-2)$$

From (4-2), for a given wave vector K_i , the following expression for the disturbance energy E is obtained

$$E \propto \Omega/\{JC_z^G\}, \quad (4-3)$$

where J is the Jacobian of the transformation from K_i to x_i , which is the determinant of the transformation matrix $\partial x_i/\partial K_j$. The matrix $\partial x_i/\partial K_j$ is obtained by differentiating Eq.(3-7) with respect to

K_j , and J is calculated by the following formula.

$$J = (1/2)\varepsilon_{ij}\varepsilon_{kl}(\partial x_i/\partial K_k)(\partial x_j/\partial K_l). \quad (4-4)$$

5. 2-dimensional non-rotating case

In this section, we review the well-known 2-dimensional ($\{U_1, U_2\} = \{U, 0\}$ and $\{K_1, K_2\} = \{k, 0\}$) non-rotating ($f = 0$) problem. In this case, the vertical group velocity (3-4) and the ray trajectory equation (3-7) are respectively reduced to the following simple forms.

$$C_z^G = (N^2 - \Omega^2)^{1/2} \Omega / \{kN^2\}, \quad (5-1)$$

$$x = \int_0^z dz \Omega / (N^2 - \Omega^2)^{1/2}, \quad (5-2)$$

where $\Omega = \mathbf{K} \cdot \mathbf{U}(z) = kU(z)$. Differentiating Eq.(5-2) with respect to k , the Jacobian J of the transformation from k to x is obtained as

$$J = dx/dk = \int_0^z dz N^2 U / (N^2 - \Omega^2)^{3/2}. \quad (5-3)$$

Substituting (5-1) into (4-3) gives

$$E \propto kN^2 / \{J(N^2 - \Omega^2)^{1/2} \Omega\}. \quad (5-4)$$

When $\Omega = kU(z)$ behaves like $\propto |z - z_C(\mathbf{K})|^n$ ($n = 1, 2, \dots$) as z approaches the critical level $z = z_C(\mathbf{K})$, then Eq.(5-4) implies that E behaves like $\propto |z - z_C(\mathbf{K})|^{-n}$.

$$E \propto |z - z_C(\mathbf{K})|^{-n}, \quad n = 1, 2, \dots, \\ \text{as } z \uparrow z_C(\mathbf{K}). \quad (5-5)$$

From Eq.(5-1), the vertical group velocity becomes zero at $z = z_C(\mathbf{K})$. From Eq.(5-2), the horizontal distance travelled by the ray trajectory remains finite. As a result, the wave energy accumulates at $z = z_C(\mathbf{K})$ to become infinite, as shown in (5-5).

6. 2-dimensional rotating case

In this section, we consider the 2-dimensional ($\{U_1, U_2\} = \{U, 0\}$ and $\{K_1, K_2\} = \{k, 0\}$) rotating ($f \neq 0$) case, in order to see how the above mentioned 2-dimensional non-rotating result is altered by the Coriolis effect.

For the meso-scale phenomena, $(f/N)^2$ and $(\Omega/N)^2$ are usually extremely smaller than unity. By neglecting $(f/N)^2$ and $(\Omega/N)^2$ compared to unity (equivalent to taking the hydrostatic approximation), the qualitative singular structure of both the vertical velocity equation (3-4) and the ray trajectory equation (3-7) do not alter. So, for the sake of the mathematical simplicity, here and hereafter,

we neglect these terms. Then, in the 2-dimensional rotating case, Eq.(3-4) and Eq.(3-7) are respectively reduced to

$$C_z^G = (\Omega^2 - f^2)^{3/2} / \{NK\Omega\}, \quad (6-1)$$

$$x = \int_0^z dz N f^2 / (\Omega^2 - f^2)^{3/2}. \quad (6-2)$$

Differentiating (6-2) with respect to k , the Jacobian J is calculated as

$$J = dx/dk \\ = -3 \int_0^z dz N k f^2 U^2 / (\Omega^2 - f^2)^{5/2}. \quad (6-3)$$

Substituting (6-1) into (4-3) gives

$$E \propto kN\Omega^2 / \{J(\Omega^2 - f^2)^{3/2}\}. \quad (6-4)$$

When $\Omega - f = kU(z) - f$ behaves like $\propto |z - z_C(\mathbf{K})|^n$ ($n = 1, 2, \dots$) as z approaches the critical level $z = z_C(\mathbf{K})$, then Eq.(6-4) implies that E behaves like $\propto |z - z_C(\mathbf{K})|^{n-1}$.

$$E \propto |z - z_C(\mathbf{K})|^{n-1}, \quad n = 1, 2, \dots, \\ \text{as } z \uparrow z_C(\mathbf{K}). \quad (6-5)$$

From Eq.(6-1), the vertical group velocity becomes zero at $z = z_C(\mathbf{K})$. From Eq.(6-2), the horizontal distance travelled by the ray trajectory becomes infinite. The latter effect dominates the former (2 effects cancel each other for $n = 1$). As a result, the wave energy becomes zero (remains finite for $n = 1$) at $z = z_C(\mathbf{K})$ as shown in (6-5).

7. 3-dimensional non-rotating case

In the 3-dimensional non-rotating ($f = 0$) case, the vertical group velocity (3-4) and the ray trajectory equation (3-7) are respectively reduced to

$$C_z^G = \Omega^2 / \{KN\}, \quad (7-1)$$

$$x_i = \int_0^z dz N K U_i / \Omega^2 - \int_0^z dz N K_i / \{K\Omega\}. \quad (7-2)$$

Noticing that $\Omega K_i / K^2 = (\mathbf{K} \cdot \mathbf{U}) K_i / K^2$ is the projection of the basic velocity vector U_i on the unit vector K_i / K , Eq.(7-2) can be rewritten as proportional to the unit vector $\varepsilon_{ij} K_j / K$,

$$x_i = - \int_0^z dz N (\mathbf{K} \times \mathbf{U}) \varepsilon_{ij} K_j / \{K\Omega^2\}. \quad (7-3)$$

Differentiating Eq.(7-3) with respect to K_i , the Jacobian matrix of the transformation from K_i to x_i has the following form.

$$\partial x_i / \partial K_j = A \varepsilon_{ij} + (\partial A / \partial K_j) \varepsilon_{ik} K_k, \\ A = - \int_0^z dz N(\mathbf{K} \times \mathbf{U}) / \{K \Omega^2\}. \quad (7-4)$$

Substituting (7-4) into the Jacobian formula (4-4), the Jacobian J is calculated as

$$J = - \left[\int_0^z dz N(\mathbf{K} \times \mathbf{U}) / \{K \Omega^2\} \right]^2. \quad (7-5)$$

Substituting (7-1) into (4-3) gives

$$E \propto KN / \{J \Omega\}. \quad (7-6)$$

There are 3 cases that $\Omega = \mathbf{K} \cdot \mathbf{U}(z)$ vanishes. Firstly, we consider the case that the direction of $U_i(z)$ becomes perpendicular to K_i at $z_C(\mathbf{K})$, and that the magnitude of $U_i(z)$ does not vanish at $z_C(\mathbf{K})$.

$$U_i(z) \perp K_i, \quad U_i(z) \neq 0 \quad \text{at} \quad z = z_C(\mathbf{K}). \quad (7-7)$$

When $\Omega = \mathbf{K} \cdot \mathbf{U}(z)$ behaves like $\propto |z - z_C(\mathbf{K})|^n$ ($n = 1, 2, \dots$) as z approaches the critical level $z = z_C(\mathbf{K})$, then Eq.(7-6) implies that E behaves like $\propto |z - z_C(\mathbf{K})|^{3n-2}$.

$$E \propto |z - z_C(\mathbf{K})|^{3n-2}, \quad n = 1, 2, \dots, \\ \text{as} \quad z \uparrow z_C(\mathbf{K}). \quad (7-8)$$

Secondly, we consider the case that the direction of $U_i(z)$ becomes perpendicular to K_i at $z_C(\mathbf{K})$, and that at the same time the magnitude of $U_i(z)$ vanishes at $z_C(\mathbf{K})$.

$$U_i(z) \perp K_i, \quad U_i(z) = 0 \quad \text{at} \quad z = z_C(\mathbf{K}). \quad (7-9)$$

When $\Omega = \mathbf{K} \cdot \mathbf{U}(z)$ behaves like $\propto |z - z_C(\mathbf{K})|^{m+n}$ and $\mathbf{K} \times \mathbf{U}(z)$ like $\propto |z - z_C(\mathbf{K})|^n$ ($m, n = 1, 2, \dots$) as z approaches the critical level $z = z_C(\mathbf{K})$, then Eq.(7-6) implies that E behaves like $\propto |z - z_C(\mathbf{K})|^{3m+n-2}$.

$$E \propto |z - z_C(\mathbf{K})|^{3m+n-2}, \quad m, n = 1, 2, \dots, \\ \text{as} \quad z \uparrow z_C(\mathbf{K}). \quad (7-10)$$

Finally, we consider the case that the magnitude of $U_i(z)$ vanishes at $z_C(\mathbf{K})$, and that the direction of $U_i(z)$ does not become perpendicular to K_i at $z_C(\mathbf{K})$.

$$U_i(z) = 0, \quad \{\mathbf{K} \cdot \mathbf{U}(z)\} / \{\mathbf{U}(z) \cdot \mathbf{U}(z)\}^{1/2} \neq 0 \\ \text{at} \quad z = z_C(\mathbf{K}). \quad (7-11)$$

When $\Omega = \mathbf{K} \cdot \mathbf{U}(z)$ and $\mathbf{K} \times \mathbf{U}(z)$ behave like $\propto |z - z_C(\mathbf{K})|^n$ ($n = 1, 2, \dots$) as z approaches the critical level $z = z_C(\mathbf{K})$, then Eq.(7-6) implies that

E behaves like $\propto |z - z_C(\mathbf{K})|^{-1} (\log |z - z_C(\mathbf{K})|)^{-2}$ for $n = 1$, like $\propto |z - z_C(\mathbf{K})|^{n-2}$ for $n \geq 2$.

$$E \propto |z - z_C(\mathbf{K})|^{-1} (\log |z - z_C(\mathbf{K})|)^{-2} \\ \text{for} \quad n = 1 \quad \text{as} \quad z \uparrow z_C(\mathbf{K}). \quad (7-12)$$

$$E \propto |z - z_C(\mathbf{K})|^{n-2} \\ \text{for} \quad n \geq 2 \quad \text{as} \quad z \uparrow z_C(\mathbf{K}). \quad (7-13)$$

From Eq.(7-1), the vertical group velocity becomes zero at $z = z_C(\mathbf{K})$. From Eq.(7-3), the ray horizontally travels away to infinity in the direction perpendicular to K_i . In most cases, the latter effect dominates the former. As a result, the wave energy is attenuated and becomes zero as $z \uparrow z_C(\mathbf{K})$, as shown in (7-8), (7-10) and (7-13). Only in the case (7-11), the wave energy becomes infinite (for $n = 1$) or remains finite (for $n = 2$), as shown in (7-12) or (7-13).

8. 3-dimensional rotating case

In the 3-dimensional rotating ($f \neq 0$) case, the vertical group velocity (3-4) and the ray trajectory equation (3-7) become

$$C_z^G = (\Omega^2 - f^2)^{3/2} / \{KN\Omega\}, \quad (8-1)$$

$$x_i = \int_0^z dz N K \Omega U_i / (\Omega^2 - f^2)^{3/2} \\ - \int_0^z dz N K_i / \{K(\Omega^2 - f^2)^{1/2}\}. \quad (8-2)$$

Noticing that $\Omega K_i / K^2 = (\mathbf{K} \cdot \mathbf{U}) K_i / K^2$ is the projection of the basic velocity U_i on the unit vector K_i / K , Eq.(8-2) can be rewritten as

$$x_i = - \int_0^z dz N \Omega (\mathbf{K} \times \mathbf{U}) \varepsilon_{ij} K_j / \{K(\Omega^2 - f^2)^{3/2}\} \\ + \int_0^z dz N f^2 K_i / \{K(\Omega^2 - f^2)^{3/2}\}. \quad (8-3)$$

Differentiating Eq.(8-3) with respect to K_i , the Jacobian matrix of the transformation from K_i to x_i has the following form.

$$\partial x_i / \partial K_j = A \varepsilon_{ij} + (\partial A / \partial K_j) \varepsilon_{ik} K_k + B \delta_{ij} \\ + (\partial B / \partial K_j) K_i, \\ A = - \int_0^z dz N \Omega (\mathbf{K} \times \mathbf{U}) / \{K(\Omega^2 - f^2)^{3/2}\}, \\ B = \int_0^z dz N f^2 / \{K(\Omega^2 - f^2)^{3/2}\}. \quad (8-4)$$

Substituting (8-4) into the Jacobian formula (4-4), the Jacobian has the following form

$$J = -A^2 - 3B \int_0^z dz N f^2 \Omega^2 / \{K(\Omega^2 - f^2)^{5/2}\} \\ - 3K \varepsilon_{ij} (\partial A / \partial K_i) \int_0^z dz N f^2 \Omega U_j / (\Omega^2 - f^2)^{5/2}. \quad (8-5)$$

Near the critical level $z = z_C(\mathbf{K})$ (i.e., $\Omega - f \sim 0$), the Jacobian in Eq.(8-5) is estimated as

$$J \sim -\{N\Omega(\mathbf{K} \times \mathbf{U})/K\}^2 \left\{ \int_0^z dz / (\Omega^2 - f^2)^{3/2} \right\}^2 \\ + (3N^2 f^2 \Omega^2 / K^2) (\Omega^2 - f^2) \left\{ \int_0^z dz / (\Omega^2 - f^2)^{3/2} \right\} \\ \times \left\{ \int_0^z dz / (\Omega^2 - f^2)^{5/2} \right\} \\ + 3f^2 \varepsilon_{ij} \left\{ \int_0^z dz N(\mathbf{K} \times \mathbf{U}) U_i / (\Omega^2 - f^2)^{3/2} \right\} \\ \times \left\{ \int_0^z dz N \Omega U_j / (\Omega^2 - f^2)^{5/2} \right\} \\ - 9f^2 \varepsilon_{ij} \left\{ \int_0^z dz N \Omega^2 (\mathbf{K} \times \mathbf{U}) U_i / (\Omega^2 - f^2)^{5/2} \right\} \\ \times \left\{ \int_0^z dz N \Omega U_j / (\Omega^2 - f^2)^{5/2} \right\}. \quad (8-6)$$

When $\Omega - f = \mathbf{K} \cdot \mathbf{U}(z) - f$ behaves like $\propto |z - z_C(\mathbf{K})|^n$ ($n = 1, 2, \dots$) as z approaches the critical level $z = z_C(\mathbf{K})$, and when $d^m U_i(z_C(\mathbf{K})) / dz^m \neq 0$, $d^{m-1} U_i(z_C(\mathbf{K})) / dz^{m-1} = 0, \dots, dU_i(z_C(\mathbf{K})) / dz = 0$, then $m \leq n$, and then the last term in (8-6) becomes dominant as $z \uparrow z_C(\mathbf{K})$. So, from (8-6), the Jacobian J behaves like $\propto |z - z_C(\mathbf{K})|^{2+m-5n}$,

$$J \propto |z - z_C(\mathbf{K})|^{2+m-5n} > |z - z_C(\mathbf{K})|^{2-4n}, \\ n = 1, 2, \dots, \quad \text{as } z \uparrow z_C(\mathbf{K}). \quad (8-7)$$

Substituting (8-1) and (8-7) into (4-3) gives

$$E \propto K N \Omega^2 |z - z_C(\mathbf{K})|^{5n-m-2} / (\Omega^2 - f^2)^{3/2}. \quad (8-8)$$

Since $\Omega - f = \mathbf{K} \cdot \mathbf{U}(z) - f$ behaves like $\propto |z - z_C(\mathbf{K})|^n$ ($n = 1, 2, \dots$) as z approaches the critical level $z = z_C(\mathbf{K})$, Eq.(8-8) implies that E behaves like $\propto |z - z_C(\mathbf{K})|^{7n/2-m-2} < |z - z_C(\mathbf{K})|^{5n/2-2}$.

$$E \propto |z - z_C(\mathbf{K})|^{7n/2-m-2} < |z - z_C(\mathbf{K})|^{5n/2-2}, \\ n = 1, 2, \dots, \quad \text{as } z \uparrow z_C(\mathbf{K}). \quad (8-9)$$

From Eq.(8-1), the vertical group velocity becomes zero at $z = z_C(\mathbf{K})$. From Eq.(8-3), the ray horizontally travels away to infinity. The latter effect dominates the former. As a result, the wave energy is attenuated to become zero as $z \uparrow z_C(\mathbf{K})$, as shown in (8-9).

9. Conclusions

In this note, we examined the behaviour of topographically generated wave energy (projected onto a wave vector) at the lowest critical level (corresponding to the wave vector), in a linearized 3-dimensional rotating system. The Coriolis parameter f is constant. The basic flow has both shears of direction and magnitude. Strictly speaking, the directionally sheared basic flow can not be a steady free solution in a rotating system. However, it may be regarded as steady in the gravity wave time scale. The basic flow was assumed to be slowly varying, and the ray tracing method was applied.

Considering meso-scale phenomena with the Rossby number greater than unity, we may assume that $\mathbf{K} \cdot \mathbf{U}(0) - f$ is positive. Here $\mathbf{K} \cdot \mathbf{U}(z)$ is the inner product of the wave vector and the basic flow, and $z = 0$ indicates the ground. On this assumption, the lowest critical level is where $\mathbf{K} \cdot \mathbf{U}(z) = f$. In the non-rotating case (ignoring the Coriolis effect), this is reduced to $\mathbf{K} \cdot \mathbf{U}(z) = 0$.

The obtained results show the following. The group velocity becomes zero at the critical level. At the same time, the ray horizontally travels away to infinity, as approaching the critical level. In almost all cases, the latter effect dominates the former, and the wave energy projected onto the wave vector becomes zero (or remains finite). The wave energy becomes infinite only at such a critical level that the magnitude of the basic flow vanishes there as a linear function of the vertical coordinate, and that the direction of the basic flow does not become perpendicular to the wave vector, and only when the Coriolis effect is ignored. The results are summarized in Table 1.

The well-known 2-dimensional non-rotating result that the wave energy becomes infinite at the critical level, is by no means a typical case for real phenomena.

Table 1. Wave energy E near the critical level $z = z_C(K)$. The critical level is where $G(z) = 0$. $m, n = 1, 2, 3, \dots$. The first case is read as follows. In 2D (2-dimensional) $f = 0$ (non-rotating) system, the critical level is where $kU(z) = 0$. When $G(z) = kU(z)$ behaves like $\propto |z - z_C(K)|^n$ ($n = 1, 2, 3, \dots$) as $z \uparrow z_C(K)$, then E behaves like $\propto |z - z_C(K)|^{-n}$.

	$G(z)=0$, equation determining $z=z_C(K)$	behaviour of $G(z)$ as $z \uparrow z_C(K)$	behaviour of E as $z \uparrow z_C(K)$
2D $f=0$	$kU(z)=0$	$G(z) \propto z - z_C(K) ^n$	$E \propto z - z_C(K) ^{-n}$
2D $f \neq 0$	$kU(z) - f = 0$	$G(z) \propto z - z_C(K) ^n$	$E \propto z - z_C(K) ^{n-1}$
3D $f=0$	$\{\mathbf{K} \cdot \mathbf{U}(z)\}/ \mathbf{U}(z) =0, \mathbf{U}(z) \neq 0$	$G(z) \propto z - z_C(K) ^n$	$E \propto z - z_C(K) ^{j^{n-2}}$
	$ \mathbf{U}(z) =0, \{\mathbf{K} \cdot \mathbf{U}(z)\}/ \mathbf{U}(z) \neq 0$	$G(z) \propto z - z_C(K) ^n$	$E \propto z - z_C(K) ^{-1} (\log z - z_C(K))^2$ for $n=1$ $E \propto z - z_C(K) ^{n-2}$ for $n \geq 2$
	$\{\mathbf{K} \cdot \mathbf{U}(z)\}/ \mathbf{U}(z) =0, \mathbf{U}(z) =0$	$G(z) \propto z - z_C(K) ^{m+n}$	$E \propto z - z_C(K) ^{j^{m+n-2}}$
3D $f \neq 0$	$\mathbf{K} \cdot \mathbf{U}(z) - f = 0$	$G(z) \propto z - z_C(K) ^n$	$E \propto z - z_C(K) ^{j^{n/2-2}}$

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Appendix

Wave action theorem

In this appendix, we briefly review the derivation of wave action theorem (for full discussion, see Andrews and McIntyre 1978a, 1978b). Linearizing about a slowly varying basic flow $U_\lambda = U_\lambda(x_\mu, t)$, the Boussinesq equations become

$$\begin{aligned} Du_\lambda + u_\mu \partial_\mu U_\lambda + \partial_\lambda p - \delta_{\lambda 3} b + \varepsilon_{\lambda \mu \nu} f_\mu u_\nu &= 0, \\ Db + u_\lambda \partial_\lambda B &= 0, \\ \partial_\lambda u_\lambda &= 0, \\ D &= \partial_t + U_\lambda \partial_\lambda, \end{aligned} \quad (\text{A-1})$$

where the subscripts $\lambda, \mu, \dots$, which run from 1 to 3, represent spatial directions, 1 and 2 are the horizontal, and 3 the vertical, with the convention of summation, e.g., $\partial_\lambda u_\lambda = \sum_{\lambda=1}^3 \partial_\lambda u_\lambda$. ∂_t and ∂_λ are respectively the temporal and spatial derivatives $\partial/\partial t$ and $\partial/\partial x_\lambda$, where t and x_λ are respectively the time and spatial coordinates. $f_\lambda = \delta_{\lambda 3} f$ is the Coriolis parameter in vector form. $\varepsilon_{\lambda \mu \nu}$ is defined as $\varepsilon_{\lambda \mu \nu} = 1$ if (λ, μ, ν) is an even permutation of $(1, 2, 3)$, $\varepsilon_{\lambda \mu \nu} = -1$ if (λ, μ, ν) is an odd permutation of $(1, 2, 3)$ and $\varepsilon_{\lambda \mu \nu} = 0$ otherwise. B is the buoyancy of the basic flow. The slowly varyingness means that the derivatives of the basic flow are small.

$$\begin{aligned} \partial_t U_\lambda &= O(\varepsilon), \quad \partial_\mu U_\lambda = O(\varepsilon), \quad \partial_t B = O(\varepsilon), \\ \partial_\lambda B &= C \delta_{\lambda 3} + O(\varepsilon), \quad C = \text{constant}, \\ \varepsilon &\sim 10^{-1}. \end{aligned} \quad (\text{A-2})$$

Let $L = L(\psi, D\psi, \partial\psi, \partial U, \partial B)$ be the lagrangian, where $\psi = \psi_n$ ($n = 1, 2, \dots$) are the disturbance variables, and $\partial\psi, \partial U$ and ∂B stand for spatial derivatives of ψ, U and B , respectively. Eq.(A-1) are recovered by the Euler equations for the Lagrangian L . Since the basic flow is assumed slowly varying, the following local sinusoidal wave solution is possible,

$$\psi_n = \Psi_n \cos \theta, \quad (\text{A-3})$$

where Ψ_n is slowly varying in space and time, and θ is fast varying. The derivatives of θ , which are the local wave number K_λ and frequency ω , are slowly varying.

$$\partial_\lambda \theta = K_\lambda \quad \text{and} \quad \partial_t \theta = -\omega. \quad (\text{A-4})$$

The averaged Lagrangian Λ is a function of $\Psi, D\Psi, \partial\Psi, D\theta, \partial\theta, \partial U, \partial B$,

$$\begin{aligned} (1/2\pi) \int_0^{2\pi} d\theta L &= \Lambda \\ &= \Lambda(\Psi, D\Psi, \partial\Psi, D\theta, \partial\theta, \partial U, \partial B). \end{aligned} \quad (\text{A-5})$$

If the basic flow is uniform, i.e., $\partial U = \partial\Psi = D\Psi = 0$, $\partial B = \text{constant}$, then the lagrangian Λ is reduced to a function of $\Psi, D\theta$ and $\partial\theta$,

$$\Lambda^{(0)} = \Lambda^{(0)}(\Psi, D\theta, \partial\theta). \quad (\text{A-6})$$

By a coordinate transformation $\{x_\lambda \rightarrow x_\lambda - U_\lambda t, t \rightarrow t\}$, $\Lambda^{(0)}$ becomes $\Lambda^{(0)}(\Psi, \partial_t \theta, \partial \theta)$. From this lagrangian, a conserved quantity E is constructed as

$$E = (\partial_t \theta) \delta \Lambda^{(0)}(\Psi, \partial_t \theta, \partial \theta) / \delta \partial_t \theta - \Lambda^{(0)}(\Psi, \partial_t \theta, \partial \theta). \quad (\text{A-7})$$

Since the disturbance is on a quiescent basic flow in the new coordinates system, the conserved quantity is identified with the disturbance energy. The disturbance energy is expressed in the original coordinates system as

$$E = (D\theta) \delta \Lambda^{(0)}(\Psi, D\theta, \partial \theta) / \delta D\theta - \Lambda^{(0)}(\Psi, D\theta, \partial \theta). \quad (\text{A-8})$$

The Euler equations for the averaged Lagrangian (A-5) are

$$\delta \Lambda / \delta \Psi_n - \partial_t (\delta \Lambda / \delta \partial_t \Psi_n) - \partial_\lambda (\delta \Lambda / \delta \partial_\lambda \Psi_n) = 0, \quad (\text{A-9})$$

$$\partial_t (\delta \Lambda / \delta \partial_t \theta) + \partial_\lambda (\delta \Lambda / \delta \partial_\lambda \theta) = 0. \quad (\text{A-10})$$

Because of the slowly varyingness, the leading order approximations of (A-9) and (A-10) are

$$\delta \Lambda^{(0)} / \delta \Psi_n = 0, \quad (\text{A-11})$$

$$\partial_t (\delta \Lambda^{(0)} / \delta \partial_t \theta) + \partial_\lambda (\delta \Lambda^{(0)} / \delta \partial_\lambda \theta) = 0. \quad (\text{A-12})$$

By the linearity, $\Lambda^{(0)}$ is a bilinear form of Ψ_n . Then Eq.(A-11) implies that

$$\Lambda^{(0)} = \sum_{mn} D(D\theta, \partial \theta)_{mn} \Psi_m \Psi_n = \Xi \sum_n \Psi_n \Psi_n, \\ \Xi = \Xi(D\theta, \partial \theta) = 0. \quad (\text{A-13})$$

Noticing Eq.(A-4), the second equation of (A-13) gives the dispersion relation. Using (A-8) and (A-13), Eq.(A-12) can be rewritten as

$$\partial_t (E/D\theta) + \partial_\lambda \{ (E/D\theta) (\delta \Xi / \delta \partial_\lambda \theta) / (\delta \Xi / \delta \partial_t \theta) \} = 0. \quad (\text{A-14})$$

Since $\Xi = 0$ is the dispersion relation, $(\delta \Xi / \delta \partial_\lambda \theta) /$

$(\delta \Xi / \delta \partial_t \theta) = -(\delta \Xi / \delta K_\lambda) / (\delta \Xi / \delta \omega) = \partial \omega / \partial K_\lambda$ is the group velocity C_λ^G . So, Eq.(A-14) becomes

$$\partial_t (E/\Omega) + \partial_\lambda \{ C_\lambda^G (E/\Omega) \} = 0, \\ \Omega = \partial_t \theta + U_\lambda \partial_\lambda \theta = U_\lambda K_\lambda - \omega. \quad (\text{A-15})$$

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