NOTES AND CORRESPONDENCE

Comparison of the Extended Kalman Filter and the Ensemble Kalman Filter Using the Barotropic General Circulation Model

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Abstract

In this study, we compare assimilation techniques of the full-rank extended Kalman filter (EKF) and the ensemble Kalman filter (EnKF), using a barotropic general circulation model, called barotropic S-model, under the perfect model configuration. We investigate the accuracy of the EnKF in reference to the direct computation of the EKF and examine the influence of the localization for EnKF. The barotropic S-model is based on the primitive equations and predicts the vertical mean state of the atmosphere. Although it has the predictability comparable to the operational prediction models, the direct computation of the EKF is possible. Therefore, we can assess the accuracy of the EnKF as a function of the ensemble members. In this study, the convergence of the EnKF to the EKF is examined using various ensemble members of 20, 50, 100, 410, and 1000. The EKF and EnKF directly assimilate the observation in the spectral space, and the observational elements are model variables.

According to the result of the root mean square error (RMSE), the EnKF converges to the full-rank EKF filter when the ensemble member is increased to more than 50. It is demonstrated that the 20 ensemble members are insufficient with respect to the convergence. An empirical orthogonal function (EOF) analysis is conducted using the covariance matrices of analysis error for both filters. The structure of the first EOF (EOF-1) indicates the characteristics of the baroclinic instability waves in mid-latitudes in both filters, showing the same geographical distributions when it has converged. Interestingly, another large analysis error is detected in the Arctic region. Furthermore, the influence of the localization is examined by introducing the local ensemble transform Kalman filter (LETKF), which assimilates the observations in the physical space. The observations which are assimilated by the LETKF are retrieved from the spectral space to the physical space. It is found that the analysis error of the non-localized EnKF in the spectral space is smaller than that of the LETKF in the physical space.

It is concluded from the comparison of the RMSE that more than 50 ensemble members are required for the non-localized EnKF to converge to the full-rank EKF for the practical assimilation in the spectral space under the perfect model configuration of the barotropic general circulation model of the atmosphere.

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1. Introduction

Data assimilation is one of the most important techniques in numerical weather predictions. Many operational centers use 3D-Var (Parrish and Derber

1992), which is an economical and accurate statistical interpolation scheme, but the background error covariance is time-independent. In some operational centers (ECMWF, France, UK, NCEP, Japan, and Canada), 4D-Var (e.g., H. Liu and X. Zou 2001, and Pierre and Jean-Noël 2001) has been used recently, and its accuracy is much higher than that of the 3D-Var. In addition to its high accuracy, the 4D-Var allows the assimilation of many observations, for example, asynchronous observations such as satellite radiances at their correct observation time. However, the computational cost of the 4D-Var implementation is very expensive because the 4D-Var requires a linear tangent model and its adjoint model.

Kalman filter (Kalman 1960, hereafter KF) is one of the data assimilation schemes and gives an optimum solution of linear systems. Jazwinski (1970) extended the KF to an extended Kalman filter (hereafter, EKF) to apply the KF to nonlinear systems. However, it is known that the KF may not be always stable in the case of being applied to the nonlinear systems because the KF is based on a linear theory (e.g., Pham 2001). In the case of applying the EKF to the general circulation models (GCMs), their degrees of freedom are very large, and their orders are at least $O(10^6)$. So, it is impossible to implement the EKF, which inverts the full-rank covariance matrix with the size of at least $O(10^6)$ by $O(10^6)$. In addition to that problem, the EKF requires a tangent linear model of the nonlinear model and its adjoint model. The costs for calculating the both models are very expensive.

Then, Evensen (1994) approximated the covariance matrix of the KF using ensemble predictions, which was called an ensemble Kalman filter (hereafter EnKF). The early EnKFs are classified as perturbed observation filters, but the recent EnKFs are classified as ensemble square root filters, for example, an ensemble adjustment Kalman filter (EAKF, Anderson 2001), an ensemble transform Kalman filter (ETKF, Bishop et al. 2001) and a local ensemble Kalman filter (LEKF, Ott et al. 2002, 2004). Whitaker and Hamill (2002) showed the possibility that the ensemble square root filter is more accurate than the perturbed observation filters because in the step of ensemble update the perturbed observation filters introduce a random error to the observation as a perturbation in order to obtain the analysis perturbation. On the other hand, the ensemble square root filter does not introduce a random error to the observation. The analysis perturbation is

obtained directly by solving the square root of the Kalman filter analysis error covariance. Hunt (2005) and Hunt et al. (2007) developed a local ensemble transform Kalman filter (hereafter LETKF) by applying ETKF to LEKF. The LETKF has an important advantage in which the observations are assimilated in each local patch. Because of this advantage, LETKF has higher performance to implement in parallel computers. Miyoshi et al. (2007a) applied the LETKF to AFES (AGCM (atmospheric general circulation model) for the Earth Simulator, Ohfuchi et al. 2004) with a T159L48 resolution to perform an experimental reanalysis and investigated the stability of the LETKF.

There are a number of studies about the comparison of the EnKF, 3D-Var or 4D-Var with various numerical models. For example, Whitaker et al. (2008) showed that the EnKF outperformed 3D-Var with the NCEP global forecast system, Fertig et al. (2007) compared the LETKF and 4D-Var with the Lorenz-96 system (Lorenz 1996), and Kalnay et al. (2007) discussed which of the EnKF and 4D-Var outperformed. On the other hand, the comparison between the EKF and the EnKF is hardly studied with the primitive equation models because of the problem with the huge matrix inversion of the EKF. Then Zang and Malanotte-Rizzoli (2003) compared the reduced-rank EKF and EnKF with the ocean model. They confirmed that for the strongly nonlinear case, the EnKF outperforms the reduced-rank EKF, and for the weak nonlinear case the performances of the reduced-rank EKF and EnKF are similar. Their reduced-rank EKF assumes a steady state filter. So the reduced-rank EKF can not treat the atmospheric states of flow dependence.

In this study, we compare the EnKF with the EKF under a perfect model scenario using a barotropic S-model (Tanaka 2003), which is based on primitive equations and has only 410 degrees of freedom. So, it is possible to implement the EKF at a full rank. The full-rank EKF can treat the atmospheric states of flow dependence completely. We investigate the accuracy of the EnKF in comparison to the EKF by using the root mean square error (RMSE) of both filters. In addition to comparing accuracy of the both filters, we investigate how much the eigenvalues and the eigenvector of the analysis covariance matrices coincide with each other for the EnKF and EKF. In Section 2, we describe the model, the assimilation system, and our experimental design. In Section 3, we describe

our numerical results of the assimilation. It will be found that the EnKF can approximate the EKF by using ensemble predictions when there are enough ensemble members. Finally, the conclusion is summarized with discussion in Section 4.

2. Methods and experimental settings

2.1 Barotropic S-model

The model equations used in this study are the 3D spectral primitive equations on a sphere (Tanaka 2003). A system of primitive equations with a spherical coordinate of longitude λ , latitude θ , pressure p , and time t may be reduced to three prognostic equations of horizontal motions and thermodynamics for three dependent variables (u, v, ϕ) . Here, u and v are the horizontal velocity. The variable ϕ is a departure of the local isobaric geopotential from the global mean reference state. By expanding the state variables in 3D normal mode functions, we obtain a system of 3D spectral primitive equations in terms of the spectral expansion coefficients:

$$\frac{dw_i}{d\tau} = -i\sigma_i w_i - i \sum_{jk} r_{ijk} w_j w_k + f_i, \quad i = 1, 2, 3, \dots, \quad (1)$$

where τ is a dimensionless time, σ_i is the eigenfrequency of the Laplace's tidal equation, w_i is the expansion coefficient of the state variables, f_i is the expansion coefficient of the external forcing of viscosity and diabatic heating rate, and r_{ijk} is the interaction coefficient for nonlinear wave-wave interactions.

In the 3D spectral representation, the vertical expansion basis functions may be divided into barotropic ($m = 0$) and baroclinic ($m \neq 0$) components. In this study, we use only the barotropic components ($m = 0$) of Rossby modes, by truncating all the baroclinic modes and high-frequency gravity modes (Kasahara 1977). Such a model is equivalent to predicting the vertical average of meteorological variables. The zonal and meridional wave truncation of the present model is equivalent to rhomboidal 20 with an equatorial wall. Because w_i is a complex-valued state variable, only the positive (and zero) zonal wave numbers are considered. Therefore, the degree of freedom of the system is reduced enormously to only $N = 410$:

$$\begin{aligned} \frac{dw_i}{d\tau} &= -i\sigma_i w_i - i \sum_{jk} r_{ijk} w_j w_k + f_i + g_i, \\ i &= 1, 2, 3, \dots \end{aligned} \quad (2)$$

The spectral equation for such a barotropic model Eq. (2) has the same form as that for the baroclinic model Eq. (1), except for the fact that the barotropic-baroclinic interaction g_i appears on the right-hand side. Henceforth, we combine $f_i + g_i$ and designate it as the external forcing s_i of the barotropic model.

The parameterization of the external forcing $s_i = f_i + g_i$ in Eq. (2) is presented by Tanaka (1991, 1998) considering the dynamical processes of the baroclinic instability, the topographic forcing, the biharmonic diffusion, the zonal surface stress, and the Ekman pumping. Since those parameterizations are fundamentally linear approximations to the true forcing and will never be perfect for the real atmosphere, we have attempted in this study to obtain the best linear fit of the forcing s_i based on the 56 years of the NCEP/NCAR reanalysis from 1950 to 2005 as in Tanaka and Nohara (2001):

$$s_i = \tilde{s}_i + s'_i = \tilde{s}_i + \mathbf{A}_{ij} w_j + \mathbf{B}_{ij} w_j^* + \varepsilon_i, \quad (3)$$

where $\tilde{s}_i$ and s'_i are the climate and anomaly of s_i , respectively. The asterisk for w_i represents a complex conjugate of w_i . The linear matrices $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ can be obtained by the standard method of the least square fitting for s'_i to minimize the regression residual ε_i . However, the system matrices $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ mostly represent topographic forcing (TF). For this reason, we have added the energy source in terms of the parameterized baroclinic instability (BC) described by Tanaka (1998). The increased energy source at the synoptic scale requires an additional energy sink, so the biharmonic diffusion (DF), the zonal surface stress (DZ), and the Ekman pumping (DE) are also included in reference to the resulting energy spectrum of the model climate. The final form of the external forcing s_i is expressed as

$$\begin{aligned} s_i &= \tilde{s}_i + \mathbf{A}_{ij} w_j + \mathbf{B}_{ij} w_j^* + (BC)_{ij} w_j + (DF)_{ij} w_j \\ &\quad + (DZ)_{ij} w_j + (DE)_{ij} w_j. \end{aligned} \quad (4)$$

This model with such a forcing is called “barotropic S-model” since a part of forcing is obtained statistically from observation, solving the inverse problem.

2.2 Extended Kalman filter

The KF algorithm suggested by Kalman (1960) cannot be applied to nonlinear systems. So, Jazwinski (1970) extended the KF to the EKF for the nonlinear model. A nonlinear model M is expressed

using a linear tangent model $\mathbf{M}$ around $\mathbf{x}_t$.

$$\mathbf{M} = \left. \frac{\partial M}{\partial \mathbf{x}} \right|_{\mathbf{x}_t}, \quad (5)$$

where $\mathbf{x}_t$ denotes an N -dimensional state variable at time t . In the nonlinear scenario the dynamical system is integrated by the forecast step from a time $t-1$ to time t followed by an analysis step at the time t , in such a way to estimate the most likely system state. In the forecast step the EKF propagates the model solution from time $t-1$ to time t , and in the analysis step the EKF combines the information provided by the observations at that time with the propagated information from the prior observations.

In the forecast step, an analysis at time $t-1$, $\mathbf{x}_{t-1}^a$ and an associated error covariance matrix, $\mathbf{P}_{t-1}^a$, have been produced using the forecast model. Then, the forecasts of $\mathbf{x}_t^f$ and $\mathbf{P}_t^f$ are obtained by:

$$\mathbf{x}_t^f = M(\mathbf{x}_{t-1}^a), \quad (6)$$

$$\mathbf{P}_t^f = \mathbf{M}\mathbf{P}_{t-1}^a\mathbf{M}^\top + \mathbf{Q}, \quad (7)$$

where superscript $\top$ denotes the matrix transpose. $\mathbf{M}^\top$ is an adjoint model of $\mathbf{M}$, and $\mathbf{Q}$ is the model error covariance. Under the linear model, the Eqs. (6) and (7) describe how the model propagates the state variable and the covariance to the future.

The analysis step is a process minimizing the estimated error. The estimated analysis state $\mathbf{x}_t^a$ is obtained by combining the observations $\mathbf{y}_t^o$ and the state estimate $\mathbf{x}_t^f$:

$$\mathbf{x}_t^a = \mathbf{x}_t^f + \mathbf{K}_t(\mathbf{y}_t^o - \mathbf{H}\mathbf{x}_t^f), \quad (8)$$

where $\mathbf{K}_t$ is a matrix called Kalman gain, which describes the weight for the observations and the state estimate, and the matrix $\mathbf{H}$ is an observational operator. The $\mathbf{H}$ in Eq. (8) is obtained by linearizing H around $\mathbf{x}_t^f$, where H is the nonlinear observational operator which maps the model state vector to the observational vector. Likewise, the analysis covariance $\mathbf{P}_t^a$ which minimizes the estimated error by using the observations is obtained by:

$$\mathbf{P}_t^a = [\mathbf{I} - \mathbf{K}_t\mathbf{H}]\mathbf{P}_t^f, \quad (9)$$

where $\mathbf{I}$ denotes a unit matrix.

The Kalman gain matrix $\mathbf{K}_t$ yields the optimum solution of the EKF which minimizes the estimated error. Using $\mathbf{P}_t^f$ and $\mathbf{R}$, the $\mathbf{K}_t$ is evaluated as

$$\mathbf{K}_t = \mathbf{P}_t^f\mathbf{H}[\mathbf{H}\mathbf{P}_t^f\mathbf{H}^\top + \mathbf{R}]^{-1}, \quad (10)$$

where $\mathbf{R}$ represents an observational error covari-

ance matrix at the analysis time. The algorithm of the EKF completes by the Eqs. (6) to (10).

2.3 Ensemble Kalman filter

It is noteworthy that the evaluation of $\mathbf{K}_t$ in Eq. (10) requires an inverse of a huge matrix, which is impossible for the general AGCMs, since the degree of freedom n of the AGCMs is at least $O(10^6)$. The implementation of EKF costs much computational resources to compute the covariance matrices of $n \times n$. In addition, the EKF uses the linearized tangent model $\mathbf{M}$ and its adjoint model $\mathbf{M}^\top$ to compute the covariance matrix, which is not easy and is so difficult to compute for the AGCMs, and if the nonlinear model M is updated, it is necessary that $\mathbf{M}$ and $\mathbf{M}^\top$ are updated.

In contrast, the EnKF does not require the computation of $\mathbf{M}$ and $\mathbf{M}^\top$ of $n \times n$ matrices. Moreover, the EnKF can use the nonlinear model M and nonlinear operator H in stead of $\mathbf{M}$, $\mathbf{M}^\top$ and $\mathbf{H}$. The EKF is approximated by the EnKF using the technique of ensemble predictions. The main idea of the EnKF is as follows. First, we set an ensemble of the initial conditions at time $t-1$ which spread around $\mathbf{x}_{t-1}^a$. The analysis error covariance $\mathbf{P}_{t-1}^a$ can be estimated from the ensemble perturbations. Secondly, the nonlinear model propagates each ensemble member to time t . Then we can compute $\mathbf{P}_t^f$ based on perturbations of the propagated ensemble members at time t .

There are many possible choices of the analysis ensemble. A variety of the EnKFs have been proposed, and one of the main differences among them is how the analysis ensemble is chosen. In this study, we choose the LETKF approach. In the LETKF we compute the analysis step in the space spanned by the ensemble perturbations.

Let $\mathbf{X}$ denote an $n \times m$ matrix composed of m ensemble members of an n -dimensional variables in the local patch. That is,

$$\mathbf{X} = [\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(m)}], \quad (11)$$

where $\mathbf{x}^{(i)}$ denotes the i -th ensemble member. Predictions are performed for each ensemble member according to the nonlinear model,

$$\mathbf{x}_t^f = M(\mathbf{x}_{t-1}^a). \quad (12)$$

In order to estimate the forecast error covariance, we evaluate the ensemble of the forecast errors:

$$\begin{aligned} \mathbf{E}_t^f &= [\mathbf{x}^{f(1)} - \bar{\mathbf{x}}^f, \mathbf{x}^{f(2)} - \bar{\mathbf{x}}^f, \dots, \mathbf{x}^{f(m)} - \bar{\mathbf{x}}^f], \\ &= \mathbf{X}_t^f - \bar{\mathbf{X}}_t^f, \end{aligned} \quad (13)$$

where $\bar{\mathbf{x}}_t^f$ and $\bar{\mathbf{X}}_t^f$ denote the ensemble mean vector and the matrix of the forecast, respectively. The forecast error covariance $\mathbf{P}_t^f$ may be estimated as,

$$\mathbf{P}_t^f = \frac{\mathbf{E}_t^f (\mathbf{E}_t^f)^\top}{m-1} = \mathbf{E}_t^f \tilde{\mathbf{P}}_t^f (\mathbf{E}_t^f)^\top, \quad (14)$$

where, $\tilde{\mathbf{P}}_t^f = \frac{\mathbf{I}}{m-1}$. The $\tilde{\mathbf{P}}_t^f$ is a covariance matrix in the space spanned by the ensemble forecast errors in the m -dimensional space. Similarly, the analysis error covariance matrix $\mathbf{P}_t^a$ may be computed by

$$\mathbf{P}_t^a = \mathbf{E}_t^f \tilde{\mathbf{P}}_t^a (\mathbf{E}_t^f)^\top, \quad (15)$$

where $\tilde{\mathbf{P}}_t^a$ is given by

$$\begin{aligned} \tilde{\mathbf{P}}_t^a &= [(m-1)\mathbf{I}/\rho + (\mathbf{H}\mathbf{E}_t^f)^\top \mathbf{R}^{-1} (\mathbf{H}\mathbf{E}_t^f)]^{-1}, \\ &= [\mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top]^{-1}, \\ &= \mathbf{V}\mathbf{\Lambda}^{-1}\mathbf{V}^\top, \end{aligned} \quad (16)$$

where ρ is a covariance inflation parameter, which inflates the analysis error covariance matrix to avoid underestimating the error covariance and filter divergence. The ρ is slightly larger than 1, and the covariance inflation is expressed by multiplying the square root of ρ by the $\mathbf{E}^f$. The analysis error covariance $\tilde{\mathbf{P}}_t^a$ can be calculated by solving the eigenvalue problem of $(\tilde{\mathbf{P}}_t^a)^{-1}$, where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues, and $\mathbf{V}$ is a matrix containing eigenvectors with the orthonormal conditions $\mathbf{V}^\top \mathbf{V} = \mathbf{I}$. We assume that the observational errors are uncorrelated. Therefore, $\mathbf{R}$ is diagonal, and the inverse is trivial. In the model space, the analysis ensemble mean is evaluated as

$$\bar{\mathbf{x}}_t^a = \bar{\mathbf{x}}_t^f + \mathbf{E}_t^f \tilde{\mathbf{P}}_t^a (\mathbf{H}\mathbf{E}_t^f)^\top \mathbf{R}^{-1} (\mathbf{y}_t^o - \mathbf{H}\bar{\mathbf{x}}_t^f), \quad (17)$$

and the analysis perturbation may be obtained as,

$$\begin{aligned} \mathbf{E}_t^a &= \mathbf{E}_t^f [(m-1)\tilde{\mathbf{P}}_t^a]^{1/2}, \\ &= \mathbf{E}_t^f \sqrt{m-1} \mathbf{V}\mathbf{\Lambda}^{-1/2} \mathbf{V}^\top. \end{aligned} \quad (18)$$

Finally, the analysis ensemble $\mathbf{X}_t^a$ is prepared using Eq. (18) as,

$$\mathbf{X}_t^a = \bar{\mathbf{X}}_t^a + \mathbf{E}_t^a, \quad (19)$$

where $\bar{\mathbf{X}}_t^a$ is the ensemble mean matrix of the analysis. Furthermore, we can use the nonlinear observational operator H in place of the linear observational operator $\mathbf{H}$ as

$$\mathbf{H}\mathbf{E}^f \simeq H(\mathbf{X}^f) - H(\bar{\mathbf{X}}^f), \quad (20)$$

$$\mathbf{H}\bar{\mathbf{x}}^f \simeq H(\bar{\mathbf{x}}^f). \quad (21)$$

Hence, the LETKF has been completed without using the Kalman gain matrix and the linear operators, explicitly.

2.4 Experimental settings

The EKF and the LETKF are implemented to the above-mentioned barotropic S-model under the perfect model scenario.

First, the EKF and LETKF assimilate the observations every 6 hours in the spectral space. In the physical space with the small ensemble size, the spurious covariance among distant points due to sampling errors can be reduced by covariance localization (Houtekamer and Michell 2001; Hamill et al. 2001). According to Szunyogh et al. (2005), there is no clear difference between the ensemble sizes 40 and 80 in the case of optimum localization. However, in the spectral space, the error covariance of the spectral expansion coefficient w_i cannot be appropriately localized, because interactions between different coefficients are highly non-local. So, the LETKF in this study does not employ the localization when the assimilation is conducted in the spectral space. The LETKF experiments employ 20, 50, 100, 410 and 1000 ensemble members. The period of the experimental assimilation is from 00Z 1 January 1990 to 00Z 31 January 1990. The covariance inflation parameter $\rho_{nonlocal}^m$ ($m = 20, 50, 100, 410, 1000$) is changed for each experiment to minimize the estimated errors ranging from 1.005 to 1.10. Specifically, $\rho_{nonlocal}^{20} = 1.10$, $\rho_{nonlocal}^{50} = 1.03$, $\rho_{nonlocal}^{100} = 1.01$, $\rho_{nonlocal}^{410} = 1.005$, $\rho_{nonlocal}^{1000} = 1.005$. The truth of w_i^t is the model output from the control run, and the observation w_i^o is made by adding random vectors to the truth. The random vectors are generated from the normal distribution having the observational error statistics at the gravity wave level, which is a white noise of 50 J/m² in its energy level (see the observational analysis by Tanaka 1985). The noise in its energy level is equivalent to about 5.9 m in barotropic height and 1.0 m/s in wind speed. For simplicity, we assume that all model variables are observed, i.e., the dimension of the observation w_i^o is the same as the dimension of the truth w_i^t . So, H and its tangent linear operator $\mathbf{H}$ are equal to the unit matrix $\mathbf{I}$. We assume that the model error covariance $\mathbf{Q}$ is 0 because the perfect model has been assumed. At each analysis time, we evaluate the analysis error using the RMSE between the true state and the analysis ensemble mean and compute the ensemble spread. The ensemble spread is mea-

sured using the trace of the analysis covariance matrix in (15).

Second, the LETKF experiment is conducted in the physical space for the confirmation purpose in order to investigate the impacts of the localization. This LETKF is the non-local patch version (Miyoshi et al. 2007b). The ensemble member is chosen to be 20, 50, 100 and 410 and the observations are assimilated every 6 hours. To assimilate the same observations of the experiments as in the spectral space, the observations in the spectral space are transformed to the observations in the physical space, and the observational covariance matrix is computed using the observations in the physical space. The observational elements are barotropic height and horizontal wind components. The physical space for the barotropic model is constructed on the 2-dimensional space, with 72 grid points in longitudes and 30 grid points in Gaussian latitudes in the Northern Hemisphere. The observations cover about 17% grid points of the entire 2-dimensional grid space. It is assumed that the thinned observational errors are not correlated with each other. So the observational error covariance matrix is diagonal. In this study, the Gaussian-like fifth order polynomial function (Gaspari and Cohn 1999) is adopted for the localization, by multiplying its inverse to the diagonal elements of the observational covariance matrix. The localization scale is defined by one standard deviation. The horizontal localization scales for 20, 50, 100, and 410 members are 900 km, 900 km, 1200 km and 1200 km, respectively. The Gaussian-like function drops to zero at about 3300 km to 4400 km. The covariance inflation parameter ρ_{local}^m ($m = 20, 50, 100, 410$) is changed for each experiment to minimize the estimated errors ranging from 1.07 to 1.08. Specifically, $\rho_{local}^{20} = 1.08$, $\rho_{local}^{50} = 1.07$, $\rho_{local}^{100} = 1.08$, $\rho_{local}^{410} = 1.08$. The other settings are the same with the non-localized LETKF experiments.

In this study, the non-localized LETKF is called “EnKF” which assimilates the observations in the spectral space, and the localized LETKF is called “LETKF” which assimilates the observations in the physical space.

2.5 Initial settings

The initial ensemble for the first step of the assimilation experiments is prepared by the method of lagged average forecast (LAF) ensemble. First, we conduct many forecasts starting from different observational times. Let $\mathbf{x}_t$ be the initial value at

the valid time t . If we write the forecast at time t starting from one day before as $\mathbf{x}_t^{f(1)}$, then the forecast with the initial value at k days before may be denoted as $\mathbf{x}_t^{f(k)}$. The forecast errors $\delta\mathbf{x}_t^{f(i)}$ up to $i = 1$ to k are then orthogonalized by analyzing the principal components. In this experiment, the norms of ensemble perturbations are set to about 4 times of the observational error. By cycling the data assimilation, the norm of the ensemble perturbation becomes smaller. The predictions up to k days provide $m = 2k$ ensemble members of the initial perturbation by the pair of positive and negative signs of the vector. Thus, we can compute the initial perturbations by this method up to $m = 2N = 820$. When we need more initial perturbations, we repeat the same procedure using different years of the winter data, although those are not orthogonal. However, there is no problem because the EnKF updates these ensemble members to reflect the analysis error covariance at every analysis cycle.

The initial $\mathbf{P}^a$ of the EKF is set as 2 times of the observational covariance matrix $\mathbf{R}$. This assumption does not have any problem because the $\mathbf{P}^a$ is updated by the EKF at every analysis cycle.

3. Results

Figure 1 shows the eigenvalues of the analysis covariance matrices for the EKF and the EnKFs on 00Z 31 January 1990, after the filters have converged to the truth. There are only 19, 49, and 99 eigenvalues for the EnKF of 20, 50, and 100 ensemble members, respectively. The eigenvalues for the analysis covariance show the variance spectrum of the analysis error. The variance spectra of the EnKF for 20 ensemble members differ from those of the EKF, but the spectra are approximately the same as those of the EKF for the 50 ensemble members. Interestingly, the magnitudes of eigenvalues are smaller than those of the EKF for the ensemble members 100 and more. The difference may come from the assumption of the linearization in the EKF, which is not assumed in the EnKF. It is found from the result that the EnKF has almost converged when the ensemble member has increased to 100. The magnitude of the 100th eigenvalue for the EKF is about $1/10^5$ of that of the first eigenvalue, suggesting that the matrix almost degenerates.

Figure 2 illustrates the first eigenvectors of the analysis covariance matrices for the EKF and the EnKFs on the same day as in Fig. 1, describing

Eigenvalue for EKF and EnKF

Fig. 1. Eigenvalues of the analysis covariance matrix for EKF and EnKFs (ensemble members: 20, 50, 100, 410 and 1000) on 00Z 31 January 1990.

the most dominant pattern of the analysis error. We can see the waves of baroclinic instability over the east Pacific Ocean, the Northern American continent, and the Atlantic Ocean in the cases of the EKF and the EnKFs of 50, 100, 410 and 1000 members. On the other hand, these waves are absent in the cases of EnKFs of 20 members. In all cases there is another peak of error over the Arctic. We can confirm that both of the EKF and the EnKF have converged to the same error pattern for 50, 100, 410 and 1000 members. The second eigenvectors (not shown) are not the same in the EKF and EnKF probably due to the assumption of the linearization in the EKF.

Figure 3 shows the time series of the RMSEs against the truth of the state variables w_i and the analysis ensemble spreads for the EKF and the EnKFs during one month. The observational error is about 2.25×10^{-4} . The analysis errors for both the EKF and the EnKFs are reduced by assimilating the observations iteratively. In the case of the EnKFs with 50, 100, 410 and 1000 members, the RMSEs are comparable to the ensemble spread, respectively. In the case of the EnKF of 20 members,

however, the RMSE is larger than the spread. This shows that the EnKF of 20 members may underestimate the forecast error covariance and does not assimilate the observations. The RMSEs show daily variations, but the ensemble spreads show less daily variations. The RMSEs generally have the same order as the ensemble spreads except for the EnKF of 20 members in this study. The degree of the ensemble spreads can be adjusted by the covariance inflation, which is a constant but can be tuned so that the RMSEs are minimized and become comparable to the ensemble spreads.

Figure 4 illustrates the analysis fields of barotropic height (contours) and the analysis error against the truth on 00Z 31 January 1990. The analysis error of the EnKF for 20 members is larger than that of the EKF. That of the EnKF of 50 members is almost the same as that of the EKF. Moreover, the analysis error of the EnKF for 410 members is very similar to that of the EnKF for 1000 members. The error patterns of the EKF and EnKF of more than 50 members are similar, and are characterized by the baroclinic instability. These results indicate that the accuracy of the EnKF of 50 members and above is almost the same as that of the EKF.

In order to increase the statistical confidence of the result, we repeated the same experiments for 10 cases in different seasons and different years. Figure 5 shows the time series of the mean RMSEs against the truth of the state variables w_i for the average of the 10 cases. The analysis errors decrease for both the EKF and the EnKFs by assimilating the observations iteratively. The analysis RMSE of EnKF for 20 members is the largest, and that for 410 and 1000 members is the smallest. Interestingly, the analysis RMSEs for 100, 410 and 1000 members are smaller than that of the EKF, probably due to the error involved by the model linearization of the EKF. The magnitude of the analysis RMSE for the EKF is almost the same as that of the EnKF for a sufficient number of the members. We see that the EnKF has converged to the EKF for more than 50 members.

Finally, Fig. 6 shows the time series of the RMSEs against the truth of the state variables w_i for the LETKF evaluated in the physical space with localization. The results are compared with the EnKF of 50 ensemble members during one month. The observational error is about 2.25×10^{-4} as before. The analysis RMSE of each LETKF decreases with time, and the RMSE becomes

Eigenvector for Analysis Cov. EKF vs EnKF

(a) EnKF ($m = 20$) (b) EnKF ($m = 50$) (c) EnKF ($m = 100$) (d) EnKF ($m = 410$) (e) EnKF ($m = 1000$)

(f) EKF

Fig. 2. First eigenvectors of the analysis covariance matrix for EKF and EnKFs on 00Z 31 January 1990. (a) EnKF with 20 ensemble members, (b) EnKF with 50 ensemble members, (c) EnKF with 100 ensemble members, (d) EnKF with 410 ensemble members, (e) EnKF with 1000 ensemble members and (f) EKF.

Fig. 3. Time series of RMSEs (solid lines) against truth of the barotropic expansion coefficient w_i of the state variables (u, v, ϕ) and analysis ensemble spreads (dashed lines) for EKF and EnKF for 1 month from 00Z 1 January 1990 to 00Z 31 January 1990 (0 of the horizontal axes corresponds to 00Z 1 January 1990). (a) EnKF with 20 ensemble members, (b) EnKF with 50 ensemble members, (c) EnKF with 100 ensemble members, (d) EnKF with 410 ensemble members, (e) EnKF with 1000 ensemble members and (f) EKF. The observational error is about 2.25×10^{-4} .

smaller than the observational error in each experiment. Moreover, the analysis ensemble spreads are comparable to the RMSEs in each experiment (not shown). It seems that the analysis RMSEs of the LETKF has converged for 20 members and above. However, the RMSE of the LETKF is much larger than that of the EnKF. In the experiments, we attempted to expand the localization scale up to 1500 km. However, the RMSEs are not improved. These results indicate that the localization applied in the LETKF works appropriately in the physical space, and the assimilation has converged as discussed by Szunyogh et al. (2005). However, it is found by the present study that the RMSE of the LETKF in the physical space is not converging to the EnKF in the spectral space where the locali-

zation is not assumed. The discrepancy should be investigated in detail as a future work.

4. Conclusion and discussion

In this study we investigate the accuracy of the analysis fields by the full-rank EKF, EnKF and LETKF under the perfect model scenario with the barotropic S-model. The accuracies of the EKF, EnKF and LETKF are evaluated using the analysis error covariance matrix, RMSE, and the ensemble spread.

First, we compare the analysis error covariance matrix for the EKF and EnKFs by the eigenvalue decomposition of the variance of the analysis error. The magnitude of the 100th eigenvalue for the EKF is about $1/10^5$ of that of the first eigenvalue,

Fig. 4. Analysis fields (contour) and the differences against truth (shaded) of the barotropic height (in m) on 00Z 31 January 1990. (a) EnKF with 20 ensemble members, (b) EnKF with 50 ensemble members, (c) EnKF with 100 ensemble members, (d) EnKF with 410 ensemble members, (e) EnKF with 1000 ensemble members and (f) EKF.

Fig. 5. Average of RMSEs against truth of the barotropic expansion coefficient w_i of the state variables (u, v, ϕ) for EKF and EnKFs (ensemble members: 20, 50, 100, 410 and 1000). The number of the sample is 10, respectively. The observational error is about 2.25×10^{-4} .

Fig. 6. Time series of RMSEs against truth of the barotropic expansion coefficient w_i of the state variables (u, v, ϕ) for LETKF and EnKF (member = 50) for 1 month from 00Z 1 January 1990 to 00Z 31 January 1990 (0 of the horizontal axes corresponds to 00Z 1 January 1990). The observational error is about 2.25×10^{-4} .

suggesting that the covariance matrix is almost degenerate. It is found that the eigenvalue spectrum of the EnKF is smaller than that of the EKF, probably due to the assumption of the linearization in the EKF. However, the most dominant error pattern represented by the EOF-1 of the analysis covariance matrix of the EnKF has converged to the EKF when the ensemble members increased to 100 and above. The spatial pattern represents the characteristics of the parameterized baroclinic instability which yields the largest error growth. Another rapid error growth is detected over the Arctic probably due to the nonlinear interactions. It is concluded from the results that the ensemble size of the EnKF of more than 50 is required to converge to the EKF for the specific example of the barotropic S-model.

Next, the RMSEs and the ensemble spread are compared to the EKF and EnKF. The RMSEs are

comparable to the ensemble spread for most of the experiments except for 20 members of the EnKF, although the experiment for the EnKF with 20 members does not result in filter divergence. The EnKF requires more than 50 ensemble members to accurately estimate the analysis error. The daily variations of the ensemble spreads are smaller than those of the RMSE. The former represents the characteristics of the model which does not strongly depend on the observation, whereas the latter is influenced by the introduced random noise in the time series. The accuracy of the EKF and the EnKFs is examined by the analysis error fields. The error in the analysis fields of the EnKFs is very similar to that of the EKF. The more the number of members increases, the smaller the differences become. The results are confirmed from the comparison of the RMSEs for the EKF and the EnKFs.

In order to increase the statistical confidence of the result, we have repeated 10 experiments to investigate the average of the results. The magnitude of the analysis RMSE in the EKF is the same as that of the EnKF for the member size of more than 50. The RMSEs of the EnKFs for 100, 410 and 1000 members are smaller than that of the EKF, probably due to the error involved by the linearization of the EKF. However, the EnKF does not require the linearization of the model. Furthermore, the RMSE of the EnKF for 410 members is almost the same as that of 1000 members. These results suggest that the EnKF outperforms the EKF. In the barotropic S-model the error growth is relatively small because the dynamic instability of the model is weak (Tanaka and Nohara 2001). In Zang and Malanotte-Rizzoli (2003), in the weakly nonlinear case the performances of the reduced-rank EKF and EnKF are similar. In this study, the performance of the full-rank EKF is also similar to that of EnKF.

Finally, the influence of the localization of the LETKF is investigated. In the physical space, the analysis errors of the LETKF become smaller than the observational errors. So the LETKF works appropriately. It seems that the analysis RMSEs of the LETKF have converged for the member size more than 20. This result agrees with that by Szunyogh et al. (2005). However, the comparison of the LETKF in the physical space and EnKF in the spectral space indicates that the RMSEs of the LETKF are much larger than that of EnKF. As the first reason, it is considered that the localization influences the analysis field. Because the degree of freedom of the barotropic S-model is very small in the spectral space, the EnKF can assimilate all the observations even for the small ensemble size. On the other hand, the degree of the freedom is much larger in the physical space than in the spectral space for the barotropic S-model. For the physical space, it is necessary to apply the localization in order to reduce the spurious covariance among distant points due to the sampling errors, so that the filter will not diverge. In this study the influence of the localization is examined by applying the LETKF to the physical space of the barotropic S-model. It is found that the LETKF can capture the analysis errors of the regional scale such as synoptic disturbances. However, it can not capture the analysis errors of the global scale such as teleconnection or zonally symmetric errors. As the second reason, it is considered that the observations are different

between the spectral space and the physical space. The observations in the physical space are retrieved from the observations in the spectral space. So, it is considered that the observations in the physical space have the local information and involve the retrieved errors. It is not guaranteed that the observational errors in physical space are equivalent to those in the spectral space. On the other hand, the EnKF assimilates the observations whose elements are model variables in the spectral space. So, it is considered that the observations have the information of the hemispheric scale. Therefore, it is necessary to investigate the influence of the localization, the discrepancy of the assimilated observations and the error correlation scale of the barotropic S-model in detail as a future work.

As is demonstrated in this study, it is found that the EnKF has a good performance compared to the full-rank EKF for the barotropic S-model in which the dynamic instability is weak. Although the EnKF performs better as the ensemble members increase from 20 to 50, the performance has converged as the members increase from 50 to 1000.

It is concluded from the comparison of the full-rank EKF and the non-localized EnKF that the ensemble members of more than 50 is required for the non-localized ensemble Kalman filter to converge to the extended Kalman filter for the practical assimilation in the spectral space under the perfect model setting of the barotropic general circulation model of the atmosphere.

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