The Effect of the Horizontal Component of the Angular Velocity of the Earth’s Rotation on Inertia-Gravity Waves

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Abstract

By using a linearized Boussinesq model on the tangent plane in the mid-latitudes, how the effect of the horizontal component of the angular velocity of the Earth’s rotation ($f_H$-effect) modifies the characteristics of inertia-gravity waves is examined. The $f_H$-effect widens the range of the intrinsic wave frequency. A physical interpretation of this modification is made in terms of restoring forces. There are cases in which the rotational direction of the hodograph with time is anticlockwise (clockwise) even in the Northern (Southern) Hemisphere, unlike the case of $f_H = 0$. Considering the form stress over the potential temperature surface, the sign of the vertical group velocity can be the same as that of the vertical phase speed, unlike the case of $f_H = 0$, because the $f_H$-effect can reverse the direction of the form stress through the vertical force balance. The minimum frequency increases with the buoyancy frequency ($N$) for $f_H \neq 0$, when the latitude and the direction of the wavevector are fixed. This fact indicates that waves trapped in a weakly stratified layer (WSL) exist, where $N$ is low. Using an idealized vertical profile of $N$ in the form of a square well, the trapped wave solution is derived. The solution is composed of two plane waves in the WSL, while it decays exponentially outside. Using operational radiosonde data in Japan, it is shown that there is a persistent WSL slightly below the tropopause where the climatological minimum value of $N (N_{\text{min}})$ is about a half times lower than the typical tropospheric value ($\sim 0.01 \text{ s}^{-1}$). The $N_{\text{min}}$ value is not sufficiently small to form trapped waves having wavelengths in a realistic range within a few days, because the condition of $N_{\text{min}} < 0.001 \text{ s}^{-1}$ is necessary. Thus, such trapped waves are rarely observed in the WSL slightly below the tropopause.

Keywords: inertia-gravity wave; traditional approximation; angular velocity of the Earth’s rotation; tropopause; hodograph; form stress

1. Introduction

In many geophysical problems, only a component of the Coriolis force that is proportional to the Coriolis parameter $f_r (\equiv 2\Omega \sin \phi)$ is considered, where $\Omega$ is the magnitude of the angular velocity of the Earth’s rotation $\Omega$ and $\phi$ is the latitude. In this paper, we call this manipulation the traditional approximation (TA). The TA is equivalent to the assumption that the angular velocity of the Earth’s rotation is locally vertical, since $f_r$ is proportional to the vertical component of $\Omega$. When the hydrostatic balance or shallow approximation is used in the equations of motion, the TA needs to be used to satisfy the conservation laws of energy, potential vorticity, and angular momentum (Holton 2004; White et al. 2005). In this paper, we discuss the effect of another component of the Coriolis force proportional to $f_H (\equiv 2\Omega \sin \phi)$, which is two times as large as the horizontal component of $\Omega$. Hereafter, the
In the 1970s, several studies were reported on the effects of \( f_H \)-terms (\( f_{ir} \)-terms). By using a linearized Boussinesq model in the ocean, Saint-Guily (1970) showed that the range of internal wave frequency is enlarged and that there are waves vertically trapped in the pycnocline due to the tilt of \( \Omega \) from the vertical.

Recently, the \( f_H \)-effect has been revisited for the development of high-resolution weather prediction models of atmosphere and ocean including small-scale phenomena. Thuburn et al. (2002) and Kasahara (2003) analytically examined wave modes in the ocean bounded vertically at the top and bottom with Cartesian coordinates on the tangent plane. They obtained a solution of a new wave mode, which exists because of the \( f_H \)-effect, in addition to well-known solutions of two modes, acoustic mode and inertia-gravity mode, which are slightly modified by the \( f_H \)-effect. It was shown that this new mode is identical to the mode obtained by Saint-Guily (1970). This new mode is called the sub-inertial mode, while the traditional inertia-gravity mode is called the super-inertial mode. The wave frequencies of the former are slightly lower than the inertial frequency (Coriolis parameter) \( f_I \), while those of the latter are higher than \( f_I \). The sub-inertial mode is mainly present in the weakly stratified layer (WSL) where the buoyancy frequency \( N \) is low, while the super-inertial mode is mainly present in the strongly stratified layer (SSL) (Gerkema and Shrira 2005a; Kasahara and Gary 2006). The reason for the sub-inertial mode presence is that the \( f_{ir} \)-effect is related to vertical motion, which is large in the WSL. At first sight, the sub-inertial mode looks different from the super-inertial mode.

More precise interpretation was given by Durran and Bretherton (2004): The sub- and super-inertial modes are simply the superposition of two plane waves. When \( f_H = 0 \), to satisfy the boundary condition that the vertical velocity is zero at top and bottom boundaries, there is one way of superposing two plane waves with frequencies higher than \( f_I \), which, respectively, propagate energy upward and downward. On the other hand, when \( f_H \neq 0 \), there is another possible way of superposing waves with frequencies lower than \( f_I \). The reason for the latter is that the \( f_{ir} \)-terms modify the dispersion relation of the plane wave. In this paper, the modification of the dispersion relation is discussed by examining the vertical group velocity \( c_{gr} \) in terms of the form stress.

Unlike in the ocean, there is no upper boundary in the atmosphere. This fact means that atmospheric waves are usually not “modes” that are formed between the two boundaries. Thus, it is valuable to examine the \( f_{ir} \)-effect on atmospheric waves. In this paper, it is shown that the sub-inertial waves vertically trapped in the WSL can exist. According to Durran and Bretherton (2004), the necessary condition for the existence of the sub-inertial modes in the vertically bounded domain is that the frequency takes the minimum \( \omega_{\text{min}} \) at a finite vertical wavenumber. However, this condition is not sufficient for the existence of trapped waves in the WSL. As shown later, it is also necessary that \( \omega_{\text{min}} \) is a monotonically increasing function of \( N \). For the ocean, Gerkema and Shrira (2005a) briefly noted the possibility of similar sub-inertial waves vertically trapped in the WSL between the seasonal thermocline and the permanent pycnocline. However, existence of the trapped wave solution in the vertically unbounded domain has not been exactly shown. Thus, it is meaningful to examine the \( f_{ir} \)-effect in the WSL. The purpose of this study is to examine the \( f_{ir} \)-effect on the inertia-gravity waves in the atmosphere.

In Section 2, plane wave solutions in a linearized Boussinesq model containing the \( f_{ir} \)-terms on the mid-latitudes tangent plane are examined. In particular, the modifications to the dispersion relation and the hodograph by the \( f_{ir} \)-effect are discussed in terms of the force balance. It is shown that taking opposite signs of the vertical group velocity \( c_{gr} \) and vertical phase speed \( c_{ph} \) is a general wave characteristic for internal plane waves and equatorial waves in the linearized Boussinesq and primitive equations for \( f_H = 0 \). Furthermore, how this characteristic is modified by the \( f_{ir} \)-effect is examined. In Section 3, analytic solutions trapped in the WSL are examined for an idealized distribution of \( N \) having the form of a square well. In Section 4, by using radiosonde data in Japan, the structure of \( N \) in the meridional cross section is shown. Using the climatological minimum value of \( N \), the \( f_{ir} \)-effect on inertia-gravity waves in the real atmosphere is examined. Summary and concluding remarks are discussed in Section 5.

## 2. Characteristics of plane wave solutions in the linearized Boussinesq model with \( f_H \neq 0 \)

### 2.1 Formulation and plane wave solutions

In the Cartesian coordinates \((x, y, z)\) with \(x\), \(y\), and \(z\) directed eastward, northward and upward, respectively, the components \((k, l, m)\) of the wavevector \(k\) of a plane wave solution are expressed as

\[
(k, l, m) = (\kappa \cos \beta \cos \alpha, \kappa \cos \beta \sin \alpha, \kappa \sin \beta),
\]  
(2.1)
where $\kappa$ is the magnitude of $k$, and $\alpha$ and $\beta$ are the angles of $k$ from the x axis and the horizontal plane, respectively (Fig. 1). The $xyz$ coordinates are rotated by the angle $\alpha$ around the z axis so as to let a new $x$ axis point to the horizontal wavevector $k_H$. These coordinates are referred to as $x'y'z$ coordinates. In the $x'y'z$ coordinates, the equations of the linearized Boussinesq model containing the $f_{ir}$-terms for the plane wave solution are expressed in the following (Gerkema and Exarchou 2008).

\[
\frac{\partial \tilde{u}}{\partial t} - f_\text{s} \tilde{v} + f_{ir} \cos \alpha \, w + \frac{1}{\rho_0} \frac{\partial p}{\partial x} = 0, \tag{2.2a}
\]
\[
\frac{\partial \tilde{v}}{\partial t} + f_{ir} \cos \alpha \, \tilde{u} - f_\text{s} \sin \alpha \, w = 0, \tag{2.2b}
\]
\[
\frac{\partial \tilde{w}}{\partial t} - f_\text{s} \cos \alpha \, \tilde{u} + f_{ir} \sin \alpha \, \tilde{v} - b + \frac{1}{\rho_0} \frac{\partial p}{\partial z} = 0, \tag{2.2c}
\]
\[
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{w}}{\partial z} = 0, \tag{2.2d}
\]
\[
\frac{\partial p}{\partial t} + N^2(z)w = 0. \tag{2.2e}
\]

Here a motionless and stably stratified basic state is considered. The dependent variables are the velocity components ($\tilde{u}$, $\tilde{v}$, $w$) in ($x'$, $y'$, $z'$) together with $p$ and $b$, which are the perturbation pressure and the buoyancy potential temperature from the basic state components considered. The dependent variables are the velocity $\tilde{u}$, $\tilde{v}$, $w$, $\tilde{p}$, $\tilde{b}$, $\tilde{q}$, where $\theta$ is the perturbation potential temperature from the basic state $\theta_0$ and $\bar{\theta}_0$ is an average of $\theta_0$. The constant basic density is $\rho_0$. The square of the buoyancy frequency is defined as $N^2 \equiv \frac{\partial \theta}{\partial z}$. To focus on the $f_{ir}$-effect, the latitudinal gradient of $f_\text{s}$ is ignored. In this paper, the case of $N > f_\text{s} > 0$ is considered, which is usually valid for the real atmosphere in the mid-latitudes in the Northern Hemisphere. Note that similar theories shown below can be derived for the Southern Hemisphere.

In the $x'y'z$ coordinates, $\tilde{k}$ is expressed as

\[
(\tilde{k}, 0, m) = (\kappa \cos \beta, 0, \kappa \sin \beta), \tag{2.3}
\]

where $\tilde{k}$ ($>0$) is the $x'$ component of $k$. The fluctuation components ($\tilde{u}$, $\tilde{v}$, $w$, $b$, $p$) are expressed as

\[
(\tilde{u}, \tilde{v}, w, b, p) = (\tilde{u}_0, \tilde{v}_0, w_0, b_0, p_0) \exp i(\tilde{k}x' + mz - \tilde{\omega}t), \tag{2.4}
\]

where subscripts 0 denote constant complex numbers. The frequency $\tilde{\omega}$ is taken to be positive without losing generality. By considering a constant buoyancy frequency $(N(z) = N_0 (= \text{const}))$ and substituting (2.4) into (2.2a)-(2.2e), the dispersion relation is obtained:

\[
\tilde{\omega}^2 = \frac{k^2N_0^2 + (f_{ir} \sin \alpha + mf_\text{s})^2}{k^2 + m^2} = \frac{(k^2 + l^2)N_0^2 + (f_{ir} + mf_\text{s})^2}{k^2 + l^2 + m^2}. \tag{2.5}
\]

The right-hand side of (2.5) is the expression in the $xyz$ coordinates.

Figure 2a shows dispersion curves, i.e., $\tilde{\omega}$ as a function of $m$ normalized by $\tilde{k}$ for $f_{ir} = 0$ (a long dashed curve) and $f_{ir} = 0$ (a solid curve). Here we take $\alpha = \pi/2$, $\phi = 45^\circ$, and $N_0 = 1.5 f_\text{s}$ as an example. It is seen that while $\tilde{\omega}_{\text{min}} = f_\text{s}$ and $\tilde{\omega}_{\text{max}} = N_0$ for $f_{ir} = 0$, the range of $\tilde{\omega}$ is wider and $\tilde{\omega}$ does not show symmetry around $m = 0$ for $f_{ir} = 0$. The inclination of the dispersion curve for $f_{ir} = 0$ indicates that the vertical group velocity $c_\text{g}$ ($\equiv \partial \tilde{\omega}/\partial m$) is positive (negative) for $m < 0$ ($m > 0$) and zero for $m = 0$. For $f_{ir} = 0$, the dispersion curve is modified so that $c_\text{g}$ is negative when $m < m_{\text{min}} < 0$ and positive when $0 \leq m < m_{\text{max}}$, respectively, where $m_{\text{min}}$ and $m_{\text{max}}$ are defined as $\tilde{\omega}_0 (m_{\text{min}}) = \tilde{\omega}_{\text{min}}$ and $\tilde{\omega}_0 (m_{\text{max}}) = \tilde{\omega}_{\text{max}}$ for a given $\alpha$. In other words, the energy propagation is not always upward (downward) when the phase propagation is downward (upward) for $f_{ir} = 0$. These features are always seen when $0 < \alpha < \pi$.

Here, although we considered an unusual situation of the ratio of $f_\text{s}$ and $k_\text{n}$ to clarify the modification by the $f_{ir}$-effect, similar nature is observed even when $N_0 \gg f_\text{s}, f_\text{i}$ as in the real atmosphere.

Figure 2b shows dispersion curves for $\alpha = \pm \pi/2$, $\phi = 45^\circ$, and $N_0 = 1.5 f_\text{s}$.
When \( \pi < \alpha < 2\pi \) or \( f_v < 0 \) (i.e., in the Southern Hemisphere), the dispersion curve for \( f_H = 0 \) takes a reflected shape about \( m = 0 \). When either \( \alpha = 0 \) or \( \alpha = \pi \) (i.e., \( k_H \) points purely eastward or westward), the dispersion curve for \( f_H = 0 \) accords with that for \( f_H = 0 \). The \( f_H \)-effect does not modify the wave characteristics in this case because the effect appears in the form of \( f_H \sin \alpha \) in (2.5). The range of \( \hat{\omega} \) is the most wide for \( \alpha = \pm \pi/2 \) (Fig. 2b) for the same horizontal wave number.

### 2.2 Intuitive interpretation on the modification of the maximum and minimum of \( \hat{\omega} \) by the \( f_H \)-effect

Gerkema and Exarchou (2008) mathematically showed that the range of \( \hat{\omega} \) becomes wider for \( f_H = 0 \). However, its physical explanation has not been clarified yet. In this subsection, the modifications to the maximum and minimum frequencies (\( \hat{\omega}_{\max} \) and \( \hat{\omega}_{\min} \)) by the \( f_H \)-effect are examined from the viewpoint of restoring forces. In this paper, the case of \( 0 < \alpha < \pi \) is mainly discussed, but a similar discussion can be easily made for the other case of \( \pi < \alpha < 2\pi \).

A new \( x \) axis pointing to the \( k \) direction is taken,
which corresponds to the procedure to rotate the \( x'y'z' \) coordinates by the angle \(-\beta\) around the \( y'\) axis (see Fig. 1). These new coordinates in which \( k \) is expressed as \((k (> 0), 0, 0)\) are hereafter referred to as \( x'y'z'' \) coordinates (Fig. 3). The linearized Boussinesq model equations are expressed in the following.

\[
-F_x\mathbf{\hat{v}} + F_y\mathbf{\hat{w}} - b \sin \beta + \frac{1}{\rho_c} \frac{\partial \rho}{\partial x'} = 0, 
\]

(2.6a)

\[
\frac{\partial \mathbf{\hat{v}}}{\partial t} = F_x\mathbf{\hat{v}} = 0, 
\]

(2.6b)

\[
\frac{\partial \mathbf{\hat{w}}}{\partial t} + F_x\mathbf{\hat{v}} - b \cos \beta = 0, 
\]

(2.6c)

\[
\frac{\partial \mathbf{\hat{u}}}{\partial t} = 0, 
\]

(2.6d)

\[
\frac{\partial \mathbf{\hat{w}}}{\partial t} + N_0 \cos \beta \mathbf{\hat{v}} = 0, 
\]

(2.6e)

where \((\mathbf{\hat{u}}, \mathbf{\hat{v}}, \mathbf{\hat{w}})\) are the velocity components for \((x', y', z'')\), and the \( i \)th \((1-3)\) components of \(2\boldsymbol{\Omega}\):

\[
F_x = 2\boldsymbol{\Omega} \cdot e_x = f_r \sin \beta + f_{II} \sin \alpha \cos \beta, 
\]

(2.7a)

\[
F_y = 2\boldsymbol{\Omega} \cdot e_y = f_{II} \cos \alpha, 
\]

(2.7b)

\[
F_z = 2\boldsymbol{\Omega} \cdot e_z = f_r \cos \beta - f_{II} \sin \alpha \sin \beta, 
\]

(2.7c)

where \(e\) is the basis vector of the \( i \)th direction. The continuity equation (2.6d) indicates \( \mathbf{\hat{u}} = 0 \), which means that the parcel motion is only in the \( y'z'' \) plane (an equiphase surface) (Fig. 3). In other words, the inertia-gravity waves for \( f_{II} = 0 \) are transverse waves as in the case of \( f_r \). The \( x' \) momentum equation (2.6a) determines the distribution of \( \rho \) so as to keep the tilt of the parcel motions. The restoring forces for parcels are the Coriolis force including the \( f_{II} \)-terms and the buoyancy force. The equation for \( \mathbf{\hat{w}} \) and the dispersion relation in the \( x'y'z'' \) coordinates become

\[
\frac{\partial^2 \mathbf{\hat{w}}}{\partial t^2} = -(F_{x''} + N_0^2 \cos^2 \beta)\mathbf{\hat{w}}, 
\]

(2.8)

\[
\omega^2 = F_{x''} + N_0^2 \cos^2 \beta. 
\]

(2.9)

Note that (2.9) is identical to (2.5). The equations (2.6b) and (2.8) imply that parcels rotate clockwise with time in the \( y'z'' \) plane, as seen from the \( \boldsymbol{\Omega}_a \) direction (Fig. 3), where \( \boldsymbol{\Omega}_a \) is the projection of \( \boldsymbol{\Omega} \) onto the \( k \) direction.

First, the case of \( \omega = \omega_{max} \) is considered. For \( f_{II} = 0 \), \( \omega_{max} \) is \( N_0 \), independent of \( \alpha \) and latitude. For \( f_r \neq 0 \), this is not the case. When \( 0 \leq \beta < \pi/2 \), \( F_r > f_r \sin \beta \geq 0 \) because of non zero \( f_r \) (i.e., the tilt of \( \boldsymbol{\Omega} \)), as is evident from (2.7a) and Fig. 4a. This means that the \( f_r \)-effect facilitates the parcel motions. Thus, \( \omega \) takes its maximum, which is higher than \( N_0 \). Since the Coriolis force is not perpendicular to the buoyancy force, unlike the case for \( f_{II} \neq 0 \) (see Figs. 5a, b), \( \omega_{max} \) depends on \( N_0 \), \( \alpha \), and latitude. Note that both buoyancy and \( f_r \)-effects increase \( \omega \). For convenience of discussion below, we define \( \beta_{max} \) as \( \omega (\beta_{max}) = \omega_{max} \) for a given \( \alpha \). Note that \( m_{max} \) in Fig. 2 is identical to \( k \sin \beta_{max} \).

Second, the case of \( \omega = \omega_{min} \) is considered. For \( f_{II} = 0 \), \( \omega_{min} \) is \( f_r \), independent of \( \alpha \) and \( N_0 \). For \( f_{II} \neq 0 \), this is also not the case. When \( -\pi/2 < \beta < \beta_{max} \) \( |f_r| > |f_{II} \sin \beta| \) because of non zero \( f_{II} \) (i.e., the tilt of \( \boldsymbol{\Omega} \)), as is evident from (2.7a) and Fig. 4b, where

\[
\tan \beta_{max} = -\frac{f_{II} \sin \alpha}{2f_r}, 
\]

(2.10)

for \( -\pi/2 < \beta_{max} < 0 \). As \( k \) becomes vertical (\( \beta \rightarrow \pm \pi/2 \)), \( \omega \) approaches \( f_r \) because the buoyancy and \( f_r \)-effects become weaker. Since the Coriolis force is not perpendicular to the buoyancy force, unlike the case for \( f_{II} \neq 0 \) (Figs. 5a, b), \( \omega_{min} \) depends on both \( N_0 \) and \( f_{II} \). The \( f_r \)-effect decreases \( \omega \), while the buoyancy effect increases it. Thus, the magnitudes of the two effects have to be compared. The buoyancy effect, which is proportional to \( \cos^2 \beta \) as in (2.9), becomes weak more rapidly than the \( f_r \) effect as \( \beta \rightarrow -\pi/2 \). Thus, \( \omega \) takes its minimum, which is lower than \( f_r \), dependent of \( N_0 \),
(a) $0 < \beta < \pi/2$

(b) $-\pi/2 < \beta < \beta_{th}$

Fig. 4. Magnitudes of $F_x (= |2\Omega_\alpha|$) for $f_H = 0$ (TA) and $f_H \neq 0$ (nonTA) (a) when $0 < \beta < \pi/2$ and (b) when $-\pi/2 < \beta < \beta_{th}$. $2\Omega_\alpha$ is the projection of $2\Omega$ onto the $x'z$ plane, and $f_{Hz} = f_H \sin \alpha$. Note that $0 < \alpha < \pi$.

Fig. 5. Illustration of parcel oscillations and forces acting on the parcel in the $y$-$z$ cross section for (a) $f_H = 0$, (b) $f_H \neq 0$ and $\hat{\omega} = \hat{\omega}_{max}$, (c) $f_H \neq 0$ and $\hat{\omega} = \hat{\omega}_{min}$, and (d) $f_H \neq 0$ and the displacements are parallel to $\Omega$. Here we assume the background pressure field is not disturbed by the parcel’s motion (i.e. parcel method).

latitude and $\alpha$. For convenience of discussion below, we define $\beta_{min}$ as $\hat{\omega} (\beta_{min}) = \hat{\omega}_{min}$ for a given $\alpha$. Note that $m_{min}$ is identical to $\kappa \sin \beta_{min}$.

The exact values of $\hat{\omega}_{min}$ and $\hat{\omega}_{max}$ are calculated (see Appendix for details) as
2.3 Modification of the hodograph by the $f_H$-effect

As a function of the hodograph with time, as well as its shape, is this paper, modification of the rotational direction of one. Gerkema and Exarchou (2008) showed that the major axis of the hodograph is not always parallel to $k_H$ for $f_{it} \neq 0$. However, they only argued about the shape of the hodograph using characteristic coordinates and did not discuss the rotational direction, which gives important information on wave propagation (e.g., Sato 1994). In this paper, modification of the rotational direction of the hodograph with time, as well as its shape, is examined as a function of $\dot{\omega}$ and $m_{\perp}$ by using the $x'y'z$ coordinates, where the $x'$ direction is the same as $k_H$.

The polarization of horizontal velocity for $f_{it} \neq 0$ is

$$\dot{\omega}^2_{max/min} = \frac{N_0^2 + f_{it}^2 + f_{ita}^2 \pm \sqrt{(N_0^2 - f_{it}^2 + f_{ita}^2)^2 + 4f_{ita}^2f_{iv}^2}}{2},$$

(2.11)

$$f_{ita} = f_{it}\sin \alpha,$$

(2.12)

and those for $\beta_{max}$ and $\beta_{min}$ are calculated as

$$\begin{cases} 
\beta_{max} = \gamma \to \\
\beta_{min} = -\pi/2 + \gamma,
\end{cases}$$

(2.13)

respectively, where

$$\gamma = \tan^{-1} \left( \frac{2f_{ita}f_{iv}}{(N_0^2 + f_{it}^2 - f_{it}^2)^2 + 4(N_0^2 + f_{ita}^2 + f_{iv}^2)f_{it}^2} \right).$$

(2.14)

Note that $\tan \gamma = m_{max}/k$ and $-1/\tan \gamma = m_{min}/k$. The formula (2.11) is equivalent to the expression obtained by Gerkema and Shrir (2005a) and Gerkema and Exarchou (2008). The meaning of (2.13) is that the wavenumber vector for $\dot{\omega}_{max}$ is perpendicular to that for $\dot{\omega}_{min}$.

Figure 6 shows the range of the internal wave frequency as a function of $N_0$ by shading for $\phi = 45^\circ$ and $\alpha = \pi/2$ as an example. It is important that $\dot{\omega}_{min}$ monotonically increases with $N_0$ for $f_{it} \neq 0$, whereas it is constant ($= f_v$) for $f_{it} = 0$. In contrast, $\dot{\omega}_{max}$ is a monotonically increasing function of $N_0$ whether $f_{it} = 0$ or not. This is because the $f_H$-effect weakens as $N_0$ increases, because stability suppresses the vertical motion. From Fig. 6 it is seen that waves whose frequencies are in the range of $f \leq \dot{\omega} \leq \dot{\omega}_{max}$ can oscillate in any stratification (Gerkema and Exarchou 2008). Another important implication of Fig. 6 is the existence of a wave solution trapped in the WSL, which has $\dot{\omega}$ higher than $\dot{\omega}_{min}$ in the WSL and lower than $\dot{\omega}_{min}$ outside the WSL.

2.3 Modification of the hodograph by the $f_H$-effect

The hodograph is frequently analyzed to examine the characteristics of the inertia-gravity waves. Gerkema and Exarchou (2008) showed that the major axis of the hodograph is not always parallel to $k_H$ for $f_{it} \neq 0$. However, they only argued about the shape of the hodograph using characteristic coordinates and did not discuss the rotational direction, which gives important information on wave propagation (e.g., Sato 1994). In this paper, modification of the rotational direction of the hodograph with time, as well as its shape, is examined as a function of $\dot{\omega}$ and $m_{\perp}$ by using the $x'y'z$ coordinates, where the $x'$ direction is the same as $k_H$.

The polarization of horizontal velocity for $f_{it} \neq 0$ is

$$\ddot{v} = -i \frac{f_v}{\dot{\omega}} \left[ \frac{f_v \sin \alpha \ k_H}{m_{\perp} + 1} \right] \ddot{u},$$

(2.15)

$$\ddot{v} = -i \frac{f_v}{\dot{\omega}(\alpha, \beta)} \left[ \frac{\sin \alpha \ \tan \phi \ \tan \beta}{m_{\perp} + 1} \right] \ddot{u},$$

Here $\dot{\omega}$ is a function of $\alpha$ and $\beta$ (see (2.3) and (2.5)). For $f_{it} \neq 0$, (2.15) is valid when $m \neq 0$ (i.e., $\beta \neq 0$). When $m = 0$, $\ddot{u} = 0$, and $\ddot{v} \neq 0$ ((2.2b) and (2.2d)) for $f_{it} \neq 0$, the polarization is linear, while there is no inertia-gravity wave solution at this limit for $f_{it} = 0$, but the hodograph approaches linear when $m \to 0$.

The shape and rotation of the hodograph with time are modified for $f_{it} \neq 0$, as illustrated in Fig. 2c, as a function of $m$. The hodograph is also linear when $k \perp \Omega$, because the Coriolis force does not act on parcels (Fig. 5d). The direction of the linear hodograph is parallel to $k_H$. The vertical wave number ($m_{\perp}$) for $k \perp \Omega$ is expressed as

$$m_{\perp} = k \sin \left[ \tan^{-1} \left( \frac{-\sin \alpha}{\tan \phi} \right) \right].$$

(2.16)

An important characteristic is that the rotational direction of the hodograph with time depends on $\phi$ and $k$ for $f_{it} \neq 0$. The rotation for $f_{it} = 0$ in NH is always clockwise because $\Omega_H$ points to the negative $k$ direction for $m < 0$. The rotational direction of the hodograph with time for $f_{it} \neq 0$ is anticlockwise in a range of $m_{\perp} < m < 0$ and clockwise in the other ranges. This anticlockwise rotation for $m_{\perp} < m < 0$ corresponds to
the situation in which \( \Omega \) points to the \( \mathbf{k} \) direction due to the \( f \)-effect (i.e., the tilt of \( \mathbf{\Omega} \)) even when \( m \) is negative.

For \( f_{ii} = 0 \), the long axis of the hodograph is always parallel to the \( \mathbf{u} \) axis, i.e., \( \mathbf{k}_H \). However, this is not the case for \( f_{ii} \neq 0 \) (Gerkema and Exarchou 2008). The long axis of the hodograph is parallel to the \( \mathbf{v} \) axis (i.e., perpendicular to \( \mathbf{k}_H \)) for \( \omega > N_0 \), while it is parallel to the \( \mathbf{u} \) axis (i.e., parallel to \( \mathbf{k}_H \)) for \( \omega < N_0 \). When \( \tilde{k} = 0 \) (i.e., \( \mathbf{k} \) is vertical), \( \omega = f_r \) and the hodograph is a circle similarly to the case for \( f_{ii} = 0 \). When \( \omega = N_0 \), the hodograph is also a circle (Gerkema and Exarchou 2008), whose rotational direction is clockwise (anticlockwise) when \( m > 0 \) (\( m < 0 \)). As the rotational direction changes at \( m = m_\perp \), in which the hodograph is linear, \( m_\perp \) does not exceed \( m \) at which \( \omega = N_0 \). Note that \( m_\perp \) approaches zero with \( f_{ii} \sin \alpha \).

2.4 Dependence of \( c_{wp} \) on \( c_{wp} \) modified by the \( f \)-effect

From Fig. 2a it is seen that when \( f_{ii} = 0 \), the inertia-gravity waves exhibit the property that the sign of the vertical group velocity \( c_{wp} \) is always opposite to that of vertical phase speed \( c_{wp} \). However, this is not the case for \( f_{ii} \neq 0 \). Although the reason is mathematically clear (\( 0 < \beta_{\text{max}} < \pi/2 \), \( -\pi/2 < \beta_{\text{min}} < 0 \) and \( \omega \rightarrow f_r \) as \( \beta \rightarrow \pm \pi/2 \)), the physical mechanism is not clearly explained so far. Thus, an intuitive illustration is given in this study in terms of the form stress over the potential temperature surface (\( \Theta \) surface).

a. Consideration in terms of energy flux

The vertical energy flux in the Boussinesq model is related to \( c_{wp} \) as\(^3\)

\[
\bar{p}w = E c_{wp},
\]
(2.17)

where an the overbar denotes a horizontal average over the wavelength (e.g., Vallis 2006), and \( E \) is

\[
E = \frac{\rho_0}{2} \left[ u'^2 + v'^2 + w'^2 + (b/N)^2 \right].
\]
(2.18)

Thus, \( \text{sgn}(c_{wp}) \) is equal to that of \( \bar{p}w \). Note that (2.17) holds also for equatorial waves after integrating with respect to latitude (Andrews et al. 1987). An examination is made on the \( x'y'z' \) coordinates, in which the form stress has the \( x' \) component only. Using \( w \equiv D\xi/Dt = \partial\xi/\partial t \), where \( \xi \) is vertical displacement of the parcels, the vertical energy flux is written as

\[
E_{cp} = \bar{p}w = \left( \frac{\omega}{k} \right) \left( -\frac{\partial p}{\partial x} \right) = \hat{e} \left( -\frac{\partial p}{\partial x} \right),
\]
(2.19)

where \( \hat{e} (>0) \) is the horizontal phase speed. The right-hand side of (2.19) expresses the rate of work done by the form stress over the wavy \( \Theta \) surface moving toward the \( x' \) direction with a speed of \( \hat{e} \).

1) Case of \( f_{ii} = 0 \)

As space is horizontally isotropic for \( f_{ii} = 0 \), the form stress does not depend on the horizontal direction (\( \alpha \) in Fig. 1). The relation between \( p \) and \( \xi \) is obtained by (2.2c) and (2.2e) when \( m = 0 \):

\[
p = \frac{\rho_0}{km} \left( N_0^2 - \omega^2 \right) \frac{\partial \xi}{\partial x}. \]
(2.20)

Thus, the vertical energy flux becomes

\[
\bar{p}w = -\hat{e} \frac{\rho_0}{km} \left( N_0^2 - \omega^2 \right) \frac{\partial \xi}{\partial x} = -c_{wp} \frac{\rho_0}{k} \left( N_0^2 - \omega^2 \right) \frac{\partial \xi}{\partial x}^2.
\]
(2.21)

Because \( \omega < N_0 \), (2.21) indicates that \( \text{sgn}(\bar{p}w) \), and hence \( \text{sgn}(c_{wp}) \), is opposite to \( \text{sgn}(c_{wp}) \). If the hydrostatic balance can be assumed, \( \omega \) in (2.20) and (2.21) can be neglected.

More importantly, it should be noted here that this relation generally holds for all plane waves in linearized Boussinesq and primitive equation (using the log \( p \) height) systems on beta planes, because (2.21) is derived only from the vertical momentum and thermodynamic equations. Moreover, this relation holds also for the equatorial waves, because a meridional structure is not assumed to derive (2.21), and (2.21) can be integrated with respect to the meridional direction. Note that the relation \( \text{sgn}(c_{wp}) = -\text{sgn}(c_{wp}) \) has been derived using the dispersion relation of each wave so far.

When \( m = 0 \), the relation between \( p \) and \( \xi \) is derived from the horizontal momentum equations, depending on the models. However, considering the continuity of the dependence of the vertical energy flux on \( m \), it is clear that \( c_{wp} = 0 \) when \( m = 0 \) for the Boussinesq fluid.

2) Case of \( f_{ii} \neq 0 \)

As space is not horizontally isotropic for \( f_{ii} \neq 0 \), the form stress depends on the \( \mathbf{k}_H \) direction (i.e., \( \alpha \)). The energy flux is separately examined for \( m = 0 \) and \( m = 0 \). For \( m = 0 \), using (2.2c), (2.2d), and (2.2e),

\[
\text{Even when } f_{ii} = 0, \text{ the horizontal energy flux } (pu, pv) \text{ of Rossby waves is not proportional to the horizontal group velocity because of the ageostrophic velocity components, whose magnitudes are proportional to } \gamma \text{ on the beta plane (Pedlosky 1987).} \]
The vertical energy flux is expressed as
\[ \overline{\rho w} = -\frac{c}{k}\frac{N_0^2 - \omega^2}{\overline{\rho w}} \left[ \frac{\partial^2}{\partial x^2} \right] + f_{\text{hi}} \sin \alpha \frac{\partial \rho}{\partial x} \frac{\partial x}{\partial x} (2.22) \]

Using (2.2b) and (2.2d),
\[ \nu = \left( f_{\text{hi}} \sin \alpha + \frac{f_m}{k} \right) \zeta. \] (2.23)
Thus, (2.22) is written as
\[ \overline{\rho w} = -\frac{c}{k} \frac{N_0^2 - \omega^2 + f_{\text{hi}} \sin \alpha \left( f_{\text{hi}} \sin \alpha + \frac{f_m}{k} \right)}{\partial \zeta^2}. \] (2.24)

For \( 0 < \alpha < \pi \) (i.e., \( \sin \alpha > 0 \)), (2.24) indicates that \( \text{sgn} (\overline{\rho w}) \), i.e., \( \text{sgn} (c_{\nu}) \), is the same as \( \text{sgn} (c_{\nu}) \) when \( m < m_{\text{th}}(\omega, \bar{k}, \alpha) \), where
\[ m_{\text{th}}(\omega, \bar{k}, \alpha) = -\frac{\zeta}{N_0^2 - \omega^2 + (f_{\text{hi}} \sin \alpha)^2}{f_{\text{hi}} \sin \alpha}. \] (2.25)

Using (2.9), it is shown that \( \beta \) which satisfy \( m = m_{\text{th}}(\omega, \bar{k}, \alpha) \) are \( \beta_{\text{max}} \) and \( \beta_{\text{min}} \). The condition \( m < m_{\text{min}}(\omega, \bar{k}, \alpha) \) is equivalent to \( m < m_{\text{in}} < 0 \) or \( 0 < m < m_{\text{max}} \). Thus, the reason for the changes of \( \text{sgn} (c_{\nu}) \) at \( \omega_{\text{in}} \) and \( \omega_{\text{max}} \) is that the \( f_{\text{hi}} \)-effect reverses the direction of the form stress at these frequencies. For \( \pi < \alpha < 2\pi \) (i.e., \( \sin \alpha < 0 \)), the same results are obtained except that the condition \( m < m_{\text{th}}(\omega, \bar{k}, \alpha) \) is replaced with \( m > m_{\text{th}}(\omega, \bar{k}, \alpha) \).

Next, for \( m = 0 \), the relation between \( p \) and \( \zeta \) is obtained by using the \( \bar{x} \) momentum equation instead of the vertical one. In this case, \( \bar{u} = 0 \) from (2.2d). From (2.2a), we obtain
\[ \frac{\partial p}{\partial \zeta} = \rho_{\text{fj}} \nu - \rho_{\text{hi}} \cos \alpha \frac{\partial \zeta}{\partial \bar{x}}. \] (2.26)

The vertical energy flux is expressed as
\[ \overline{\rho w} = c \left( \rho_{\text{fj}} \nu \bar{\zeta} \right). \] (2.27)
Using (2.23), the vertical energy flux is written as
\[ \overline{\rho w} = c \left( \rho_{\text{fj}} \nu \sin \alpha \bar{\zeta} \right). \] (2.28)
The equation (2.28) indicates that \( c_{\nu} \) is positive (negative) when \( \sin \alpha > (\sin \alpha) 0 \) (Fig. 2a).

b. Intuitive illustration

In this subsection, the relation between \( \text{sgn} (c_{\nu}) \) and \( \text{sgn} (c_{\nu}) \) is examined in terms of the vertical force balance (2.2c). First, the case of \( f_{\text{hi}} = 0 \) is considered. In this case, \( \bar{w} < N_0 \), which means that the magnitude of the buoyancy force is always greater than that of \( -\partial w/\partial t \). Thus, the vertical pressure gradient force is always opposite to the buoyancy force to keep the balance (Fig. 7a). For \( m < 0 \) (i.e., \( c_{\nu} < 0 \), constant phase lines tilt against the \( \bar{x} \) direction with height. Because of the direction of the vertical pressure gradient force, \( p \) is distributed as illustrated in Fig. 7a. As the regions of negative \( p \) and positive \( \partial \zeta/\partial \bar{x} \) accord, the form stress points to the \( \bar{x} \) direction. The \( \Theta \) surface does positive work on the fluid above, and hence, \( c_{\nu} \) is positive. Similarly, it is understood that \( c_{\nu} \) is negative for \( c_{\nu} > 0 \).

Second, the case of \( f_{\text{hi}} = 0 \) and \( m = 0 \) is considered. For simplicity, we set \( \alpha = \pi/2 \) (i.e., \( k_{\text{hi}} \) points northward) because the \( f_{\text{hi}} \)-effect is the largest in this case, namely \( f_{\text{hi}} \cos \alpha \bar{u} = 0 \) and \( f_{\text{hi}} \sin \alpha \nu = f_{\text{hi}} \nu \) in (2.2c). The cases of \( m < m_{\text{min}} \) and \( 0 < m < m_{\text{max}} \) are mainly considered because the cases of \( m_{\text{min}} < m < 0 \) and \( m_{\text{max}} < m \) are similar to the cases of \( m < 0 \) and \( m > 0 \) for \( f_{\text{hi}} = 0 \), respectively. For \( m < m_{\text{min}} \) in which the parcel trajectories are nearly horizontal, the balance between the Coriolis force and the vertical pressure gradient force is dominant. Note that the parcel moves clockwise, seen from the \( \Omega_k \) direction as in Fig. 7b, illustrating the distribution of \( \nu \) on the \( \Theta \) surface. The \( \Theta \) surface does negative work on the fluid above, indicating that \( c_{\nu} \) is negative.

For \( 0 < m < m_{\text{max}} \) in which the parcel trajectories are nearly vertical, the magnitude of \( -\partial w/\partial t \) is greater than that of the sum of the buoyancy and Coriolis forces due to the \( f_{\text{hi}} \)-effect. Thus, the vertical pressure gradient force must be opposite to \( -\partial w/\partial t \) (Fig. 7c). The \( \Theta \) surface does positive work on the fluid above, indicating that \( c_{\nu} \) is positive.

Third, the case of \( f_{\text{hi}} = 0 \) and \( m = 0 \) is considered (Fig. 7d). Because there is no vertical pressure gradient, the horizontal force balance determines \( c_{\nu} \) and the distribution of \( p \). In this case, \( \bar{u} = 0 \) in (2.2d), and hence, \( \nu \) is induced by the \( f_{\text{hi}} \)-effect ((2.2b) and (2.2c)). The Coriolis force \( f_{\text{hi}} \nu \) is opposite to the horizontal pressure gradient \( -1 \bar{p}_{\zeta} / \partial \bar{x} \) in (2.2a). The \( \Theta \) surface does positive work on the fluid above. Thus, \( c_{\nu} \) is positive. Note that \( \nu \) is always zero for \( f_{\text{hi}} = 0 \) and hence, \( p \) is also zero. Thus, the \( \Theta \) surface does no work on the fluid above, and hence, \( c_{\nu} \) is always zero.

When \( \alpha = -\pi/2 \), the directions of the form stress are opposite to those in Figs. 7b–7d. Thus, \( \text{sgn} (c_{\nu}) \) is also opposite in all cases. When \( \alpha = \pm \pi/2 \), \( f_{\text{hi}} \cos \alpha \bar{u} \) in (2.2c) and \( f_{\text{hi}} \cos \alpha \nu \) in (2.2a) are not zero. However, \( \bar{u} \) and \( w \) are zero at the top and bottom of the wavy \( \Theta \) surface (that is, the minimum and maximum phases of \( \zeta \)). Thus, the same explanation as for \( \alpha = \pm \pi/2 \) can be given.
Fig. 7. Force balance on the $\Theta$ surface (a) when $f_H = 0$, $m < 0$, (b) when $f_H \neq 0$, $m < m_{\text{min}} < 0$, (c) when $f_H \neq 0$, $0 < m < m_{\text{max}}$, and (d) when $f_H \neq 0$, $m = 0$. Right figures show $y'$ cross sections of the $\Theta$ surfaces and the force balances, and left figures show the parcel motions projected onto the $y'z''$ planes (equiphase surfaces). The arrow labeled $\hat{c}$ for each figure shows the velocity of the $\Theta$ surface. Note that scales of arrows showing the buoyancy force are different for respective figures.
3. Presence of trapped inertia-gravity waves in the WSL

3.1 Necessary conditions for the existence of trapped solutions

According to Durran and Bretherton (2004), the necessary condition for the existence of sub-inertial modes in the vertically bounded domain is that the frequency takes its minimum at a finite vertical wave number in the dispersion curve. The sub-inertial modes consist of two plane waves whose \( \hat{\omega} \) (\(< \hat{\omega}_{\text{min}} \)) are the same but the signs of \( c_p \) are opposite, although magnitudes of the two \( c_p \) may be different. However, this condition is not sufficient for the existence of trapped waves in the WSL in the fluid. The condition that the minimum frequency is a monotonically increasing function of \( N_0 \) is also necessary, because the amplitude of wave must decrease outside the WSL, as shown in Subsection 2.2.

3.2 Trapped solution in the idealized WSL

An analytic solution trapped in the WSL is examined for an idealized distribution of \( N \) having a form of a square well with a width of \( 2a \), as shown in Fig. 8. Considering the solutions with the shape of

\[
(u, v, w, p/p_0) = [U(z), iV(z), iW(z), b(z), P(z)] \text{ exp}(kx + ly - \hat{\omega}t),
\]

where \( \hat{\omega} \neq f_V \) is assumed. Note that the original Cartesian coordinates \((xyz)\) are used. Following Kasahara (2003), \( W(z) \) is written as follows.

\[
W(z) = \psi(z) \text{exp} i\Gamma z,
\]

where \( \psi(z) \) is the envelope function of \( W \). Defining \( E \) and \( V(z) \) as

\[
E \equiv \Gamma + \frac{\hat{\omega}^2 (k^2 + f_V^2) - \hat{f} f_i^2}{f_i^2 - \hat{\omega}^2},
\]

\[
V(z) \equiv N(z)^2 \frac{k^2 + f_V^2}{f_i^2 - \hat{\omega}^2},
\]

respectively, the equation of \( \psi(z) \) is obtained from

\[
-\frac{d^2}{dz^2} \psi + V(z) \psi = E \psi,
\]

which is similar to Schrödinger’s equation in a square well potential \( V(z) \). Note that \( E \) does not explicitly depend on \( N(z) \) but implicitly does through \( \hat{\omega} \). For the trapped solution, \( \hat{\omega} \) is decided by the boundary conditions at \( z = \pm a \). The equation (3.6) indicates that \( \psi \) oscillates where \( E > V(z) \) and decays exponentially where \( E < V(z) \).

Wave solutions trapped in the WSL are present when \( V(N_T) > E \), \( V(N_S) > E \), and \( V(N_T) < E \). The solutions in the respective ranges are

\[
\psi(z) = \begin{cases} 
A \exp \left( \frac{z}{H_T} \right) & (z < -a), \\
B \exp(i\mu z) + C \exp(-i\mu z) & (-a \leq z \leq a), \\
D \exp \left( -\frac{z}{H_S} \right) & (a < z), 
\end{cases}
\]

where \( A, B, C, \) and \( D \) are constant complex numbers, and \( H_T, H_S, \) and \( \mu \) are, respectively, defined as

\[
H_T \equiv \left[ V(N_T) - E \right]^{-1/2}, \\
H_S \equiv \left[ V(N_S) - E \right]^{-1/2}, \\
\mu \equiv \left[ E - V(N_T) \right]^{1/2}.
\]

By using boundary conditions that pressure and vertical velocity, i.e., \( \psi \) and \( d\psi/dz \), are continuous at \( z = \pm a \) and evanescent for \( |z| \rightarrow \infty \), we obtain...
Table 1. Values of $\hat{\omega}/f_r$, $2\pi/\mu$, $2\pi/\Gamma$, $H_s$, $H_r$, and $c_\psi$ for (a) $\lambda_1 = 500$ km, (b) $\lambda_1 = 1000$ km, and (c) $\lambda_1 = 3000$ km for the trapped wave solution of $S_l = 1$-4, assuming that $a = 1.5$ km, $k = 0$, $\phi = 45^\circ$, $N_s = 200 f_r$, $N_r = 100 f_r$, and $N_p = 50 f_r$. $c_\psi$ is vertical group velocity of plane waves that comprise the trapped waves.

(a) $\lambda_1 = 500$ km

<table>
<thead>
<tr>
<th>SI</th>
<th>$\hat{\omega}/f_r$</th>
<th>$2\pi/\mu$ (km)</th>
<th>$2\pi/\Gamma$ (km)</th>
<th>$H_s$ (km)</th>
<th>$H_r$ (km)</th>
<th>$c_\psi$ (m/day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9998002</td>
<td>6.053</td>
<td>-0.200</td>
<td>8.215E-03</td>
<td>1.837E-02</td>
<td>3.99E-03 / -3.49E-03</td>
</tr>
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<td>-0.199</td>
<td>8.202E-03</td>
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<td>8.46E-03 / -6.50E-03</td>
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<tr>
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<td>-0.198</td>
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<td>1.832E-02</td>
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</tr>
<tr>
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<td>-0.197</td>
<td>8.154E-03</td>
<td>1.827E-02</td>
<td>1.88E-02 / -1.11E-02</td>
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(b) $\lambda_1 = 1000$ km

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<th>SI</th>
<th>$\hat{\omega}/f_r$</th>
<th>$2\pi/\mu$ (km)</th>
<th>$2\pi/\Gamma$ (km)</th>
<th>$H_s$ (km)</th>
<th>$H_r$ (km)</th>
<th>$c_\psi$ (m/day)</th>
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(c) $\lambda_1 = 3000$ km

<table>
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<th>SI</th>
<th>$\hat{\omega}/f_r$</th>
<th>$2\pi/\mu$ (km)</th>
<th>$2\pi/\Gamma$ (km)</th>
<th>$H_s$ (km)</th>
<th>$H_r$ (km)</th>
<th>$c_\psi$ (m/day)</th>
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<tr>
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<td>4.662E-02</td>
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<td>4.205E-02</td>
<td>9.988E-02</td>
<td>8.73E-01 / -7.74E-02</td>
</tr>
</tbody>
</table>

The equation for $\hat{\omega}$ is

$$\tan(2\mu a) = -\frac{H_r + H_s}{1/\mu - \mu H_s H_r}.$$  \hspace{1cm} (3.10)

Note that $H_s$, $H_r$, and $\mu$ are functions of $\hat{\omega}$. Table 1 shows $\hat{\omega}$ and other related parameters for solution indexes (SI) of 1-4 and for a few meridional wavelengths $\lambda_1$ (\(\equiv 2\pi/l\)) = 500, 1000, and 3000 km, where SI is defined as the number indicating the order of $\hat{\omega}$ for the solution and is approximately equal to the number of crests (or troughs) of $\psi$. Here we took $a = 1.5$ km, $k = 0$, $\phi = 45^\circ$, $N_s = 200 f_r$, $N_r = 100 f_r$, and $N_p = 50 f_r$ as an example. Note that the vertical wavelength of $w$ is $2\pi/\Gamma$, while that of $\psi$ is $2\pi/\mu$. Figure 9 illustrates $w$ in the $y-z$ section and envelope $\psi$ as a function of $z$ for SI = 1-4.

Several important characteristics of the trapped wave solutions in the WSL are described below.

- Frequencies of the solutions satisfy $\hat{\omega} \lesssim f_r$.
- Values of $\hat{\omega}$ are discrete because of the boundary conditions on the edges of the WSL.
- As $\hat{\omega}$ increases, $\mu$ and SI increase.
- Inclination of phase lines with respect to the $y$ axis is positive, independent of $k$ and $l$:

$$-\frac{l}{f_r^2 - \omega^2} > 0.$$  \hspace{1cm} (3.11)

- Solutions generally have fine vertical structures. The vertical wavelength ($2\pi/\Gamma$) decreases when the $k_\mu$ direction approaches the $x$ axis.

According to (3.7) and (3.9), the solution is composed of two plane waves having vertical wave numbers of $\Gamma - \mu$ and $\Gamma + \mu$ in the WSL, which have the same frequencies and are lower than $f_r$. These plane waves have finite $c_\psi$ even at $z = \pm a$ so that modes are formed in the WSL.
4. The $f_H$-effect on inertia-gravity waves in the real atmosphere

In this section, we discuss the degree of importance of the $f_H$-effect on the inertia-gravity waves in the real atmosphere. First, the structure of $N$ in the meridional cross section is shown from the observation. Next, using the observed $N$ structure around the tropopause, the $f_H$-effect on the hodograph and the possibility of the presence of the trapped inertia-gravity waves are examined. Note that the $f_H$-effect weakens as $N$ increases.

4.1 Structure of $N$ in the meridional cross section

Twice-daily radiosonde data obtained operationally by the Japan Meteorological Agency (JMA) are used for the analysis of the $N$ structure. The vertical intervals are approximately 200 m for temperature. The accuracy of temperature measurement is approximately 0.1 K (Japan Meteorological Agency 1995). The data were interpolated at a uniform vertical interval of 200 m using the cubic spline method to make further analysis easier.

Figure 10a shows the meridional cross section of $N$ averaged for a summer period from June to August in 2007. Radiosonde data at stations distributed in the latitude range of 24.3°N–43.3°N are used following the study by Sato and Dunkerton (2002). Crosses denote the tropopause heights at respective stations. Figures 10b and 10c show the mean vertical profiles of $N$ for...
the same period at low latitude (Naze; 28.4°N, 129.5°E) and high latitude (Akita; 39.7°N, 140.1°E) stations. Horizontal bars show the standard deviation. It is clear that there is a persistent WSL slightly below the tropopause. A similar WSL is observed for the summer periods of the other years (not shown). Figure 10a also indicates that the WSL is wider, and the minimum $N$ values are smaller at lower latitudes. In the winter period from December to February, the WSL is narrower in the south region of Japan (24.3°N–30.0°N), and sometimes obscure in the north region (30.0°N–43.3°N) (not shown).

As the tropopause height changes with time, the mean $N$ structure shown in Fig. 10 may be smeared. Thus, we obtained the minimum of $N^2$ that exists in the height range of 8 km below the tropopause for each observation, which is referred to as $(N_{\text{min}})^2$. Figures 11a and 11b show the frequency distributions of $N_{\text{min}}/f_v$ at Naze and Akita, respectively. It is seen that the frequency $N_{\text{min}}/f_v$ is distributed around 80 ($N_{\text{min}} \approx 0.005 \text{ s}^{-1}$) at Naze and around 65 ($N_{\text{min}} \approx 0.006 \text{ s}^{-1}$) at Akita. Here $N_{\text{min}}$ is defined as the most frequent value of $N_{\text{min}}$. 

Fig. 10. (a) Latitude–height cross section of buoyancy frequency $N$ averaged for June through August in 2007. (b) Vertical profiles averaged over the same period at Naze (28.4°N, 129.5°E, station number 47909) and (c) at Akita (39.7°N, 140.1°E, station number 47582), as typical examples for stations at low and high latitudes. Labels on the top axis in (a) denote the locations of respective stations (station number 47xxx). Contour intervals are 0.003 s$^{-1}$ in (a). Crosses in all figures denote the tropopause heights at respective stations. Horizontal bars in (b) and (c) show the standard deviation of $N/f_v$. 
Note that $N_{\text{inst}}$ in the mid-latitudes is about a half times as low as a typical tropospheric value ($\approx 0.01 \text{ s}^{-1} \approx 100 f_V$). It is also seen that $\left(N_{\text{inst}}/f_V\right)^2$ is sometimes zero and even negative, indicating that convective instability occurs slightly below the tropopause. The instantaneous vertical profile of $N$ may contain disturbances whose periods are approximately one day and less. To reduce their amplitudes, $\left(N_{\text{inst}}^{1.5\text{day}}/f_V\right)^2$ is also defined as the minimum of the 1.5-day running mean vertical profile of $N^\perp$ in the height range of 8 km below the tropopause. Figures 11c and 11d show the frequency distributions of $N_{\text{inst}}^{1.5\text{day}}/f_V$ at Naze and Akita, respectively. It is seen that $N_{\text{inst}}^{1.5\text{day}}$ values are sometimes close to $N_{\text{min}}$. Thus, $N_{\text{min}} (\approx 50 f_V)$ is used as the possible minimum value of the background $N$ in the WSL for the disturbances for further analysis.

4.2 The $f_H$-effect on the hodograph in the real atmosphere

As shown in Subsection 2.3 and 2.4, the characteristics of the hodograph for $f_H \neq 0$ are different from those of the hodograph for $f_H = 0$ where $\omega > N_0$, $0 < m < m_{\text{max}} (m_{\text{max}} < m < 0)$, $m < m_{\text{min}} < 0 (0 < m_{\text{min}} < m)$, or $m_{\perp} < m < 0 (0 < m < m_{\perp})$ for $0 < \alpha < \pi (\pi < \alpha < 2\pi)$. Figure 12 shows $\omega$ as a function of $m$ normalized by $k$ for $f_H \neq 0$, $N_0 = 50 f_V$, indicating that $\omega_{\text{max}} \approx N_0$ and $m_{\text{max}} \approx 0$ in the real atmosphere. Thus, the area where $\omega > N_0$ and $0 < m < m_{\text{max}}$ is quite narrow. On the other
hand, $m_{\text{min}}$ and $m_\perp$ may be important parameters to consider the $f_{r\perp}$-effect in the real atmosphere. Figures 13a and 13b, respectively, show $\lambda_{\text{min}} = 2\pi/m_{\text{min}}$ and $\lambda_\perp = 2\pi/m_\perp$ as a function of $\lambda_H = 2\pi/|k_H|$ and latitude for $\alpha = 90^\circ$ and $N_0 = 0.005$ s$^{-1}$ (i.e., $N_0 \approx 50 f_r$ at $\phi = 45^\circ$). Note that $\lambda_\perp$ does not depend on $N_0$ as in (2.16). Figure 13b indicates that $\lambda_\perp$ is comparable to $\lambda_H$, which may be rare in the real atmosphere. On the other hand, there are possible cases in the real atmosphere, e.g., $\lambda_{\text{min}} = 0$ (100 m) for $\lambda_H = 0$ (100 km). In such cases, we need to be careful for the treatment because they show different characteristics from those analyzed neglecting the $f_{r\perp}$-effect where $\text{sgn} (c_g) = \text{sgn} (c_\perp)$. Figures 11c and 11d show that $N_{\text{min}}^{1.5\text{day}}$ can be less than 50 $f_r$. Thus, it is necessary to examine the magnitude of $N$ for the hodograph analysis.

4.3 Possibility of the presence of trapped inertia-gravity waves in the real atmosphere

It was shown that there is a persistent WSL slightly below the tropopause. Thus, the possibility of the presence of the inertia-gravity waves trapped there is examined in terms of the magnitude of $c_g$. Table 1 shows the values of $c_g$ of the plane waves that comprise the trapped wave. As the magnitude of $c_g$ is smaller than 1 m day$^{-1}$ in all cases, it takes quite a long time to form the trapped wave. If the trapped wave is formed within a few days, it is necessary that the characteristic magnitude of $c_g$ is greater than 100 m day$^{-1}$ for the characteristic width of the WSL of 1 km. Using $c_g$ derived from (2.5), $N_0$ should be smaller than 0.001 s$^{-1}$ ($\approx 10 f_r$):

$$c_g = \frac{(f_{id}+f_id)[f_i(k^2+m^2)-f_{id}m] - N_0^2 m(k^2+l^2)}{(k^2+l^2+m^2)^{1/2}}.$$  \hspace{1cm} (4.1)

for typical scales $k, l \sim 2\pi/100$ km, $m \sim 2\pi/1$ km, $\omega \sim f_r$, and $f_{id} \sim f_r \sim 10^{-4}$ s$^{-1}$. Thus, it is concluded that in the WSL with $N_{\text{min}} \approx 50 f_r$ slightly below the tropopause, such trapped waves are rarely observed.

5. Summary and concluding remarks

By using the linearized Boussinesq model containing $f_{r\perp}$-terms, the characteristics of the inertia-gravity waves modified by the $f_{r\perp}$-effect have been examined. The $f_{r\perp}$-effects on the inertia-gravity waves are summarized as follows.

1. For $f_{it} = 0$, $f_r < \omega < N_0$. However, for $f_{it} \neq 0$, the minimum frequency is lower than $f_r$, and the maximum is higher than $N_0$. The vertical wave number is finite at the minimum frequency

Fig. 12. Frequency $\omega$ normalized by $f_r$ as a function of the vertical wave number $m$ normalized by $k$ for $f_{it} = 0$ and $\alpha = \pi/2$. A star on the solid curve denotes the position of $m_\perp$, for which $k$ is perpendicular to $\Omega$. $\phi = 45^\circ$ and $N_0 = 50 f_r$ are taken.

(Gerkema and Shrira 2005a; Gerkema and Exarchou 2008).

2. For $f_{it} = 0$, the hodograph is elliptic and rotates clockwise with time. However, for $f_{it} \neq 0$, the rotational direction is anticlockwise when $m$ ranges from 0 to $m_\perp$. Here $m_\perp$ is the vertical wave number for which $k \perp \Omega$. The hodograph becomes linear at $m = m_\perp$ in addition to $m = 0$.

3. For $f_{it} = 0$, the sign of the vertical component of the group velocity is different from that of the vertical phase speed for all internal and equatorial waves. This relation is understood from the form stress over the $\Theta$ surface. However, for $f_{it} \neq 0$, these signs can be the same for the inertia-gravity waves. This is because the $f_{r\perp}$-effect can reverse the direction of the form stress through the vertical force balance.

4. The minimum frequency is the increasing function of $N_0$ for $f_{it} \neq 0$. This is necessary for the existence of trapped waves in the WSL.

5. The analytic solution trapped in the WSL with a form of a square well for $N$ always has a positive inclination of the phase lines with respect to the $y$ axis regardless of the sign of the meridional wavenumber. This is due to the horizontal anisotropy of the $f_{r\perp}$-effect. The trapped wave has a fine vertical structure (the vertical wavelength is
approximately 1 km) in normal atmospheric conditions.

From analysis of operational radiosonde data in Japan, it was shown that there is a persistent WSL slightly below the tropopause. The possible minimum value of the background \( N \) in the WSL \( (N_{\text{min}}) \) was 50 \( f_r \) \(( \approx 0.005 \text{ s}^{-1}) \), which is about a half times lower than a typical tropospheric value \(( \approx 100 \ f_r \) \). Using the \( N_{\text{min}} \) value, the degree of modification of the hodograph by the \( f_r \)-effect and the possibility of the presence of the trapped inertia-gravity waves were examined. The modification of the hodograph would be important when the horizontal wavelength is approximately 100 km and the vertical one is approximately 100 m, for example. As the \( f_r \)-effect increases as \( N \) decreases, it is necessary to examine the magnitude of \( N \) for the hodograph analysis. By using the scale analysis of \( c_{\text{tr}} \), it was shown that the condition of \( N_{\text{min}} < 0.001 \text{ s}^{-1} \) \(( \approx 10 \ f_r \) \) is necessary to form a trapped wave within a few days, whose wavelength is in a realistic range. Thus, such trapped waves are rarely observed in the WSL, slightly below the tropopause.

It is well known that wave-like structures are frequently observed near the tropopause (sometimes called tropopausal waves). Their frequencies are low and close to the inertial frequency, and their vertical wavelengths are only a few kilometers (Sato and Woodman 1982; Cornish et al. 1989; Sato and Dunkerton 2002). Such tropopausal waves can play an essential role in the exchange between the troposphere and stratosphere (Danielsen et al. 1991). However, the mechanism of the persistence of such tropopausal waves is not clear. As there is a persistent WSL slightly below the tropopause, the \( f_r \)-effect might facilitate the persistence of such wave disturbances as well as the background shear and/or beta effects. In the mid-latitudes, the typical magnitude of the \( f_r \)-term \((f_r w)\) is smaller than those of the vertical background shear \((w_{\text{at}} \frac{df}{dy})\) and beta \((\beta y v)\) terms in the equations of motion by only one or two orders, where \( U \) is the zonal component of the background velocity and \( \beta \) is \( \frac{df}{dy} \). At the equator where \( f_r \) is maximized, the magnitude of the \( f_r \)-term can be comparable to that of the vertical background shear term in the equations of motion. Thus, the \( f_r \)-effect might modify the properties of the equations of motion.

In this paper, to focus on the \( f_r \)-effect, we ignored the effect of the latitudinal gradient of \( f_r \) (beta effect). Kasahara and Gary (2010) showed that the super-inertial and sub-inertial modes are not largely affected by the beta effect. In particular, the beta effect may be important for the trapped waves at the equator, where \( f_r \) is zero, and hence, \( \omega_{\text{min}} \) is zero \((2.11)\), indicating that the trapped solution does not exist in the linearized Boussinesq model. Moreover, in the ocean, since the beta effect plays an important role on the transition from the super-inertial to the sub-inertial mode near the critical latitude, wave energy is accumulated in the bottom WSL by the reflection between the bottom surface and the upper SSL (Gerkema and Shrira 2005b). In the atmosphere, an interesting phenomenon may be also present near the critical latitude, which will be studied in the future.

![Diagram](image-url)
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Appendix: Exact values of $\hat{\omega}_{\min}$ and $\hat{\omega}_{\max}$

The exact values of $\hat{\omega}_{\min}$ and $\hat{\omega}_{\max}$ are derived using a quadratic form. By substituting (2.7a) into (2.9), the dispersion relation in the $x''y''z''$ coordinates is expressed as

$$\hat{\omega}^2 = (f_i \sin \beta + f_{in} \cos \beta)^2 + N_0^2 \cos^2 \beta.$$  \hfill (A.1)

As $\hat{\omega}$ satisfies $\hat{\omega} \neq 0, \infty$, $s$ can be defined as $s \equiv 1/\hat{\omega}$. Further, we, respectively, define $q$ and $r$ as

$$q \equiv s \cos \beta, r \equiv s \sin \beta.$$  \hfill (A.2)

By substituting (A.2) into (A.1), we obtain

$$1 = \begin{pmatrix} q \\ r \end{pmatrix}^T \begin{pmatrix} N_0^2 + f_{in}^2 & f_{in}f_i \\ f_{in}f_i & f_i^2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix}. \hfill (A.3)$$

Note that (A.3) is a quadratic form expressing an ellipse. The middle matrix is a symmetric matrix; thus, it can be transformed into the diagonal form:

$$1 = \begin{pmatrix} q' \\ r' \end{pmatrix}^T \begin{pmatrix} \omega_{\max}^2 & 0 \\ 0 & \omega_{\min}^2 \end{pmatrix} \begin{pmatrix} q' \\ r' \end{pmatrix}. \hfill (A.4)$$

This transformation is equivalent to the rotation of the $qr$ coordinates by an angle $\gamma$ (Fig. 14). Thus, $\omega_{\max}^2, \omega_{\min}^2$ and $\gamma$ are, respectively, obtained as follows:

$$\omega_{\max,\min}^2 = \frac{N_0^2 + f_i^2 + f_{in}^2 \pm \sqrt{(N_0^2 + f_i^2 + f_{in}^2)^2 - 4N_0^2f_i^2}}{2},$$  \hfill (A.5)

$$\gamma = \tan^{-1} \left( \frac{2f_{in}f_i}{(N_0 + f_{in} - f_i)(N_0^2 + f_{in}^2 - f_i^2) + 4f_{in}^2f_i^2} \right).$$  \hfill (A.6)

Gerkema and Exarchou (2008) noted that $\omega_{\min} < f_i$ and $N_0 < \omega_{\max}$ without showing details. This characteristic

\begin{figure}
\centering
\includegraphics[width=\textwidth]{Fig_14}
\caption{Ellipse showing the dispersion relation of inertia-gravity waves in the space of $q = \frac{1}{\hat{\omega}} \cos \beta$ and $r = \frac{1}{\hat{\omega}} \sin \beta$. The $q'r'$ coordinates are the principal axes of the dispersion curve. $\gamma$ is the angle between the $q$ and $q'$ axes, which is given as (2.14).}
\end{figure}

References


Gerkema, T., and V. I. Shira, 2005a: Near-inertial waves in the ocean: Beyond the ‘traditional approximation’. J.


