Linear Responses of Buoyancy Induced by Band-Shaped Precipitation with an End

Masanori YOSHIZAKI
Research Institute for Global Change, Japan Agency for Marine-Earth Science and technology, Yokohama, Japan

Teruyuki KATO
Meteorological Research Institute, Japan Meteorological Agency, Tsukuba, Japan

and

Kazuaki YASUNAGA
Faculty of Science, University of Toyama, Toyama, Japan

(Manuscript received 21 June 2012, in final form 18 April 2013)

Abstract

Band-shaped precipitation systems are frequently observed as back-building (BB) types with most convective cells successively initiating on the upwind side. Usually, nearly neutral stratifications of moist static energy ($h_*$) are observed in the convection areas, whereas conditionally unstable stratifications remain in the surroundings. To understand the difference in features between these two areas and the frequent occurrence of the BB type, heating, whose types are (i) a point source, (ii) an infinitely extending line, and (iii) a line with an end, is prescribed in environmental fields with uniform zonal wind ($U_*$) and damping ($\gamma_*$), and the linear responses of buoyancy are studied. The response fields for heating types (ii) and (iii) are obtained by applying a solution of buoyancy for (i) heating. Here it is assumed that the neutralization of $h_*$ occurs by temperature adjustment with a deviation having an intensity of $\Delta T_*$. In the $U_* = \gamma_* = 0$ case, the temperature deviations are $\Delta T_*/2$ at most in the influence region far from the end, because two oppositely propagating gravity waves are induced for the response. This solution corresponds to response one on the Cartesian coordinate system. It is concluded that the neutralization of $h_*$ in the convection areas and its response in the surroundings induce the different stratifications, maintaining the conditionally unstable stratifications in the surroundings even though intense precipitation occurs. In the $U_* \neq 0$ cases, frequent formations of the BB types are anticipated as one possibility, because the influence of the response on the upwind side of precipitation is small, and new convection is likely to occur.

Keywords band-shaped precipitation; response problem; end effect

1. Introduction

Heavy precipitation is produced by isolated and organized thunderstorms, mesoscale convective systems (MCSs), hurricanes/typhoons, and mesoscale convective complexes, and it sometimes causes serious disasters that disrupt the lives of human beings, such as flash floods and landslides. Gaining a deeper understanding of precipitating convective systems is urgent and is important for forecasting their occurrence and attempting their mitigation.
In the midwestern United States, Bluestein and Jain (1985) classified formation patterns of heavily precipitating squall lines. Among 40 events, 14 cases are grouped as broken-line types, in which cumulonimbus clouds simultaneously develop in lines around the convergence zone, and 13 cases are grouped as back-building (BB) types, in which most convective cells are successively initiated in the upwind edges of the squall lines. A general review on this topic is found in Houze (1993).

Similarly, MCS-scale band-shaped precipitation systems are frequently observed in Japan, and some of them evolve into heavy precipitation systems. Heavy rainfall events such as 1998-Niigata, 1999-Fukuoka/Hiroshima, 2003-Fukuoka, and 2004-Niigata/Fukushima are classified as BB types (Yoshizaki and Kato 2007), since new convective cells mainly developed on the upwind side of the prevailing winds. Kato (1998, 1999, 2006), Kato and Goda (2001), and Kato and Aranami (2005) simulated these BB-type cases and discussed their formation and maintenance mechanisms in detail. In the 1999-Fukuoka/Hiroshima case, by separating a multiscale structure into three scales—large-scale cold fronts, MCSs, and cumulonimbus clouds—Kato (2006) showed that the propagation directions and speeds are different, and the MCSs are organized into a band-shaped precipitation system.

On June 29, 2011, a BB type that looked similar to the 2004-Niigata/Fukushima case was observed in the northwestern Honshu Islands. Figure 1 shows the horizontal distributions of simulated moist static energy $h^*$, total liquid and ice water $TW^*$, and vertical distributions of vertical velocity $W^*$ and $h^*$ at 00 UTC on June 29. Here $h^*$ is defined as $c_p^* T^* + g^* Z^* + L_{v^*} q^*$, where $c_p^*$ is the specific heat at constant pressure, $T^*$ the temperature, $g^*$ the gravitational acceleration, $Z^*$ the height, $L_{v^*}$ the latent heat of vaporization, and $q^*$ the mixing ratio of water vapor. The simulation was conducted using the Japan Meteorological Agency’s non-hydrostatic model (Saito et al. 2006) with a horizontal resolution of 1 km. The BB type was aligned northwest to southeast (Figs. 1a and 1b) and stagnated for nearly 9 h, causing flash floods and landslides in many places. The $h^*$ in the intense updraft area (Fig. 1c) was nearly uniform in the vertical direction (Fig. 1d), while the stratification in the surroundings was still conditionally unstable.

At least two issues arising from the information in Fig. 1 need to be discussed; (1) the reason for the difference in the stratifications between the convection area and surroundings and (2) the reason for the frequent observation of the BB type as band-shaped precipitation systems. In the case of dry convection, the adjustment of dry static energy $s^* ( = c_p^* T^* + g^* Z^*)$ promptly occurs when the vertical stratification of $s^*$ is unstable, leading to the neutralization of $s^*$. Similarly, moist convection takes place, leading to the neutralization of $h^*$. However, two conditions should be satisfied in this case: the initial unstable stratification of $h^*$ and the occurrence of condensation. Owing to these requirements, the adjustment of $h^*$ takes place only in the cloud areas, whereas no adjustment occurs elsewhere. In the areas with precipitation, in particular, large amounts of latent heat are released which can be treated as the response of the atmosphere to this heating. Bretherton and Smolarkiewicz (1989) showed that induced buoyancy horizontally expands as gravity waves in stably stratified layers. Considering a heat source with two vertical profiles, Nicholls et al. (1991) and Mapes (1993) pointed out that the second vertical mode is responsible for the initiation of convection. The formation/maintenance processes of precipitation systems become complicated by the inclusion of environmental vertical wind shear. Stechmann and Majda (2009) stressed that gravity waves in the presence of wind shear can create a more favorable environment on one side of the preexisting convection. In cases of squall lines, Pandya et al. (1996) indicated that diabatic heating induces mesoscale circulations, such as an ascending front-to-rear flow, a midlevel rear inflow, and an upper-level rear-to-front flow. The evaporation of raindrops, which produces the gravity current, and the Coriolis force make these precipitation systems more complex. Skamarock et al. (1994) and Weisman and Davis (1998) obtained a preferred development of a cyclonic vortex, leading to asymmetric convective systems, when convergence at mid-levels enhances the Coriolis rotation.

Among various factors that are closely related to these two issues, only the impacts of a uniform zonal wind and a damping of the Rayleigh-damping type are studied to fundamentally understand the formation and influence of a precipitation band with an end. The uniform zonal wind is necessary to determine the wind directions that are essential for the formation of the BB type. Damping, which mimics the effects of radiative cooling and the leakage of the upward-propagating gravity waves, is included to represent the features of precipitation systems more realistically. In this study, we investigate the linear responses of buoyancy for heating extending in one horizontal direction after it has been suddenly imposed. The descriptions of the model and the governing equations are presented in Section 2. In Section 3, the solution of buoyancy...
induced by heating is obtained in environmental fields without uniform zonal wind or damping. In Section 4, the solution of buoyancy is derived in environmental fields with uniform zonal wind and/or damping. In Section 5, the temperature changes in and around the precipitation band are discussed using the results obtained in Sections 3 and 4. A summary is presented in Section 6. Detailed derivations of buoyancy about the point source of heating for the cases of no damping and damping are presented in Appendices A and B, respectively.

2. Model and governing equations

Heating (precipitation) is prescribed for simplicity, although realistic heating has a wide spectrum of temporal and spatial variations. A simple vertical stratification, shown in Fig. 2a, is adopted, which is similar to the environmental fields around the precipitation area in Fig. 1d. The total amount of heating is anticipated to be a shaded area in Fig. 2b, and the amplitude at the middle level corresponds to \( c_p \Delta T^* \). Usually, heating is produced by the temperature and moisture changes among three components of \( h^*; c_p T^*; g z^* \); and \( L^* q^* \). However, it is assumed that the...
neutralization of $h^*$ occurs by the adjustment of the temperature with a deviation having an intensity of $\Delta T^*$. Then, our next concern is the production of buoyancy response by this heating and the distribution of temperature deviation.

In this study, a stably stratified (positive constant static stability $N^*$) atmosphere with a depth $Z^*$ confined by two boundaries ($z^*_0 = 0, Z^*$) on a non-rotating plane is considered. Uniform basic zonal wind ($U^*$) and damping ($\gamma^*$) are assumed. In the $U^* = \gamma^* = 0$ case, three types of heating are considered: a point source, an infinitely extending line, and a line with an end semi-ininitely extending in the X*-direction from the origin ($0, 0$) at $Y^* = 0$. Here $X^*$ and $Y^*$ are the horizontal coordinates. The governing equations of radial velocity $u^*$, vertical wind $w^*$, pressure divided by density $\varphi^*$, and buoyancy $b^*$ on the cylindrical coordinate system are written as

$$\frac{\partial u^*}{\partial t^*} = - \frac{\partial \varphi^*}{\partial r^*},$$

$$0 = - \frac{\partial \varphi^*}{\partial z^*} + b^*,$$

$$0 = \frac{\partial (r u^*)}{\partial r^*} + \frac{\partial (r w^*)}{\partial z^*},$$

and

$$\frac{\partial b^*}{\partial t^*} = Q^* - w^* N^* 2,$$

where $Q^*$, $t^*$, and $r^*$ are the heating, time, and radial distance from heating (see Fig. 3a). Dimensional buoyancy is defined as

$$b = g^* \frac{\Delta \theta^*}{\Delta \theta^* 0^*} \approx g^* \frac{\Delta T^*}{T^* 0^*},$$

where $\Delta \theta^*$ and $\Delta T^*$ are deviations of the potential temperature and the temperature from the initial state, respectively. For simplicity, the basic potential temperature $\theta^* 0^*$ and basic temperature $T^* 0^*$ are assumed to be constants. Then, the transformation from buoyancy to temperature has a one-to-one correspondence. The dimensional temperature deviation will be used in Section 5.

The form of $Q^*$ in the $U^* = \gamma^* = 0$ case is assumed as

$$Q^*(r^*, z^*, t^*) = Q^* f(r^*) H(t^*) \sin \left( \frac{\pi z^*}{Z^*} \right),$$

Here the function $f$ meets the following condition: $\int_0^{\infty} f(r^*) dr^* = 1$. $H(t^*)$ is the Heaviside step function ($H = 0$ for $t^* < 0$ and $1$ for $t^* > 0$). Since a fundamental vertical mode is specified, the variables are represented as
(\(u, \varphi\)) = (\(u', \varphi'\)) \cos (\pi z*/Z*) and (\(w, b, Q\)) = (\(w', b', Q'\)) \sin (\pi z*/Z*), where variables with a dash are dependent only on \(r*\) and \(t*\). When \(Q_{0*}\) is the maximum amplitude of \(Q*, \) dimensional variables are written as
\[r* = \left(\frac{Z*}{\pi}\right) r, \quad t* = N^{-1} t, \quad (u', w') = Q_{0*} N^{-2} (u, w), \quad \varphi' = Q_{0*} (Z* / \pi) N^{-1} \varphi, \quad b' = Q_{0*} N^{-1} b, \quad Q' = Q_{0*} Q, \]
where variables without dashes are non-dimensional ones. Non-dimensional governing equations are represented as
\[
\frac{\partial u}{\partial t} - \frac{\partial b}{\partial r^*} = 0, \tag{7}
\]
\[
\frac{\partial b}{\partial t} = Q - w, \tag{8}
\]
\[
w = -\frac{1}{r} \frac{\partial (ru)}{\partial r}. \tag{9}
\]
Hereafter, the buoyancy field is described at the height of 1/2.

3. Solutions of buoyancy induced by three types of heating in the \(U = \gamma = 0\) case

a. Case of a point source of heating

Consider a point source of heating having the form of
\[
Q(r, t) = \frac{\delta(r)}{2\pi r} H(t), \tag{10}
\]
where \(\delta\) is the Dirac delta function. Point \(P_n(x_n, y)\) is the position of the observer, where the subscript \(n\) is an integer. When the point source of heating is located at \((\xi, 0)\), the buoyancy at point \(P_1(x_1, y)\) is given as

![Fig. 3. Horizontal distributions of heating of (a) a point source, (b) an infinitely extending line, and (c) a line with an end. Points \(P_n\) denoted by filled squares are the observation ones. Filled circles and open squares denote point sources of heating and points at the boundaries where the influence from the point source reaches, respectively. In (a), an influence region is shaded. Bold solid lines along the \(X\) direction at \(Y = 0\) in (b) and (c) mean lines of heating. In (c), region \(I\) (II) denotes an area that is not influenced (influenced) from the end.](image)
Vertical velocity can be estimated by

\[ \frac{\partial b}{\partial t} + \nabla b \cdot \mathbf{v} = Q \]

where \( r \) denotes the maximum distance is \( t \). Figure 4 indicates the expanding features of buoyancy and vertical velocity with \( t \). Vertical velocity can be estimated by \( \frac{\partial b}{\partial t} + Q \) from (8). The boundary between the regions influenced and not influenced moves with a speed of 1.

Since heating is put in the limited area at a constant rate, the domain-averaged buoyancy is proportional to \( t \). Indeed, it is shown using (11) as

\[ 2\pi \int_0^\infty b(r', t) \, dr' = \frac{1}{2} \int_0^\infty \frac{d\alpha}{\sqrt{t^2 - \alpha^2}} = t, \quad (12) \]

where \( \alpha = r'^2 \). Domain-averaged buoyancy is denoted by a dashed line in Fig. 4a.

\[ \mathbf{b}'(x, y, t) = \int_{-L}^L \frac{b(x, y, t) \, dx \, dy}{\pi L^2}, \quad (13) \]

where \( \xi = L\xi \). Buoyancy is entirely constant in the influence region, independent of \( x \) and \( y \). Such a simple solution was obtained by Bretherton and Smolarkiewicz (1989) as

\[ \mathbf{b}''(y, t) = \frac{1}{2} \left( H(y + t) - H(y - t) \right). \quad (14) \]

Here \( \partial(y) \)-type heating excites the gravity waves propagating with a constant speed of 1 in both the positive and the negative \( Y \) directions, and the response reaches \(|y| = t\).

c. Case of a line of heating with an end

Consider heating that semi-ininitely extends in the \( X \) direction from the origin \((0, 0)\) (Fig. 3c). Two regions (I and II) for the observation points should be counted when \( t \) and \( y \) are given. Region I is located far from the end, and the same argument can be made as for those at

\[ \mathbf{b}'(x, y, t) = \frac{1}{2} \left( H(x + t) + H(x - t) \right) \]

Similarly, Nicholls et al. (1991) obtained the non-dimensional buoyancy as

\[ \mathbf{b}''(y, t) = \frac{a}{2} \left[ \tan^{-1}\left( \frac{y + t}{a} \right) - \tan^{-1}\left( \frac{y - t}{a} \right) \right] \]

for heating \( Q''(y, t) = \frac{a}{y + \frac{t}{a}} H(t) \), where \( a \) is a half-width.
point \(P_2\) in Fig. 3b. Point \(P_3(x, y)\) is a typical point until point \(P_4\), where the influence of heating at the origin is observed. On the other hand, region II is close to the end, and point \(P_5(x, y)\) is a representative point between \(P_4\) and \(P_6\), where the influence from the origin is observed. The integral interval of point \(P_5\) is between \(-L_{\xi}^x\) and \(-x_5\) about \(\xi\), i.e.,

\[
b_{t}(x, y, t) = \int_{-L_{\xi}}^{x} \frac{1}{2\pi} \left[\sin^{-1} \zeta \right] - \zeta_i = -\frac{1}{2\pi} \sin^{-1} \zeta + \frac{1}{4},
\]

where \(x_5 = L_{\xi}^x\). Since \(\zeta\) is a function of \(x\) and \(y\), the obtained solution is dependent on them.

The numerical integration of the sum of the point sources of heating, which are located at an interval of 0.00001, is performed in the following, although the analytical calculation is available using (13) and (15). Figure 5 shows the horizontal distribution of buoyancy for heating semi-infinitely extending from the origin at \(t = 2, 4,\) and 6. Buoyancy with an intensity of 1/2 is uniform in region I, as stated in Subsection 3b. On the other hand, deviated distributions of buoyancy are observed, in region II, where its intensity changes from 0.5 to 0 in the rightward direction. At \(t = 4\) and 6, features similar to those at \(t = 2\) are observed, except for expanded features from the origin.

4. Buoyancy responses in the \(U \neq 0\) and/or \(\gamma \neq 0\) cases

Here \(\tau\) is introduced instead of \(t\), which satisfies \(\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + U_{\gamma} \frac{\partial}{\partial x} + \gamma\). The governing equations described by \((\tau, x, y)\) are the same as those in the \(U = \gamma = 0\) case, and then, we can apply the same method as in Section 3.

The values of \(U = -0.31\) and \(\gamma = 0.05\) are selected, where \(U_{\gamma}\) and \(\gamma_{\gamma}\) correspond to \(-10\) m s\(^{-1}\) and \(5 \times 10^{-4}\) s\(^{-1}\), respectively, when values are adopted in which \(N_{\gamma} = 10^{-2}\) s\(^{-1}\) and \(h_{\gamma} = 10^4\) m. Three cases are considered: (i) \(U = -0.31\) and \(\gamma = 0\), (ii) \(U = 0\) and \(\gamma = 0.05\), and (iii) \(U = -0.31\) and \(\gamma = 0.05\).

(i) The \(U = -0.31\) and \(\gamma = 0\) case

In this case, the buoyancy field obtained in Section 3 is uniformly transformed from \((x, y)\) into \((x - U t, y)\) relative to fixed heating. Figure 6 shows the horizontal distributions of buoyancy in the \(U = -0.31\) and \(\gamma = 0\) case. Since \(U < 0\) is applied, the origin \((0, 0)\) shifts leftward, and the entire areas simply move with the speed of \(|U|\).

(ii) The \(U = 0\) and \(\gamma = 0.05\) case

Here the value of \(\gamma\) is assumed to be small. Then, the buoyancy response for a point source of heating is approximated for \(t > r > 0\) as
Detailed derivations are presented in Appendix B. In Fig. 7, the deviated distributions of buoyancy shift leftward with time, and the responses become weaker than that shown in Fig. 5.

(iii) The U = −0.31 and γ = 0.05 case

Figure 8 shows the horizontal distributions of buoyancy in the U = −0.31 and γ = 0.05 case. The leftward shift is stronger than those shown in Figs. 6 and 7, and more deviated features are obtained in the downwind direction. Although the essential features are similar to those in Fig. 5, it is anticipated that the buoyancy response in Fig. 8 is typical to the buoyancy response that occurs in real precipitation bands.

In Fig. 1b, a northwesterly wind prevails at the height of 5.0 km, and a quasi-stationary BB type is observed to be aligning from the northwesterly direction. When the (x, y) axes are defined as indicated in Fig. 1b, the flow is similar to that in the U < 0 case in Fig. 8. It is shown that the non-influence region

\[
2\pi b(r, t) \approx \frac{1}{\sqrt{t^2 - r^2}} + \left[ -\frac{t}{\sqrt{t^2 - r^2}} + \frac{1}{2}\ln\left(\frac{t + \sqrt{t^2 - r^2}}{t - \sqrt{t^2 - r^2}}\right)\right] \gamma + O(\gamma^2).
\]
widely extends on the upwind side.

5. Temperature changes in and around a precipitation band

Using the results obtained in Sections 3 and 4, the temperature changes in and around a precipitation band are discussed. At the initial stage, it is assumed that no convection exists in the entire area, and the stratification is conditionally unstable uniformly in the horizontal direction (Fig. 2a). Then, convection locally takes place in a band-shaped form, and the stratification becomes neutral \( \left( \frac{dh^*}{dz^*} \rightarrow 0 \right) \) (Fig. 2b), where the dimensional temperature deviation is \( \Delta T^* \) at the middle level of the convective layer. Then, in the surroundings, the propagating gravity waves play an essential role in constructing the response field. The excited gravity waves split into two—the positive and negative \( Y^* \) directions—where the precipitation band is located in the \( X^* \) direction. Then, a half of buoyancy deviation, which corresponds to the dimensional temperature deviation \( \Delta T^*/2 \), appears in region I from Section 3 (Fig. 2c). Toward the end (region II), the response becomes smaller, and no response is observed outside the influence region. It is thus concluded that the different stratifications between the convection area and surroundings are produced by different physical processes: the neutralization of \( h^* \) in the convection areas and its response in the surroundings. It also indicates that the response stratification in the surroundings is still conditionally unstable even though intense precipitation occurs. This is an answer for the first issue presented in the introduction.

An answer for the second issue follows. Here it is assumed that the triggering of convective cells is horizontally uniform anytime and anywhere. Then, new convective cells are more likely to occur over the regions that are influenced slightly or not at all. In the \( U_* \neq 0 \) case, as compared with those in the downwind sides, the favorable areas for the new convective cells widely extend on the upwind side of the environmental winds as seen from the results in Fig. 8. Thus, the present discussion supports the possibility of preferential occurrences of BB types on the upwind side of precipitation bands.

6. Summary

To discuss the reasons for the difference in the stratifications between the convection area and surroundings and for the frequent observation of the BB type, the linear responses of buoyancy for heating were examined in environmental fields combined with uniform zonal wind \( (U_*) \) and damping \( (\gamma_*) \). Here line-shaped heating with an end is assumed to be a band-shaped precipitation system. In the \( U_* = \gamma_* = 0 \) case, buoyancy responses are obtained for an infinitely extending line of heating and a line with an end by applying a solution of buoyancy for a point source of heating. It is assumed that convection produces non-dimensional heating with an amplitude of 1 at the middle level of the convective layer, which corresponds to the dimensional temperature deviation \( \Delta T_* \). In the influence region far from the end (region I), the non-dimensional buoyancy response is \( 1/2 \), which corresponds to dimensional temperature deviation...
improving the early version of this paper. The first issue presented in the introduction is answered by the different physical processes between the convection areas and surroundings: neutralization of h in the vertical direction and buoyancy response for heating.

In the $U_\gamma \neq 0$ and/or $\gamma_\gamma \neq 0$ cases, more deviated features are attained and induce preferential occurrence areas on the upwind side of the precipitation band. If an initial conditionally unstable stratification is assumed to be horizontally uniform and the triggering mechanisms of new convection are evenly located in the entire area even after convection occurs, it is speculated that the different horizontal distributions of buoyancy response lead to the possibility of frequent formation of the BB type on the upwind side of the prevailing winds. This addresses the second issue.

However, as stated in the introduction, in addition to the effect of gravity waves, several other factors induce convection in the surroundings of old convective cells. One factor is the effect of gravity currents produced by cold domes due to the evaporation of raindrops below the precipitation areas (e.g., Browning et al. 1976). The effect of gravity currents seems important, because the formation of new convection sometimes occurs near the convergence areas between the gravity currents and environmental winds. A quantitative comparison of gravity waves and gravity current is needed to examine their roles in the formation of new convection. However, gravity currents are non-linear, and therefore irrelevant for this study. Further studies performing numerical simulations with a non-hydrostatic cloud-resolving model are required.

Acknowledgements

We would like to thank Dr. M. Kawashima, Institute of Low Temperature Science Hokkaido Univ., an editor in charge, and anonymous two referees for improving the early version of this paper.

Appendix A: Solutions of the buoyancy about

H(t) $\delta(r)/r$-type heating with no damping

The equation of the buoyancy is written as

$$[-\frac{\partial^2 b}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r}(r b)]_b = -\frac{\delta(r)}{2 \pi r} \frac{\partial H(t)}{\partial t}.$$  (A1)

The Bessel and inverse Bessel transformations about $p(r)$ and $P(R)$ are represented as

$$P(R) = \int_0^\infty p(r) r J_0 (rR) dr,$$
$$p(r) = \int_0^\infty P(R) R J_0 (rR) dR,$$  (A2)

where $J_0$ is a Bessel function of the first kind of order 0 (Moroguchi et al. 1970; Abramowitz and Stegun 1970). The Bessel function $J_0$ satisfies

$$\frac{1}{r} \frac{d}{dr} \left( r J_0 (rR) \right) + J_0 (r) = 0.$$  (A3)

The transformed buoyancy $B$ is defined as

$$B(R, s) = \int_0^\infty b(r, t) r J_0 (rR) dr \int_0^\infty e^{-\pi t/s} dt,$$  (A4)

where the first transformation on the right-hand side (RHS) is a Bessel one about $r$, the second a Laplace one about $t$, and $s$, a parameter. Using

$$\int_0^\infty \delta(r) r J_0 (rR) dr = \int_0^\infty \delta(r) J_0 (r) dr = J_0 (0) = 1,$$  (A5)

$B$ is obtained as

$$2 \pi B = \frac{1}{s^2 + R^2}.$$  (A6)

The buoyancy in the physical space can be obtained in two ways. The first way is to apply the inverse Laplace transformation and then to apply the inverse Bessel one. The solution is given as

$$2 \pi b(r, t) = \int_0^\infty \sin(\pi R) J_0 (rR) dR \left[ \frac{1}{\sqrt{t^2 - R^2}} \right]$$
$$= \begin{cases} \frac{1}{\sqrt{t^2 - R^2}} & \text{for } t > r > 0, \\ 0 & \text{for } r > t > 0. \end{cases}$$  (A7)

These formulae are found in III of Moroguchi et al. (1970) and in (11.4.37) and (11.4.38) of Abramowitz and Stegun (1970).

The second way is, first, to apply the inverse Bessel transformation and then to apply the inverse Laplace one (e.g., Bretherton and Smolarkiewics 1989). Using

$$\int_0^\infty \frac{R J_0 (rR)}{R^2 + s^2} dR = K_0 (sr),$$  (A8)

where $K_0$ is a modified Bessel function of order 0 (Moroguchi et al. 1970),
is obtained \((29.3.119)\) of Abramowitz and Stegun (1970). Here the inverse Laplace transformation of \(K_{n}(s)\) is given as

\[
2\pi K_{n}(s) = \begin{cases} 
\frac{1}{2\pi i} \int_{-\infty}^{\infty} K_{n}(s) e^{st} ds & \text{for } t > r > 0 \\
\sqrt{t-r^2} & \text{for } r > t > 0 
\end{cases}
\]

(A9)

where \(\lambda = \frac{s}{r}\), and \(\nu = 0\) is applied (Moriguchi et al. 1970).

### Appendix B: Solutions of the buoyancy about H(t) \(\delta(r)/r\)-type heating with the damping

The transformed buoyancy \(B\) in the damping case is given as

\[
2\pi B = \frac{1}{(s+\gamma)^2 + R^2} + \frac{1}{s^2 + R^2} \frac{2s}{s^2 + R^2} 
\]

\[
\approx \frac{1}{s^2 + R^2} + \left[ -2s + \frac{1}{s^2 + R^2} \right] \gamma + O(\gamma^2).
\]

(B1)

Here the second way in Appendix A is applied. First, the inverse Bessel transformation of the RHS in (B1) is written as

\[
K_{n}(sr) + \left[ -rK_{n}(sr) + \frac{K_{n}(sr)}{s} \right] \gamma + O(\gamma^2),
\]

(B2)

(III of Moriguchi et al. (1970) and (11.4.44) of Abramowitz and Stegun (1970)). Then, the inverse Laplace transformation of (B2) is given as (16). Here the inverse Laplace transformation of the second and third terms in (B2) for \(\lambda = |r| > 1\) are given by using (A10) and

\[
\lim_{\nu \to 0} (\lambda + \sqrt{\lambda^2 - 1})^\nu = (\lambda - \sqrt{\lambda^2 - 1})^\nu
\]

\[
= \lim_{\nu \to 0} \left( \frac{\lambda + \sqrt{\lambda^2 - 1}}{\lambda - \sqrt{\lambda^2 - 1}} \right)^\nu 
\]

\[
= \frac{1}{2} \ln \frac{\lambda + \sqrt{\lambda^2 - 1}}{\lambda - \sqrt{\lambda^2 - 1}},
\]

(B3)

respectively (II of Moriguchi et al. (1970)).

### References


