Forced, Balanced Model of Tropical Cyclone Intensification

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Abstract

A simplified, axisymmetric, one-layer model of tropical cyclone intensification is presented. The model is based on the Salmon wave-vortex approximation, which can describe flows with low Froude number and arbitrary Rossby number. After introducing an additional approximation designed to filter propagating inertia-gravity waves, the problem is reduced to the prediction of potential vorticity (PV) and the inversion of this PV to obtain the balanced wind and mass fields. This PV prediction/inversion problem is solved analytically for two types of forcing: a two-region model in which there is nonzero forcing in the cyclone core and zero forcing in the far-field and a three-region model in which there is nonzero forcing in both the cyclone core and the eyewall, with zero forcing in the far-field. The solutions of the two-region model provide insight into why tropical cyclones can have long incubation times before rapid intensification and how the size of the mature vortex can be influenced by the size of the initial vortex. The solutions of the three-region model provide insight into the formation of hollow PV structures and the inward movement of angular momentum surfaces across the radius of maximum wind.

Keywords tropical cyclone intensification; potential vorticity; shallow water model; wave-vortex approximation

1. Introduction

The Eliassen (1951) balanced vortex equations constitute a filtered model for an evolving, axisymmetric vortex in hydrostatic and gradient balance. This model has proven to be very useful for understanding the role of inertial stability in tropical cyclone intensification (e.g., Shapiro and Willoughby 1982; Schubert and Hack 1982) and for interpreting a number of tropical cyclone features, including eyewall tilt (e.g., Emanuel 1997; Schubert and McNoldy 2010) and the syndrome of features referred to as hub clouds, eye moats, and warm ring thermal structures (Schubert et al. 2007; Sitkowski et al. 2012; Stern and Zhang 2013). Since the balanced vortex model is a filtered model, an elliptic equation always arises, no matter

what variable is chosen as the primary prognostic variable. For example, if the prognostic variable is the azimuthal wind, the elliptic equation involves the streamfunction for the transverse circulation (Eliassen 1951). In contrast, if the transverse circulation components are eliminated during the derivation, attention is then centered on the elliptic equation for the geopotential tendency (Shapiro and Montgomery 1993; Vigh and Schubert 2009; Musgrave et al. 2012; Persing et al. 2013; Smith et al. 2014). Finally, if the prognostic variable is the potential vorticity (PV) and if the evolution equation for PV is written in potential radius and potential temperature coordinates (thus simplifying the material derivative operator), then the elliptic equation is the invertibility principle relating the PV to the balanced mass field (Schubert and Alworth 1987; Möller and Smith 1994).

Although the Eliassen balanced vortex model has provided considerable insight into many aspects of tropical cyclone dynamics, deeper understanding of tropical cyclone intensification remains elusive

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because simple analytical solutions for the entire time evolution of a balanced vortex have not been found. The purpose of this paper is to present some analytical, time-dependent solutions of an axisymmetric, one-layer, balanced vortex model that is slightly simpler than the Eliassen model. The model is based on the wave-vortex approximation recently introduced by Salmon (2014). In order to fit these balanced model analytical solutions into the more general context of the shallow water primitive equations, we begin Section 2 with a review of the shallow water primitive equations for axisymmetric flow on an f -plane. These equations constitute the “parent dynamics” of the wave-vortex approximation, which is reviewed in Section 3 and in the Appendix. Section 4 introduces a filtering approximation that leads to a balanced vortex model somewhat different than the classical Eliassen balanced vortex model. With the filtering approximation, the entire dynamics can be reduced to a PV prediction/inversion problem that can be solved analytically for both two-region forcing (core and far-field) and for three-region forcing (core, eyewall, and far-field). Solutions of the two-region model are used to illustrate the range of incubation times before rapid intensification and the range of mature vortex sizes (Section 4). Solutions of the three-region model are used to illustrate the formation and thinning of PV rings (Section 5). Concluding remarks are given in Section 6, along with an application of the theory to tornado-scale vortices.

2. Shallow water primitive equations

The axisymmetric shallow water primitive equations on an f -plane are

$$\frac{\partial u}{\partial t} - (f + \zeta)v + \frac{\partial}{\partial r} \left[gh + \frac{1}{2}(u^2 + v^2) \right] = 0, \quad (1)$$

$$\frac{\partial v}{\partial t} + (f + \zeta)u = 0, \quad (2)$$

$$\frac{\partial h}{\partial t} + \frac{\partial(ruh)}{r\partial r} = 0, \quad (3)$$

where u is the radial wind, v the tangential wind, h the fluid depth, g the acceleration due to gravity, f the constant Coriolis parameter, and $\zeta = \partial(rv)/r\partial r$ the relative vorticity. The absolute angular momentum principle, obtained from (2), is

$$\frac{D}{Dt} \left(rv + \frac{1}{2} fr^2 \right) = 0, \quad (4)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} \quad (5)$$

is the material derivative. The vorticity equation, also obtained from (2), can be written in the form

$$\frac{D(f + \zeta)}{Dt} + (f + \zeta) \frac{\partial(ru)}{r\partial r} = 0, \quad (6)$$

while the continuity Eq. (3) can be written in the form

$$\frac{Dh}{Dt} + h \frac{\partial(ru)}{r\partial r} = 0. \quad (7)$$

Eliminating the divergence between (6) with (7), we obtain the potential vorticity conservation principle

$$\frac{DP}{Dt} = 0, \quad (8)$$

where

$$P = \frac{\bar{h}}{h} \left(f + \frac{\partial(rv)}{r\partial r} \right) \quad (9)$$

is the potential vorticity, with the constant $\bar{h}$ denoting the fluid depth to which $h(r, t)$ asymptotes as $r \rightarrow \infty$.

3. The wave-vortex approximation

Salmon (2014) proposed a wave-vortex approximation of (1)–(3). He derived this approximation by first expressing the exact shallow water dynamics in the equivalent Hamiltonian form, which, as discussed in the Appendix, involves the Poisson bracket and the exact Hamiltonian (total energy). In the expression for the exact Hamiltonian, the kinetic energy in the fluid column is given by $(1/2)(u^2 + v^2)h$. In the context of the equivalent Hamiltonian form, the only approximation in the wave-vortex model is to replace the exact Hamiltonian with an approximate Hamiltonian in which the kinetic energy in the fluid column is approximated by $(1/2)(u^2 + v^2)\bar{h}$. Then, for the axisymmetric, shallow water dynamics considered here, the wave-vortex approximation takes the form

$$\frac{\partial u}{\partial t} - Pv + g \frac{\partial h}{\partial r} = 0, \quad (10)$$

$$\frac{\partial v}{\partial t} + Pu = 0, \quad (11)$$

$$\frac{\partial h}{\partial t} + \bar{h} \frac{\partial(ru)}{r\partial r} = 0, \quad (12)$$

where P is defined by (9). A detailed derivation of (10)–(12) is given in the Appendix.

Whenever we replace one set of equations, such as (1)–(3), with another approximate set, such as (10)–(12), there arises the question of identifying variables in the approximate set with variables in the original set. If we identify the variables u , v , h in the approximate dynamics (10)–(12) with the variables u , v , h in the parent dynamics (1)–(3), we conclude that all three equations (1)–(3) have been approximated. However, this is not the only interpretation, and, as noted by Salmon, this identification of variables in the approximate dynamics with variables in the parent dynamics (1)–(3) is “unavoidably ambiguous.” For example, suppose we write (10)–(12) in terms of the variable $\hat{u} = (\bar{h}/h)u$, and then make the simple notational change $\hat{u} \rightarrow u$, so that (10)–(12) become

$$\frac{\partial}{\partial t} \left(\frac{h}{\bar{h}} u \right) - Pv + g \frac{\partial h}{\partial r} = 0, \quad (13)$$

$$\frac{\partial v}{\partial t} + (f + \zeta)u = 0, \quad (14)$$

$$\frac{\partial h}{\partial t} + \frac{\partial(ruh)}{r \partial r} = 0. \quad (15)$$

Now, if we identify the variables u , v , h in (13)–(15) with the variables u , v , h in the parent dynamics (1)–(3), we conclude that (14) and (15) are exact, but (13) is an approximated form of (1). Equations (10)–(12) and (13)–(15) are equally valid forms of the wave-vortex approximation, but with somewhat different interpretations attached to the associated u fields. In the next section, we shall proceed from (13)–(15), but the analysis could equally well proceed from (10)–(12).

Because the notational change $\hat{u} \rightarrow u$ has made (14) and (15) formally identical to (2) and (3), the absolute angular momentum and potential vorticity equations derived from (14) and (15) are formally identical to (4) and (8), with the material derivative given by (5). Alternatively, if we proceed from (10)–(12), we find that the absolute angular momentum and potential vorticity equations are also given by (4) and (8), but with a slightly modified definition of D/Dt , i.e., with a definition in which the radial advecting velocity is given by $(\bar{h}/h)u$. As we shall see in Section 4, the absolute angular momentum and potential vorticity equations play a crucial role in the analysis.

It is also interesting to note that the linear geostrophic adjustment dynamics of (10)–(12) or (13)–(15) are identical to those of (1)–(3). This can be seen by simply linearizing each set about a resting basic state with constant depth $\bar{h}$, and then noting that the linearized equations from each set are identical. A

discussion of this linearized dynamics can be found in Schubert et al. (1980). Concerning the nonlinear transient adjustment dynamics of (10)–(12) or (13)–(15), they do differ from those described by (1)–(3). As can be inferred from the results of Kuo and Polvani (1997), the nonlinear transient adjustment dynamics of (1)–(3) involve the formation and decay of shocks (i.e., bores), in contrast to the approximate wave-vortex dynamics, in which the inertia-gravity waves do not steepen and form shocks. This is consistent with the notion that the approximate wave-vortex dynamics is limited to flows with small Froude number, i.e., to flows in which typical flow speeds are much less than $c = (g\bar{h})^{1/2}$.

4. Forced, balanced model

We now make two changes to the wave-vortex system (13)–(15). The first change is to include a mass sink term on the right-hand side of (15). The second change is to introduce an approximation that filters inertia-gravity waves by neglecting the local time derivative term in (13). Thus, our forced, balanced model takes the form

$$Pv = g \frac{\partial h}{\partial r}, \quad (16)$$

$$\frac{\partial v}{\partial t} + (f + \zeta)u = 0, \quad (17)$$

$$\frac{\partial h}{\partial t} + \frac{\partial(ruh)}{r \partial r} = -hS. \quad (18)$$

Because of the mass sink term, the potential vorticity equation now becomes

$$\frac{DP}{Dt} = SP, \quad (19)$$

while the total energy principle becomes

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \frac{1}{2} [v^2 \bar{h} + g(h - \bar{h})^2] r \, dr \\ = \int_0^\infty g(\bar{h} - h) h S r \, dr. \end{aligned} \quad (20)$$

In the remainder of this paper, we shall present analytical solutions of the forced, balanced model (16)–(18). It should be noted that the balanced vortex model (16)–(18) is different than the balanced vortex model of Eliassen (1951), since the latter is based on gradient wind balance rather than the balance (16). The Eliassen balanced vortex model has been used in many tropical cyclone studies, such as those of

Ooyama (1969), Shapiro and Willoughby (1982), and Schubert and Hack (1982), and should be regarded as the more general of the two models, because it does not have a small Froude number restriction on its applicability. However, a particular advantage of the system (16)–(18) is the ease with which analytical solutions can be obtained.

The mass sink $S(r, t)$ simulates the effect of diabatic heating in a continuously stratified, compressible fluid, as first discussed by Ooyama (1969) in the context of an axisymmetric tropical cyclone model having three incompressible fluid layers, and as later discussed by DeMaria and Pickle (1988) in the context of an axisymmetric tropical cyclone model having three isentropic fluid layers. The use of S as a proxy for diabatic heating has also been recently discussed in the context of a one-layer model by Bouchut et al. (2009) and in the context of a two-layer model by Lambaerts et al. (2011). Another interpretation of S in terms of tropical diabatic heating has been provided by Fulton and Schubert (1985), who performed a vertical normal mode analysis of observed vertical profiles of tropical diabatic heating. Typical vertical profiles of diabatic heating project onto the external mode ($c \approx 250\text{--}300 \text{ m s}^{-1}$) and the first two internal modes ($c \approx 50\text{--}77 \text{ m s}^{-1}$ and $29\text{--}47 \text{ m s}^{-1}$), with the exact values of c depending on the boundary conditions and static stability used in the eigenvalue calculation. Since the wave-vortex approximation has a small Froude number restriction, we have experimented with values of c between 250 and 300 m s^{-1} , with the conclusion that the results change little as c is varied in this range. For the results shown here we have used $c = 250 \text{ m s}^{-1}$. As a useful guide to interpretation, the idealized analytical solutions presented here can be considered most applicable to the vertical mean structure (i.e., the external mode) of tropical cyclones.

Now, assume that the mass sink $S(r, t)$ is a positive constant S_1 in the core region and vanishes in the region outside the core. The core region is bounded by an absolute angular momentum surface that moves inward with the inward radial flow. The position of this absolute angular momentum surface is denoted by $r_1(t)$. The function $r_1(t)$ is as yet undetermined, but will later be determined as part of the solution of the PV prediction/inversion problem. With these assumptions, the mass sink $S(r, t)$ can be written in the form

$$S(r, t) = S_1 H[r_1(t) - r] = \begin{cases} S_1 & \text{if } r < r_1(t) \\ \frac{1}{2} S_1 & \text{if } r = r_1(t) \\ 0 & \text{if } r > r_1(t), \end{cases} \quad (21)$$

where $H(x)$ is the Heaviside step function (not to be confused with the Hamiltonian functional used in the Appendix), defined as zero for $x < 0$, as unity for $x > 0$, and as $1/2$ for $x = 0$. Because of the form of the mass sink (21), an initial piecewise uniform potential vorticity remains piecewise uniform. The piecewise uniform initial condition on P is

$$P(r, 0) = P_1 H(r_{10} - r) + f H(r - r_{10}) = \begin{cases} P_1 & \text{if } r < r_{10} \\ \frac{1}{2}(P_1 + f) & \text{if } r = r_{10} \\ f & \text{if } r > r_{10}, \end{cases} \quad (22)$$

where the constant P_1 is the initial PV in the core and the constant r_{10} is the initial value of $r_1(t)$.

Given the forcing (21) and the initial condition (22), we now seek a solution of the PV equation (19) having the piecewise constant form

$$P(r, t) = \alpha_1(t) H[r_1(t) - r] + \alpha_2(t) H[r - r_1(t)] = \begin{cases} \alpha_1(t) & \text{if } r < r_1(t) \\ \frac{1}{2}[\alpha_1(t) + \alpha_2(t)] & \text{if } r = r_1(t) \\ \alpha_2(t) & \text{if } r > r_1(t), \end{cases} \quad (23)$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are as yet undetermined functions of time. Substituting (23) into (19), we obtain

$$\begin{aligned} 0 &= \frac{\partial P}{\partial t} + u \frac{\partial P}{\partial r} - SP \\ &= \left(\frac{d\alpha_1}{dt} - S_1 \alpha_1 H[r_1(t) - r] \right) H[r_1(t) - r] \\ &\quad + \alpha_1 \left(\frac{dr_1}{dt} - u \right) \delta[r_1(t) - r] \\ &\quad + \left(\frac{d\alpha_2}{dt} - S_1 \alpha_2 H[r_1(t) - r] \right) H[r - r_1(t)] \\ &\quad - \alpha_2 \left(\frac{dr_1}{dt} - u \right) \delta[r - r_1(t)] \\ &= \left(\frac{d\alpha_1}{dt} - S_1 \alpha_1 \right) H[r_1(t) - r] + \frac{d\alpha_2}{dt} H[r - r_1(t)] \\ &= \begin{cases} \frac{d\alpha_1}{dt} - S_1 \alpha_1 & \text{if } r < r_1(t) \\ \frac{d\alpha_2}{dt} & \text{if } r > r_1(t) \end{cases} \end{aligned} \quad (24)$$

for $r \neq r_1(t)$, where $\delta(x) = dH(x)/dx$ is the Dirac delta function. To obtain the third equality in (24), we have used the fact that $dr_1/dt = u(r_1, t)$ and $H[r_1(t) - r]$

$H[r - r_1(t)] = 0$ for $r \neq r_1(t)$. When $r < r_1(t)$, the step functions take on the values $H[r_1(t) - r] = 1$ and $H[r - r_1(t)] = 0$, yielding the first case in the last line of (24). Similarly, when $r > r_1(t)$, the step functions take on the values $H[r_1(t) - r] = 0$ and $H[r - r_1(t)] = 1$, yielding the second case in the bottom line of (24). Then, with the initial condition (22), the solutions for $\alpha_1(t)$ and $\alpha_2(t)$ are

$$\alpha_1(t) = f e^{\tau_1} \quad \text{and} \quad \alpha_2(t) = f, \quad (25)$$

where $\tau_1 = S_1(t + t_1)$ is a dimensionless time, and where the constant t_1 is defined by $e^{S_1 t_1} = P_1/f$, i.e., t_1 is the time that would be required to exponentially spin-up the initial vortex from a state of rest. Using (25), the solution (23) takes the form

$$P(r, t) = f e^{\tau_1} H[r_1(t) - r] + f H[r - r_1(t)] \\ = f \begin{cases} e^{\tau_1} & \text{if } r < r_1(t) \\ \frac{1}{2}(e^{\tau_1} + 1) & \text{if } r = r_1(t) \\ 1 & \text{if } r > r_1(t). \end{cases} \quad (26)$$

The diagnosis of the mass and wind fields from the potential vorticity field (26) requires solution of the invertibility principle. The invertibility principle for h can be derived by combining the definition of P with the balance condition (16) to obtain

$$\frac{\partial}{r \partial r} \left(\frac{r}{P} \frac{\partial h}{\partial r} \right) - \frac{P}{g h} h = -\frac{f}{g}. \quad (27)$$

With the solution (26) for $P(r, t)$, the invertibility principle (27) takes the form

$$\begin{cases} \frac{\partial^2 h}{\partial r^2} + \frac{\partial h}{r \partial r} - \mu_1^2 h = -\mu_1^2 \bar{h} e^{-\tau_1} & \text{if } r < r_1(t) \\ \frac{\partial^2 h}{\partial r^2} + \frac{\partial h}{r \partial r} - \mu^2 h = -\mu^2 \bar{h} & \text{if } r > r_1(t), \end{cases} \quad (28)$$

where

$$\mu_1(t) = \mu e^{\tau_1} \quad \text{and} \quad \mu = \frac{f}{c} \quad (29)$$

are the inverse Rossby lengths in the two regions. Note that the Rossby length (c/f) in the far-field is fixed, while the Rossby length $(c/f)e^{-\tau_1}$ in the core decreases exponentially with time.

In a manner analogous to that of Schubert and Hack (1982), we shall express the solution of (28) in terms of the zero order modified Bessel functions $I_0(x)$ and

$K_0(x)$, which satisfy the zero order modified Bessel equation, i.e.,

$$\frac{d^2 I_0(x)}{dx^2} + \frac{dI_0(x)}{x dx} - I_0(x) = 0, \quad (30)$$

$$\frac{d^2 K_0(x)}{dx^2} + \frac{dK_0(x)}{x dx} - K_0(x) = 0.$$

We shall also make use of the derivative relations

$$\frac{dI_0(x)}{dx} = I_1(x), \quad \frac{dK_0(x)}{dx} = -K_1(x), \quad (31)$$

$$\frac{d[xI_1(x)]}{x dx} = I_0(x), \quad \frac{d[xK_1(x)]}{x dx} = -K_0(x), \quad (32)$$

and the Wronskian

$$I_0(x)K_1(x) + K_0(x)I_1(x) = \frac{1}{x}, \quad (33)$$

where $I_1(x)$ and $K_1(x)$ are the first order modified Bessel functions. For reference, the functions $I_0(x)$, $K_0(x)$, $I_1(x)$, and $K_1(x)$ are plotted in Fig. 1.

To solve the invertibility problem (28) in the entire domain, we first solve it in each subdomain, enforcing the boundary conditions for $r = 0$ and $r \rightarrow \infty$, and then match the solutions for h and v at $r = r_1(t)$. For example, in the region $r < r_1(t)$, the solution for h consists of the sum of the particular solution $\bar{h} e^{-\tau_1}$ and the homogeneous solution proportional to $I_0(\mu_1 r)$, with the other homogeneous solution $K_0(\mu_1 r)$ discarded because it is unbounded at $r = 0$. Similarly, in the region $r > r_1(t)$, the solution for h consists of the sum of the particular solution $\bar{h}$ and the homogeneous solution proportional to $K_0(\mu r)$, with the other homogeneous solution $I_0(\mu r)$ discarded because it is unbounded as $r \rightarrow \infty$. Then, choosing the coefficients of $I_0(\mu_1 r)$ and $K_0(\mu r)$ in a way that anticipates the requirement of continuity for $v(r, t)$ at $r = r_1(t)$, the solution for the fluid depth can be written in the form

$$h(r, t) = \bar{h} \begin{cases} e^{-\tau_1} + \frac{v_1(t)}{c} \frac{I_0(e^{\tau_1} \mu r)}{I_1(e^{\tau_1} \mu r_1)} & \text{if } 0 \leq r \leq r_1(t) \\ 1 - \frac{v_1(t)}{c} \frac{K_0(\mu r)}{K_1(\mu r_1)} & \text{if } r_1(t) \leq r < \infty, \end{cases} \quad (34)$$

where the function $v_1(t)$ is as yet undetermined. In (34) and the following two equations, the first line on the right-hand side is for the core region, while the second line is for the far-field region. From the balance relation (16) and the modified Bessel function

Fig. 1. Modified Bessel functions of order zero and order one.

derivative relations (31), the corresponding solution for the azimuthal wind is

$$v(r, t) = v_1(t) \begin{cases} \frac{I_1(e^{\tau_1} \mu r)}{I_1(e^{\tau_1} \mu r_1)} & \text{if } 0 \leq r \leq r_1(t) \\ \frac{K_1(\mu r)}{K_1(\mu r_1)} & \text{if } r_1(t) \leq r < \infty. \end{cases} \quad (35)$$

Note that (35) guarantees that $v(r, t)$ is continuous at $r = r_1(t)$, thus validating the choice of the coefficients for $I_0(\mu r)$ and $K_0(\mu r)$ in (34). From (35) and the derivative relations (32), the relative vorticity is given by

$$\frac{\zeta(r, t)}{f} = \frac{v_1(t)}{c} \begin{cases} \frac{e^{\tau_1} I_0(e^{\tau_1} \mu r)}{I_1(e^{\tau_1} \mu r_1)} & \text{if } 0 \leq r < r_1(t) \\ -\frac{K_0(\mu r)}{K_1(\mu r_1)} & \text{if } r_1(t) < r < \infty. \end{cases} \quad (36)$$

Like the potential vorticity, the relative vorticity is discontinuous at $r = r_1(t)$.

We now discuss the continuity of $h(r, t)$, a requirement that introduces the essential nonlinearity of the problem and allows us to determine $r_1(t)$ and $v_1(t)$. From (34), the continuity of $h(r, t)$ at $r = r_1(t)$ requires that

$$\frac{v_1(t)}{c} = \frac{(1 - e^{-\tau_1}) I_1(e^{\tau_1} \mu r_1) K_1(\mu r_1)}{I_0(e^{\tau_1} \mu r_1) K_1(\mu r_1) + K_0(\mu r_1) I_1(e^{\tau_1} \mu r_1)}. \quad (37)$$

Because of the absolute angular momentum prin-

ciple, the edge of the mass sink $r_1(t)$ is related to the maximum tangential wind $v_1(t)$ by $r_1(t)v_1(t) + (1/2)fr_1^2(t) = (1/2)fR_1^2(t)$, where the constant R_1 is the potential radius of this angular momentum surface. This absolute angular momentum principle can be written in the form

$$\frac{v_1(t)}{c} = \frac{(\mu R_1)^2 - (\mu r_1)^2}{2\mu r_1}. \quad (38)$$

Eliminating $v_1(t)$ between (37) and (38), we obtain

$$\frac{(1 - e^{-\tau_1}) I_1(e^{\tau_1} \mu r_1) K_1(\mu r_1)}{I_0(e^{\tau_1} \mu r_1) K_1(\mu r_1) + K_0(\mu r_1) I_1(e^{\tau_1} \mu r_1)} = \frac{(\mu R_1)^2 - (\mu r_1)^2}{2\mu r_1}. \quad (39)$$

For a given value of τ_1 and a given value of μR_1 , (39) is a transcendental equation for μr_1 , the solution of which allows determination of $v_1(t)$ from (38). From solutions of this transcendental equation, we have constructed Table 1 and Fig. 2, which show $r_1(t)$ and $v_1(t)$ for the special case of an initial condition at rest, i.e., $P_1 = f$, $t_1 = 0$, $v_{10} = 0$, and $R_1 = r_{10}$. Four different cases are shown, with the four initial conditions $r_{10} = 100, 200, 300, 400$ km. The five thin black lines in Fig. 2 connect points with the same values of the dimensionless time $S_1 t$, i.e., $S_1 t = 1, 2, 3, 4, 5$. To label the points along each of the four colored curves in terms of the actual time t requires that we choose a value of S_1 on each curve. The simplest way of doing this is to assume that S_1 is the same on each curve, in which case the lines of constant $S_1 t$ in Fig. 2 are also lines of constant t , with the actual value of t depending on the choice of the single assumed numerical value of S_1 . Another way of doing this, consistent with the arguments of Gray (1981) and Schubert and Hack (1982), is as follows. Suppose a smaller initial vortex has more intense convection, or equivalently, a larger value of S_1 . In particular, assume that the initial area-averaged mass sink (averaged over a disk with radius 400 km or larger) is the same for each curve in Fig. 2, so that $S_1 r_{10}^2$ is a constant. One could then choose $S_1 = (3 \text{ h})^{-1}, (12 \text{ h})^{-1}, (27 \text{ h})^{-1}, (48 \text{ h})^{-1}$ for the four curves starting respectively at $r_{10} = 100, 200, 300, 400$ km. With this choice, the actual times t (in hours) are as indicated in the figure. Then, for example, the blue curve ($r_{10} = 200$ km) could be interpreted as a small vortex with intense forcing that becomes a Category 1 hurricane in 48 hours, while the orange curve ($r_{10} = 400$ km) could be interpreted as a larger vortex with weak forcing that is developing so

Table 1. Solutions of the transcendental Eq. (39) for the choices $c = 250 \text{ m s}^{-1}$, $f = 5 \times 10^{-5} \text{ s}^{-1}$, $\tau_1 = S_1 t$, and $R_1 = r_{10}$ (initial state of rest), with four different choices for r_{10} .

$S_1 t$	P/f (core)	$r_{10} = 100 \text{ km}$		$r_{10} = 200 \text{ km}$		$r_{10} = 300 \text{ km}$		$r_{10} = 400 \text{ km}$	
		$r_1(t)$ (km)	$v_1(t)$ (m s^{-1})	$r_1(t)$ (km)	$v_1(t)$ (m s^{-1})	$r_1(t)$ (km)	$v_1(t)$ (m s^{-1})	$r_1(t)$ (km)	$v_1(t)$ (m s^{-1})
0.0	1.0	100.0	0.0	200.0	0.0	300.0	0.0	400.0	0.0
0.5	1.6	77.9	1.3	155.9	2.5	233.9	3.8	312.2	5.0
1.0	2.7	60.7	2.6	121.4	5.2	182.4	7.8	243.6	10.3
1.5	4.5	47.3	4.1	94.6	8.2	142.2	12.3	190.0	16.3
2.0	7.4	36.8	5.9	73.7	11.7	110.9	17.5	148.3	23.3
2.5	12.2	28.7	8.0	57.5	16.0	86.5	23.9	115.8	31.7
3.0	20.1	22.3	10.6	44.8	21.2	67.5	31.7	90.5	41.9
3.5	33.1	17.4	13.9	34.9	27.8	52.7	41.4	70.9	54.6
4.0	54.6	13.6	18.1	27.3	36.0	41.3	53.5	55.7	70.4
4.5	90.0	10.6	23.4	21.3	46.4	32.4	68.6	44.1	89.7
5.0	148.4	8.2	30.1	16.7	59.5	25.6	87.3	35.2	112.8

Fig. 2. $v_1(t)$ versus $r_1(t)$ for four vortices that evolve from resting initial states with $r_{10} = 100, 200, 300, 400 \text{ km}$. Labels on the right stand for tropical depression (TD), tropical storm (TS), and Category 1–4 hurricanes. Along the five thin black lines, the dimensionless time takes on the respective values $S_1 t = 1, 2, 3, 4, 5$, while the fractional volume removed takes on the respective values $F = 0.632, 0.865, 0.950, 0.982, 0.993$. The numbers along each curve denote the time (in hours) under the assumption that the respective values of S_1 are $(3 \text{ h})^{-1}, (12 \text{ h})^{-1}, (27 \text{ h})^{-1}, (48 \text{ h})^{-1}$, which means that $S_1 r_{10}^2$ is the same on each colored curve.

slowly that it has only reached tropical storm strength in 96 hours. Although the orange case does become a hurricane, it has a long incubation time.

Another way to interpret position along the four colored curves in Fig. 2 is in terms of the volume of

fluid that has been removed by the mass sink between the initial time and time t . This volume can be found from the solution (34) by computing the area integral of $\bar{h} - h(r, t)$. The process involves radial integrals of $rI_0(\mu_1 r)$ and $rK_0(\mu r)$, which are easily evaluated with

the aid of the derivative relations (32). The result is

$$2\pi \int_0^\infty [\bar{h} - h(r, t)] r dr = \pi r_{10}^2 \bar{h} \mathcal{F}, \quad (40)$$

where $\mathcal{F} = 1 - e^{-S_1 t}$. The left-hand side of (40) is the total volume of fluid that has been removed between the initial time and time t . Since $\pi r_{10}^2 \bar{h}$ is the initial volume of fluid inside radius r_{10} , the factor $\mathcal{F}$ can be interpreted as the fraction of this volume that has been removed between the initial time and time t . The five thin black lines in Fig. 2 are also labeled in terms of the corresponding value of $\mathcal{F}$. Note that a large fraction of the initial volume must be removed for a vortex to reach hurricane strength. For example, even with $\mathcal{F} = 0.865$, all four vortices remain less than 23 m s^{-1} in strength. It is also important to note that, even though a large fraction of the initial mass inside the angular momentum surface R_1 has been removed, this does not imply that $h \ll \bar{h}$ in the vortex core. In fact, the fluid depth in the core is reduced by only a few hundred meters (see Fig. 3a), as the removed fluid is replaced through the radial inflow associated with the inward movement of the angular momentum surfaces.

Mindful of the Froude number restriction, the graphical results in Fig. 2 have been terminated at 70 m s^{-1} and the numerical results in Table 1 have been terminated at $S_1 t = 5$. However, it is possible to obtain solutions for $S_1 t \rightarrow \infty$. From such time asymptotic calculations, we can conclude that a point vortex is not produced, i.e., $r_1(t)$ does not asymptotically approach zero. As $S_1 t \rightarrow \infty$ we find that $v_1(t) \rightarrow c$ and $r_1(t)v_1(t) \rightarrow (1/2)fR_1^2$, which implies that $r_1(t) \rightarrow (1/2)fR_1^2/c$. Thus, for the four colored curves in Fig. 2, the time asymptotic value of $v_1(t)$ is 250 m s^{-1} and the time asymptotic values of $r_1(t)$ are 1, 4, 9, and 16 km, respectively. Note that a final collapse of $r_1(t)$ to zero is prevented by the factor h on the right hand side of (18) and the fact that $h(0, t) \rightarrow 0$ as $S_1 t \rightarrow \infty$. In any event, interpretation of the results as $S_1 t \rightarrow \infty$ is problematic because $v_1(t) \rightarrow c$ and the small Froude number restriction of wave-vortex theory is violated.

Another way to display the two-region model results is given in Figs. 3 and 4, which show $h(r, t)$ and $v(r, t)$, computed from (34) and (35) using a resting initial state with $r_{10} = 200 \text{ km}$ and $S_1 = (12 \text{ h})^{-1}$, which corresponds to the blue curve in Fig. 2. As can be seen, between $t = 0$ and $t = 60 \text{ h}$, the fluid depth at the vortex center falls from 6371 m to slightly less than 6000 m, while the maximum tangential wind increases to 59.5 m s^{-1} and the radius of maximum wind collapses from 200 km to 16.7 km. As

Fig. 3. (a) Isolines of the depth anomaly $\bar{h} - h(r, t)$; (b) Isolines of the tangential wind $v(r, t)$, with the 50 m s^{-1} contour unlabeled. These results are for a two-region vortex that evolves from a resting initial state with $r_{10} = 200 \text{ km}$ and $S_1 = (12 \text{ h})^{-1}$. The thick solid lines show the inward movement of the absolute angular momentum surface $r_1(t)$. The value of $\bar{h}$ is 6371 m, so the contour for the 300 m depth anomaly corresponds to a fluid depth of 6071 m.

shown in the insert of Fig. 4, the PV in the core exponentially increases to $148.4 f$ at $t = 60 \text{ h}$.

The results shown in Figs. 2–4 have certain similarities with many tropical cyclone modeling studies. For example, using a three-layer, axisymmetric, balanced model, Ooyama (1969) studied the sensitivity of tropical cyclogenesis to different initial vortex sizes and

Fig. 4. Tangential velocity $v(r, t)$ versus r for a vortex that evolves from a resting initial state with $r_{10} = 200$ km and $S_1 = (12 \text{ h})^{-1}$. The five radial profiles of $v(r, t)$ correspond to $t = 12, 24, 36, 48, 60$ h, with kinks occurring at the absolute angular momentum surface $r = r_1(t)$. The figure insert shows $P(t)/f$ in the core region $0 \leq r < r_1(t)$.

strengths. While many of his experiments resulted in intense cyclones with very similar maximum tangential wind, there were great differences in the length of the “incubation” period before rapid intensification commenced. In addition, the size of the mature cyclone was influenced by the size of the initial vortex. These properties are also present in the analytical solutions presented here. For example, under the assumption that $S_1 r_{10}^2$ is the same on each colored curve, then, at the $v_1 = 10 \text{ m s}^{-1}$ stage, the smaller vortices are better preconditioned for rapid intensification. In passing, we also note that the vortices evolving along the blue and green lines near $S_1 t \approx 1$ are qualitatively similar to the long-incubation time vortices that emerge in the recent numerical simulations of Davis (2015).

5. Formation and thinning of PV rings

In order to better understand the effects of eyewall formation, we now generalize the two-region model of Section 4 to a three-region model. We assume that the mass sink is piecewise constant in three regions: an inner region $0 \leq r < r_1(t)$ defined as the “core”,

an annular region $r_1(t) < r < r_2(t)$ defined as the “eyewall”, and a region $r_2(t) < r < \infty$ defined as the “far-field.” The mass sink is given by

$$S(r, t) = \begin{cases} S_1 & \text{if } 0 \leq r < r_1(t) \\ S_2 & \text{if } r_1(t) < r < r_2(t) \\ 0 & \text{if } r_2(t) < r < \infty, \end{cases} \quad (41)$$

where S_1 and S_2 are constants. With the forcing (41) and the piecewise constant initial condition

$$P(r, 0) = \begin{cases} P_1 & \text{if } 0 \leq r < r_{10} \\ P_2 & \text{if } r_{10} < r < r_{20} \\ f & \text{if } r_{20} < r < \infty, \end{cases} \quad (42)$$

an analysis similar to that given in section 4 shows that the solution of (19) is

$$P(r, t) = f \begin{cases} e^{\tau_1} & \text{if } 0 \leq r < r_1(t) \\ e^{\tau_2} & \text{if } r_1(t) < r < r_2(t) \\ 1 & \text{if } r_2(t) < r < \infty, \end{cases} \quad (43)$$

where $\tau_1 = S_1(t + t_1)$, $\tau_2 = S_2(t + t_2)$, and t_1 and t_2 are defined by $e^{S_1 t_1} = P_1/f$ and $e^{S_2 t_2} = P_2/f$. With the solution (43) for $P(r, t)$, the invertibility principle (27) takes the form

$$\begin{cases} \frac{\partial^2 h}{\partial r^2} + \frac{\partial h}{r \partial r} - \mu_1^2 h = -\mu_1^2 \bar{h} e^{-\tau_1} & \text{if } 0 \leq r < r_1(t) \\ \frac{\partial^2 h}{\partial r^2} + \frac{\partial h}{r \partial r} - \mu_2^2 h = -\mu_2^2 \bar{h} e^{-\tau_2} & \text{if } r_1(t) < r < r_2(t) \\ \frac{\partial^2 h}{\partial r^2} + \frac{\partial h}{r \partial r} - \mu^2 h = -\mu^2 \bar{h} & \text{if } r_2(t) < r < \infty, \end{cases} \quad (44)$$

where

$$\mu_1(t) = \mu e^{\tau_1}, \quad \mu_2(t) = \mu e^{\tau_2}, \quad \mu = \frac{f}{c}, \quad (45)$$

are the inverse Rossby lengths in the three regions. Note that the Rossby length (c/f) in the far-field is fixed, while the Rossby length $(c/f)e^{-\tau_1}$ in the core and the Rossby length $(c/f)e^{-\tau_2}$ in the eyewall both decrease exponentially with time.

As in Section 4, we can solve the invertibility problem (44) in the entire domain by first solving it in each subdomain, enforcing the boundary conditions for $r = 0$ and $r \rightarrow \infty$, and then matching the solutions for h and v at $r = r_1(t)$ and $r = r_2(t)$. For example, in the core the solution for h consists of the

sum of the particular solution $\bar{h}e^{-\tau_1}$ and the homogeneous solution proportional to $I_0(\mu_1 r)$, with the other homogeneous solution $K_0(\mu_1 r)$ discarded because it is unbounded at $r = 0$. Similarly, in the far-field, the solution for h consists of the sum of the particular solution $\bar{h}$ and the homogeneous solution proportional to $K_0(\mu r)$, with the other homogeneous solution $I_0(\mu r)$ discarded because it is unbounded as $r \rightarrow \infty$. In the eyewall, the solution for h is the sum of the particular solution $\bar{h}e^{-\tau_2}$ and the homogeneous solution, which consists of a linear combination of $I_0(\mu_2 r)$ and $K_0(\mu_2 r)$. A convenient form of the solution for the fluid depth is

$$h(r, t) = \bar{h} \begin{cases} e^{-\tau_1} + \frac{v_1(t)}{c} \frac{I_0(e^{\tau_1} \mu r)}{I_1(e^{\tau_1} \mu r_1)} & \text{if } 0 \leq r \leq r_1(t) \\ e^{-\tau_2} + \frac{v_1(t)}{c} \frac{F(e^{\tau_2} \mu r, e^{\tau_2} \mu r_2)}{G(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r_2)} \\ - \frac{v_2(t)}{c} \frac{F(e^{\tau_2} \mu r, e^{\tau_2} \mu r_1)}{G(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r_2)} & \text{if } r_1(t) \leq r \leq r_2(t) \\ 1 - \frac{v_2(t)}{c} \frac{K_0(\mu r)}{K_1(\mu r_2)} & \text{if } r_2(t) \leq r < \infty, \end{cases} \quad (46)$$

where $v_1(t)$ and $v_2(t)$ are as yet undetermined functions and where

$$\begin{aligned} F(x, y) &= I_0(x)K_1(y) + K_0(x)I_1(y), \\ G(x, y) &= I_1(x)K_1(y) - K_1(x)I_1(y). \end{aligned} \quad (47)$$

The corresponding solution for the azimuthal wind can be found from the balance relation (16) and the modified Bessel function derivative relations (31), from which it follows that

$$\begin{aligned} \frac{\partial F(e^{\tau_2} \mu r, e^{\tau_2} \mu r_2)}{\partial r} &= e^{\tau_2} \mu G(e^{\tau_2} \mu r, e^{\tau_2} \mu r_2), \\ \frac{\partial F(e^{\tau_2} \mu r, e^{\tau_2} \mu r_1)}{\partial r} &= -e^{\tau_2} \mu G(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r). \end{aligned}$$

The solution for $v(r, t)$ is

$$v(r, t) = \begin{cases} v_1(t) \frac{I_1(e^{\tau_1} \mu r)}{I_1(e^{\tau_1} \mu r_1)} & \text{if } 0 \leq r \leq r_1(t) \\ v_1(t) \frac{G(e^{\tau_2} \mu r, e^{\tau_2} \mu r_2)}{G(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r_2)} \\ + v_2(t) \frac{G(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r)}{G(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r_2)} & \text{if } r_1(t) \leq r \leq r_2(t) \\ v_2(t) \frac{K_1(\mu r)}{K_1(\mu r_2)} & \text{if } r_2(t) \leq r < \infty. \end{cases} \quad (48)$$

Note that the continuity of $v(r, t)$ at $r = r_1(t)$ and $r = r_2(t)$ follows directly from the fact that $G(x, x) = 0$. The continuity of $h(r, t)$ at $r = r_1(t)$ and $r = r_2(t)$ leads to the relations

$$\begin{aligned} \frac{v_1(t)}{c} \left(\frac{I_0(e^{\tau_1} \mu r_1)}{I_1(e^{\tau_1} \mu r_1)} - \frac{F(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r_2)}{G(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r_2)} \right) \\ + \frac{v_2(t)}{c} \left(\frac{1}{e^{\tau_2} \mu r_1 G(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r_2)} \right) \\ = e^{-\tau_2} - e^{-\tau_1}, \end{aligned} \quad (49)$$

and

$$\begin{aligned} \frac{v_1(t)}{c} \left(\frac{1}{e^{\tau_2} \mu r_2 G(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r_2)} \right) \\ + \frac{v_2(t)}{c} \left(\frac{K_0(\mu r_2)}{K_1(\mu r_2)} - \frac{F(e^{\tau_2} \mu r_2, e^{\tau_2} \mu r_1)}{G(e^{\tau_2} \mu r_1, e^{\tau_2} \mu r_2)} \right) \\ = 1 - e^{-\tau_2}, \end{aligned} \quad (50)$$

where we have made use of the Wronskian (33) to write $F(x, x) = 1/x$.

Because of the absolute angular momentum principle, the radius $r_1(t)$ is related to the azimuthal wind $v_1(t)$ by $r_1 v_1 + (1/2) f r_1^2 = (1/2) f R_1^2$, where the constant R_1 is the potential radius of the interface between the core and the eyewall. Similarly, the radius $r_2(t)$ is related to the azimuthal wind $v_2(t)$ by $r_2 v_2 + (1/2) f r_2^2 = (1/2) f R_2^2$, where the constant R_2 is the potential radius of the interface between the eyewall and the far-field. These absolute angular momentum principles can be written in the forms

$$\frac{v_1(t)}{c} = \frac{(\mu R_1)^2 - (\mu r_1)^2}{2\mu r_1}, \quad (51)$$

$$\frac{v_2(t)}{c} = \frac{(\mu R_2)^2 - (\mu r_2)^2}{2\mu r_2}. \quad (52)$$

Table 2. Solutions of Eqs. (49)–(52) for the choices $c = 250 \text{ m s}^{-1}$, $f = 5 \times 10^{-5} \text{ s}^{-1}$, $r_{10} = 100 \text{ km}$, $r_{20} = 200 \text{ km}$, $S_1 = (16 \text{ h})^{-1}$, $S_2 = (9 \text{ h})^{-1}$, $t_1 = 28 \text{ h}$, and $t_2 = 0$. The initial vortex has a maximum wind of 11.8 m s^{-1} at a radius of 100 km .

t (h)	$r_1(t)$ (km)	$r_2(t)$ (km)	$v_1(t)$ (m s^{-1})	$v_2(t)$ (m s^{-1})	P/f (core)	P/f (eyewall)
0	100.0	200.0	11.8	5.9	5.8	1.0
6	83.0	149.4	15.2	10.8	8.4	1.9
12	68.9	112.6	19.0	16.5	12.2	3.8
18	57.3	85.8	23.6	23.2	17.7	7.4
24	47.6	66.1	28.9	31.3	25.8	14.4
30	39.6	51.5	35.1	41.0	37.5	28.0
36	33.0	40.6	42.5	52.6	54.6	54.6
42	27.6	32.4	51.1	66.4	79.4	106.3
48	23.2	26.2	61.2	82.6	115.6	207.1

Thus, for a given t , the four quantities r_1 , r_2 , v_1 , and v_2 must satisfy (49)–(52). Equations (51) and (52) can be used to eliminate v_1 and v_2 from (49) and (50), resulting in a coupled pair of transcendental equations for μr_1 and μr_2 at the given t . From solutions of these transcendental equations, we have constructed Table 2 and Figs. 5 and 6. Table 2 displays $r_1(t)$, $r_2(t)$, $v_1(t)$, and $v_2(t)$ for equally spaced values of t . In this example, we have chosen $c = 250 \text{ m s}^{-1}$, $f = 5 \times 10^{-5} \text{ s}^{-1}$, $r_{10} = 100 \text{ km}$, $r_{20} = 200 \text{ km}$, $S_1 = (16 \text{ h})^{-1}$, $S_2 = (9 \text{ h})^{-1}$, $t_1 = 28 \text{ h}$, and $t_2 = 0$. These choices result in a non-resting initial state with $P_1 = 5.75f$, $P_2 = f$, $v_{10} = 11.8 \text{ m s}^{-1}$, and $v_{20} = 5.9 \text{ m s}^{-1}$, i.e., the radius of maximum wind is initially on the R_1 surface. As can be seen from the table, $r_2(t)$ moves inward faster than $r_1(t)$ so that $v_2(t)$ grows faster than $v_1(t)$. By $t = 48$ hours, $r_2 - r_1 \approx 3 \text{ km}$ and $v_2 - v_1 \approx 21.4 \text{ m s}^{-1}$, i.e., a structure closely resembling a vortex sheet has been produced. This is further illustrated in Figs. 5 and 6. From the last two columns in Table 2 and the insert in Fig. 6, note that the core PV exceeds the eyewall PV until $t = 36 \text{ h}$, after which the eyewall PV is larger. In other words, the PV field has a monopole structure before 36 h , but a thinning PV ring evolves after 36 h . An interesting feature of Table 2 and Figs. 5 and 6 is that $v_1(t) > v_2(t)$ for $t < 19 \text{ h}$, while $v_1(t) < v_2(t)$ for $t > 19 \text{ h}$. In other words, the radius of maximum tangential wind jumps from the R_1 angular momentum surface to the R_2 angular momentum surface at $t \approx 19 \text{ h}$. In effect, angular momentum surfaces between R_1 and R_2 have moved inward across the radius of maximum wind. Another interpretation comes from consideration of the fluid that is initially between the two angular momentum surfaces R_1 and R_2 , i.e., the fluid initially in the region $r_{10} < r < r_{20}$. This fluid

Fig. 5. Isolines of the tangential wind $v(r, t)$ for the three-region model, with initial conditions given in the top row of Table 2. The two thick solid lines show the inward movement of the absolute angular momentum surfaces $r_1(t)$ and $r_2(t)$.

moves inward as its bounding R_1 and R_2 surfaces move inward, while at the same time some of the fluid is removed by the mass sink. Since $S_2 > S_1$, the PV in the region $r_1(t) < r < r_2(t)$ grows faster than the PV in the other two regions so that after 19 h , $v_2(t) > v_1(t)$, i.e., the radius of maximum wind is now on the R_2 surface. In this way, fluid particles that were outside the radius of maximum wind end up inside the radius

Fig. 6. Radial profiles of tangential wind at 6 h intervals for the three-region model. The kinks in $v(r, t)$ occur at the absolute angular momentum surfaces $r_1(t)$ and $r_2(t)$. Note that the radius of maximum wind shifts from r_1 to r_2 at $t \approx 18$ h. The figure insert shows the core PV and the eyewall PV as functions of time. Note that the vortex changes from a monotonic PV structure (highest PV in the core) to a hollow PV structure (highest PV in the eyewall) at $t = 36$ h.

of maximum wind. In a many-region model, one can picture angular momentum surfaces sweeping inward and passing through the radius of maximum wind. This scenario is consistent with the results of Stern et al. (2015), who argue that in real and model tropical cyclones, contraction of the radius of maximum wind often ceases before peak intensity is reached. The scenario is also consistent with the arguments of Smith and Montgomery (2015) that the movement of the absolute angular momentum surfaces is fundamental to understanding intensification, that there is not a one-to-one relationship between these surfaces and the radius of maximum wind, and that even if the radius of maximum wind ceases to contract, an inward movement of the absolute angular momentum surfaces must accompany vortex intensification.

Finally, using Figs. 1 and 2 of Schubert et al. (1999) as a rough guide, we would expect barotropic instability to commence at the later times shown in Table 2 and Figs. 5 and 6, because, at these times, the PV ring is narrow ($0.85 < r_1/r_2 < 0.89$) and hollow

($0.56 < P_{\text{core}}/P_{\text{eyewall}} < 0.75$). For further discussion on these stability issues in more general dynamical settings, see Montgomery et al. (2002), Nolan and Montgomery (2002), Rozoff et al. (2009), Hendricks et al. (2009, 2014), Hendricks and Schubert (2010), Naylor and Schecter (2014), and Wu et al. (2016).

6. Concluding remarks

A simplified, one-layer, balanced model of tropical cyclone intensification has been presented. The model is inviscid and is forced by a mass sink. Analytic solutions have been obtained for mass sinks that are piecewise constant over two regions and three regions, with the region boundaries being angular momentum surfaces. The angular momentum surfaces collapse as the vortex intensifies. The solutions for the two-region model are sensitive to the initial radius of the bounding angular momentum surface. When this initial radius is large, the vortex experiences a long incubation time before rapid intensification. In the three-region model, the mass sink can be partitioned between the core and the eyewall. When an eye begins to form, the mass sink in the eyewall region surrounding the nascent eye becomes larger than anywhere else. Then the two absolute angular momentum surfaces bounding this annular region, while both moving inward, start to come together. At the same time, the PV in this annular region grows faster than anywhere else. In this way, a hollow PV tower develops, and the tangential wind profile becomes U-shaped. The results presented here constitute an inviscid mechanism by which thin, hollow, unstable PV rings can be produced. It should be noted that a different mechanism exists for the production of PV rings in the frictional boundary layer, as discussed by Smith and Vogl (2008), Kepert (2010), Williams et al. (2013), and Slocum et al. (2014). This frictional boundary layer mechanism involves the production of shock-like structures through the nonlinear dynamics associated with the $u(\partial u/\partial r)$ term in the radial equation of motion (an effect not included in the wave-vortex approximation). Understanding the relative importance of these two mechanisms in real tropical cyclones remains a challenging problem.

Although the axisymmetric, filtered, barotropic model results presented here constitute a drastic simplification of tropical cyclone dynamics, they appear consistent in many respects with recent observational studies of Rogers et al. (2013, 2015) and with the results of full-physics numerical models (Nolan et al. 2013; Miyamoto and Takemi 2015; Smith et al. 2011, 2014, 2015; Montgomery and Smith 2014).

Table 3. Solutions of the transcendental Eq. (39) for the “two-region tornado” case, in which $c = 250 \text{ m s}^{-1}$, $f = 1 \times 10^{-4} \text{ s}^{-1}$, $S_1 = (3 \text{ h})^{-1}$, and $R_1 = r_{10}$ (initial state of rest), with four different choices for r_{10} .

$S_1 t$	P/f (core)	$r_{10} = 1000 \text{ m}$		$r_{10} = 5000 \text{ m}$		$r_{10} = 10000 \text{ m}$		$r_{10} = 15000 \text{ m}$	
		$r_1(t)$ (m)	$v_1(t)$ (m s^{-1})	$r_1(t)$ (m)	$v_1(t)$ (m s^{-1})	$r_1(t)$ (m)	$v_1(t)$ (m s^{-1})	$r_1(t)$ (m)	$v_1(t)$ (m s^{-1})
0	1.0	1000	0.0	5000	0.0	10000	0.0	15000	0.0
1	2.7	607	0.1	3033	0.3	6065	0.5	9098	0.8
2	7.4	368	0.1	1839	0.6	3679	1.2	5519	1.8
3	20.1	223	0.2	1116	1.1	2231	2.1	3347	3.2
4	54.6	135	0.4	677	1.8	1353	3.6	2030	5.4
5	148.4	82	0.6	410	3.0	821	6.0	1232	9.1
6	403.4	50	1.0	249	5.0	498	10.0	748	15.0
7	1096.6	30	1.7	151	8.3	302	16.5	454	24.8
8	2981.0	18	2.7	92	13.6	184	27.2	277	40.7
9	8103.1	11	4.5	56	22.5	112	44.6	170	66.3
10	22026.5	7	7.4	34	36.9	69	72.6	106	105.8

Fig. 7. $v_1(t)$ versus $r_1(t)$ for three tornado-like vortices that evolve from resting initial states with $r_{10} = 5, 10, 15 \text{ km}$. Along the six thin black lines the dimensionless time takes on the respective values $S_1 t = 5, 6, 7, 8, 9, 10$, while the fractional volume removed takes on the respective values $\mathcal{F} = 0.99326, 0.99752, 0.99909, 0.99966, 0.99988, 0.99995$.

In Sections 4 and 5, our application of the balanced theory has been to tropical cyclone scale vortices. It is difficult to resist asking if the theory could also be applied to tornado-scale vortices, although the hydrostatic assumption incorporated into wave-vortex theory is obviously less valid for tornado dynamics, which can also involve strong surface frictional inflow and additional nonlinearities associated with vortex breakdown. In any event, we have explored this possibility by computing solutions of the two-region

model for the specified parameters $c = 250 \text{ m s}^{-1}$, $f = 1 \times 10^{-4} \text{ s}^{-1}$, and $t_1 = 0$ and for the four choices $r_{10} = 1000, 5000, 10000, 15000 \text{ m}$. The results are given in Table 3 and in Figs. 7 and 8. The six thin black lines in Fig. 7 connect points with the same values of the dimensionless time $S_1 t$, i.e., $S_1 t = 5, 6, 7, 8, 9, 10$. The $r_{10} = 1000 \text{ m}$ case produces a small, weak vortex that resembles a dust devil or a weak waterspout. The other three cases, which are plotted in Fig. 7, produce larger and stronger vortices. In partic-

Fig. 8. Tangential velocity $v(r, t)$ versus r for a tornado-like vortex that evolves from a resting initial state with $r_{10} = 10$ km and $S_1 = (6 \text{ min})^{-1}$. The eight radial profiles of $v(r, t)$ correspond to $t = 18, 24, 30, 36, 42, 48, 54, 60$ min, with kinks occurring at the absolute angular momentum surface $r = r_1(t)$. The figure insert shows $P(t)/f$ in the core region $0 \leq r < r_1(t)$.

ular, the $r_{10} = 10$ km case, shown in Fig. 8, produces a maximum tangential wind of 72.6 m s^{-1} at a radius of 69 m. This vortex spins up rapidly between $t = 8/S_1$ and $t = 10/S_1$. It is interesting to note that although this strong vortex takes $t = 10/S_1$ to spin-up from rest, it takes only half that long if the vortex evolves from a mesocyclone with $r_1 = 821$ m and $v_1 = 6.0 \text{ m s}^{-1}$ (i.e., from the $t = 5/S_1$ line in Table 3). This indicates that the timing of tornado appearance is very sensitive to the preconditioning of the mesoscale environment.

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Appendix

Consider axisymmetric, shallow water dynamics expressed in the Hamiltonian form

$$\frac{dF}{dt} = \{F, H\}, \quad (53)$$

where $F[u, v, h]$ is an arbitrary functional of $u(r, t)$, $v(r, t)$, and $h(r, t)$, and where the Poisson bracket of any two functionals $A[u, v, h]$ and $B[u, v, h]$ is defined by

$$\{A, B\} = \int_0^\infty \left[\left(\frac{f + \zeta}{h} \right) \left(\frac{\delta A}{\delta u} \frac{\delta B}{\delta v} - \frac{\delta B}{\delta u} \frac{\delta A}{\delta v} \right) - \frac{\delta A}{\delta u} \frac{\partial}{\partial r} \left(\frac{\delta B}{\delta h} \right) + \frac{\delta B}{\delta u} \frac{\partial}{\partial r} \left(\frac{\delta A}{\delta h} \right) \right] r dr. \quad (54)$$

The Hamiltonian functional $H[u, v, h]$ for the axisymmetric, shallow water primitive equations is

$$H = \int_0^\infty \left[\frac{1}{2} (u^2 + v^2) h + \frac{1}{2} g (h - \bar{h})^2 \right] r dr, \quad (55)$$

and the functional derivatives of this Hamiltonian are

$$\begin{aligned} \frac{\delta H}{\delta u} &= hu, \quad \frac{\delta H}{\delta v} = hv, \\ \frac{\delta H}{\delta h} &= g(h - \bar{h}) + \frac{1}{2} (u^2 + v^2). \end{aligned} \quad (56)$$

Using (53), (54), and (56), we can express the axisymmetric, shallow water dynamics as

$$\begin{aligned} \frac{dF}{dt} &= \int_0^\infty \left[\frac{\delta F}{\delta u} \left((f + \zeta)v - \frac{\partial \left[gh + \frac{1}{2} (u^2 + v^2) \right]}{\partial r} \right) \right. \\ &\quad \left. - \frac{\delta F}{\delta v} (f + \zeta)u - \frac{\delta F}{\delta h} \frac{\partial (ruh)}{r \partial r} \right] r dr, \end{aligned} \quad (57)$$

where we have performed an integration by parts to obtain the last term in the integrand. To obtain the radial momentum equation, choose the arbitrary functional F to be

$$F[u] = \int_0^\infty \delta(r - \hat{r}) u(r, t) r dr = u(\hat{r}, t), \quad (58)$$

where the Dirac delta function $\delta(r - \hat{r})$ vanishes for $r \neq \hat{r}$ and satisfies $\int \delta(r - \hat{r}) r dr = 1$. Then,

$$\frac{\delta F}{\delta u} = \delta(r - \hat{r}), \quad \frac{\delta F}{\delta v} = 0, \quad \frac{\delta F}{\delta h} = 0,$$

so that (57) yields

$$\frac{du(\hat{r}, t)}{dt} = \left[(f + \zeta)v - \frac{\partial \left[gh + \frac{1}{2} (u^2 + v^2) \right]}{\partial r} \right]_{r=\hat{r}}, \quad (59)$$

which can also be expressed in the form (1).

To obtain the tangential momentum equation, choose the arbitrary functional F to be

$$F[v] = \int_0^\infty \delta(r - \hat{r}) v(r, t) r \, dr = v(\hat{r}, t), \quad (60)$$

so that

$$\frac{\delta F}{\delta u} = 0, \quad \frac{\delta F}{\delta v} = \delta(r - \hat{r}), \quad \frac{\delta F}{\delta h} = 0.$$

Then (53) and (57) yield

$$\frac{dv(\hat{r}, t)}{dt} = -((f + \zeta)u)_{r=\hat{r}}, \quad (61)$$

which can also be expressed in the form (2). Similarly, to obtain the continuity equation, choose the arbitrary functional F to be

$$F[h] = \int_0^\infty \delta(r - \hat{r}) h(r, t) r \, dr = h(\hat{r}, t), \quad (62)$$

so that $(\delta F/\delta u) = 0$, $(\delta F/\delta v) = 0$, and $(\delta F/\delta h) = \delta(r - \hat{r})$. Then (53) and (57) yield

$$\frac{dh(\hat{r}, t)}{dt} = -\left(\frac{\partial(ruh)}{r\partial r}\right)_{r=\hat{r}}, \quad (63)$$

which can also be expressed in the form (3). Finally, to obtain the total energy conservation relation, note that the choice $F[u, v, h] = H[u, v, h]$ leads, via (54), to $\{F, H\} = \{H, H\} = 0$, so that (53) becomes

$$\frac{dH}{dt} = 0. \quad (64)$$

For the wave-vortex approximation, the dynamics are expressed in the Hamiltonian form

$$\frac{dF}{dt} = \{F, H_{wv}\}, \quad (65)$$

where the definition of the Poisson bracket is unchanged but the Hamiltonian functional (55) is approximated by

$$H_{wv} = \int_0^\infty \left[\frac{1}{2} (u^2 + v^2) \bar{h} + \frac{1}{2} g(h - \bar{h})^2 \right] r \, dr. \quad (66)$$

The functional derivatives of the Hamiltonian for the wave-vortex model are

$$\frac{\delta H_{wv}}{\delta u} = \bar{h}u, \quad \frac{\delta H_{wv}}{\delta v} = \bar{h}v, \quad \frac{\delta H_{wv}}{\delta h} = g(h - \bar{h}). \quad (67)$$

A comparison of (67) with (56) indicates that the influence of the replacement of h by $\bar{h}$ in the kinetic

energy part of (66) has spread into all three functional derivatives. Using (65), (54), and (67), we can express the axisymmetric, wave-vortex dynamics as

$$\begin{aligned} \frac{dF}{dt} = \int_0^\infty & \left[\frac{\delta F}{\delta u} \left(Pv - g \frac{\partial h}{\partial r} \right) - \frac{\delta F}{\delta v} Pu \right. \\ & \left. - \frac{\delta F}{\delta h} \frac{\partial(ruh)}{r\partial r} \right] r \, dr. \end{aligned} \quad (68)$$

To obtain the wave-vortex approximations of the radial momentum equation, the tangential momentum equation, and the continuity equation, we sequentially choose the arbitrary functional F to be that given in (58), (60), and (62). Then, following the previous arguments, we find that (68) sequentially yields

$$\frac{du(\hat{r}, t)}{dt} = \left(Pv - g \frac{\partial h}{\partial r} \right)_{r=\hat{r}}, \quad (69)$$

$$\frac{dv(\hat{r}, t)}{dt} = -(Pu)_{r=\hat{r}}, \quad (70)$$

$$\frac{dh(\hat{r}, t)}{dt} = -\left(\frac{\partial(ruh)}{r\partial r} \right)_{r=\hat{r}}. \quad (71)$$

These last three equations can also be expressed in the forms (10)–(12). Finally, to obtain the total energy conservation relation for wave-vortex theory, note that the choice $F[u, v, h] = H_{wv}[u, v, h]$ leads, via (54), to $\{F, H_{wv}\} = \{H_{wv}, H_{wv}\} = 0$, so that (65) becomes

$$\frac{dH_{wv}}{dt} = 0. \quad (72)$$

When comparing the wave-vortex approximation (10)–(12) with the original primitive equations (1)–(3), it is interesting to note how several subtle approximations in (10)–(12) have resulted from the simple replacement of h by $\bar{h}$ in the kinetic energy part of the Hamiltonian.

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