Analytical Solutions of Vortex Rossby Waves Associated with Vortex Resiliency of Tropical Cyclones

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(Manuscript received 24 March 2017, in final form 28 October 2017)

Abstract

We analytically solve a forced linear problem of vortex Rossby waves (VRWs) associated with the vortex resiliency of tropical cyclones. We consider VRWs on a basic barotropic axisymmetric vortex. VRWs, which are initially absent, are successively forced by a vertically sheared unidirectional environmental flow. The problem is formulated in the quasigeostrophic equations, linearized about the basic vortex. The basic potential vorticity (PV) is assumed to be piecewise constant in the radial direction so that the problem can be analytically solved. The obtained solutions show the following.

When the vertical interaction (VI) between the VRWs is weak, a stationary mode (called the pseudo mode) is selectively forced and grows linearly in time, and the vortex is eventually destroyed by the environmental vertical shear. When the VI is moderate, an almost form-preserving quasi-mode (simply called the quasi mode) of the VRWs appears and precesses about a downshear-left tilt equilibrium (DSLTE). The precession does not grow and the vortex maintains vertical coherence. In particular, in the presence of the inward radial gradient of the basic PV at the critical radius, the precession damps and the quasi mode eventually approaches the DSLTE. When the VI is strong, the VRWs are simply advected by the basic angular velocity at each radius to be axisymmetrized to some extent about the DSLTE, and the vortex maintains vertical coherence.

To examine the diabatic effect near the eyewall, the solution with the basic buoyancy frequency being small in the central region and large in the outer region is also obtained. The small and large buoyancy frequencies imply strong and weak VIs, respectively. The central VRWs are simply advected by the basic vortex flow. While, the outer VRWs precess about the DSLTE just like a quasi mode, and the vortex maintains vertical coherence.

Keywords tropical cyclone; vortex Rossby wave; vortex resiliency; diabatic effect; potential vorticity

1. Introduction

In midlatitudes, tropical cyclones (TCs) are exposed to strong environmental vertical wind shear. If TCs are simply differentially advected by the environmental flow at each level, they are tilted, vertically torn apart, and sooner or later destroyed. Contrary to the naive conjecture, TCs maintain vertical coherence in the relatively strong vertical wind shear. The ability of

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J-stage Advance Published Date: 16 November 2017

TCs to resist the vertical wind shear and recover the vertical alignment is called the vortex resilience. The vortex resilience is supposed to be caused by diabatic heating in and near the eyewall and the accompanying vertical circulation. However, several recent investigations show that adiabatic conservative processes are also important for vortex resilience.

Jones (1995) examined the temporal evolution of a TC-like vortex without diabatic processes successively exposed to a vertically sheared environmental flow. The vortex is tilted by the shear. The tilt means displacement of the upper and lower potential vorticity (PV) centers. The cyclonic horizontal circulation around the upper PV center cyclonically advects the lower PV, and that around the lower PV center cyclonically advects the upper PV. As a result, the tilted vortex cyclonically precesses. The precession is not upright but about a downshear-left tilt equilibrium. This is because the downshear advection by the environmental flow and the upshear advection by the displaced upper and lower PVs are balanced in the downshear-left tilt equilibrium state. By the precession, the vortex resists successive forcing by the vertical wind shear and maintains vertical coherence.

Reasor and Montgomery (2001) reconsidered the precession mechanism of Jones (1995) in terms of vortex Rossby waves (VRWs) on a basic axisymmetric vortex. The basic vortex is initially exposed to a vertically sheared environmental flow and tilted by the shear. The tilted vortex is regarded as a superposition of the basic vortex and VRWs with the first baroclinic vertical structure and the wave number one azimuthal structure. Because of the inward radial gradient of the basic PV, the VRWs propagate (here and hereafter, the propagation is relative to the fluid) retrograde. However, because of the dominant cyclonic advection by the basic vortex flow, the VRWs move cyclonically. Together with the first baroclinic vertical structure and wave number one azimuthal structure, the cyclonic movement of the VRWs means the cyclonic precession of the vortex. The precession is upright instead of about the downshear-left tilt equilibrium. This is because the vortex is exposed to the vertically sheared environmental flow only initially instead of persistently.

Reasor and Montgomery (2001) further examined the dependence of the behavior of VRWs on the internal Rossby deformation radius l_R . When l_R is larger than the horizontal scale l of the tilted vortex, a quasi mode of the VRWs appears. The quasi mode behaves just like a true form-preserving mode of the VRWs with a constant angular phase velocity and represents

the cyclonic precession. On the other hand, when l_R is smaller than l , the quasi mode disappears, and the VRWs are essentially advected by the basic vortex flow at each radius and eventually spirally wound up by the differential rotation. As a result, the vortex recovers the upright vertical alignment.

Schecter et al. (2002) presented a damping mechanism of the quasi mode by the critical radius damping. The critical radius is the radius where the angular velocity of the basic vortex flow is equal to the angular phase velocity of the quasi mode and is usually located at the outer region for dry dynamics. Because of the same angular velocity, the quasi mode resonantly interacts with the VRW at the critical radius. By the resonant interaction, the quasi mode damps accompanied with the growth of the VRW at the critical radius. As the quasi mode damps, the vortex recovers the upright vertical alignment.

Reasor et al. (2004) considered a quasi mode successively forced by a vertically sheared environmental flow, instead of the quasi mode with the upright precession of Reasor and Montgomery (2001) and Schecter et al. (2002) which is only initially forced by the vertically sheared environmental flow. Reasor et al. (2004) presented a damping mechanism of the quasi mode, which precesses about the downshear-left tilt equilibrium, by the critical radius damping. The damping quasi mode approaches the downshear-left tilt equilibrium. The damping of the quasi mode implies that the vortex resists the successive environmental forcing and maintains vertical coherence.

Wong and Chan (2004) investigated the behavior of a TC-like vortex with diabatic processes and demonstrated the following. Although the adiabatic mechanism (i.e., the precession) also works in the moist case, the diabatic heating and the accompanying vertical circulation also contribute to the vortex resiliency.

Schecter and Montgomery (2007) derived a system of equations governing disturbances on a vortex with condensational heating and evaporative cooling, but without precipitation. In the system, the diabatic effect is represented as the reduction of the buoyancy frequency. They used this system to examine the moist VRW dynamics and showed that the growth rate of phase-locked counter-propagating VRWs in the eyewall is diminished by clouds.

On the basis of the system of Schecter and Montgomery (2007), Reasor and Montgomery (2015) examined the behavior of moist VRWs successively forced by a vertically sheared environmental flow. In the central region, where the moist buoyancy frequen-

cy vanishes, the moist VRWs are simply advected by the basic vortex flow and spirally wound up by the differential rotation. However, the VRWs in the outer region behave just like the quasi mode, which precesses about the downshear-left tilt equilibrium.

Schecter (2015) compared experiments with and without secondary vertical circulation, which accompanies diabatic heating, and showed the following. As a whole, the diabatic effects are well accounted for by the reduction of the buoyancy frequency. However, the secondary vertical circulation also has a discernible influence on the vortex resiliency at least in the eyewall region (convective momentum transport, pathway for dry air to enter the vortex core, etc.).

The above-mentioned theories were obtained mainly by numerical experiments primarily because of large Rossby number vortices. In this paper, we analytically formulate the problem in the quasigeostrophic (QG) system; we obtain analytical solutions of the VRWs associated with the vortex resiliency and confirm the theories. Of course, the QG equations cannot be applied to large Rossby number vortices like TCs. However, on the basis of the similarity between the QG and the asymmetric balance (AB) equations (Shapiro and Montgomery 1993) that can be applied to these vortices, we believe that the QG solutions are not so far from reality. Specifically, we consider VRWs on a barotropic axisymmetric basic vortex, which are successively forced by a vertically sheared environmental flow. To obtain closed form solutions, we further assume that the basic vorticity is radially piecewise constant. We analytically solve the forced linear problem, in the absence of initial VRWs.

This paper is organized as follows. In Section 2, the governing equation is derived and the analytical solution is presented. In Section 3, the dependence of the solution on the strength of the vertical interaction of VRWs is considered. In Section 4, the diabatic effect is considered. In Section 5, the concluding remarks are given.

2. Governing equation and solution

In this section, we derive the governing equation, and present the closed form analytical solution.

2.1 Quasigeostrophic and asymmetric balance (AB) equations

We consider disturbances on a barotropic axisymmetric basic vortex. It is well known that disturbances on a TC-like vortex are well described by the AB equations of Shapiro and Montgomery (1993). These are expressed in the cylindrical coordinate system

(r, θ, z) whose origin is set at the vortex center and whose azimuthal angle is measured from the x (eastward)-axis as follows.

$$\left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \theta}\right) q'_{AB} - \frac{\bar{\xi}}{\bar{q}} \frac{1}{r} \frac{d\bar{q}}{dr} \frac{\partial \psi'}{\partial \theta} = F, \quad (1)$$

$$q'_{AB} = \left(\bar{q} \bar{\xi} \frac{\partial}{\partial r} \frac{r}{\bar{q}} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\bar{q} \bar{\xi}}{N^2} \frac{\partial^2}{\partial z^2} \right) \psi'. \quad (1)$$

Here, z is the pseudoheight of Hoskins and Bretherton (1972) and t is the time. The overbarred quantities are of the basic vortex and also represent the azimuthal average of the basic vortex flow at radius r , i.e., $\bar{\omega}$ is the basic angular velocity, $\bar{q}$ is the basic PV (i.e., the basic absolute vorticity), $\bar{\xi} = f + 2\bar{\omega}$ is the inertia parameter, and f is the Coriolis parameter, which is assumed to be constant. Equation (1) are obtained from (3.10) in Shapiro and Montgomery (1993) by neglecting the terms including $\partial \bar{v} / \partial z$. Further, we assume that f is positive because we consider vortices in the northern hemisphere. The primed variables are of the disturbance, i.e., q'_{AB} is the disturbance PV of the AB system, $\psi' = \phi' / f$, ϕ' is the disturbance geopotential. N is the basic buoyancy frequency, and F represents the external forcing. Because of the r dependence of the coefficients of (1), we cannot obtain the closed form solution of (1) analytically, although some analytical investigations exist (e.g., Schecter and Montgomery 2003). To obtain the analytical closed form solution, we consider the quasigeostrophic equation (5). Of course, the QG equations cannot be applied to TC-like vortices with large Rossby number $\bar{v} / (rf)$. However, the first equation of (1) is similar to the first equation of (5), although the generation of disturbance vorticity by the radial advection of $\bar{q}$ is underestimated because $\bar{\xi} / \bar{q} > 1$ for monotonically decreasing $\bar{q}$. We replace the last term (representing the vertical interaction) of the second equation of (5) by the last term of (1). Then, the Green function (representing the interaction between perturbations) for the second equation of (1) is expected to be similar to that for the second equation of (5), although the horizontal interaction between perturbations seems to be overestimated in the central region (see Appendix B). These two effects (underestimation and overestimation in the QG analog system) seem to mitigate each other to some extent. On the basis of the similarity between (1) and (5), we believe that the solution to (5) is not so far from reality.

2.2 Basic assumptions and equations

We begin with the adiabatic and frictionless quasigeostrophic PV equations.

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \theta}\right) q = 0, \\ q = \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{f^2}{N^2} \frac{\partial^2}{\partial z^2}\right) \psi + f, \quad (2)$$

where $u = -(1/r)\partial\psi/\partial\theta$ and $v = \partial\psi/\partial r$ are the radial and azimuthal components of geostrophic velocity, respectively, ψ is the stream function, and q is the quasigeostrophic PV. We consider a barotropic axisymmetric basic vortex whose PV $\bar{q}$ is monotonically decreasing with radius in order to remove the barotropic instability, for example, eyewall region instability (e.g., Schubert et al. 1999) and outer region instability (e.g., Itano and Ishikawa 2002).

$$\bar{u} = 0, \quad \bar{v} = \bar{v}(r) > 0, \\ \bar{q} = \frac{1}{r} \frac{d}{dr}(r\bar{v}) + f, \quad \frac{d\bar{q}}{dr} < 0. \quad (3)$$

The disturbances, which are initially absent, are assumed to be successively forced by a horizontally uniform and vertically sheared environmental zonal flow.

$$U_e = U_e(z) = -U_0 \cos \frac{\pi z}{H}. \quad (4)$$

Here, U_0 is a positive constant. To obtain a closed form analytical solution, the fluid is assumed to be confined between two horizontal rigid boundaries on $z = 0$ (ground) and $z = H$ (tropopause), although the rigid boundary assumption may be allowed only for infinitesimal perturbations. The environmental flow, which is regarded as a perturbation on the basic vortex, is westward on $z = 0$ and eastward on $z = H$. Linearized about the basic vortex in (3), the PV equation in (2) becomes

$$\left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \theta}\right)(q' + q_e) - \frac{1}{r} \frac{d\bar{q}}{dr} \frac{\partial}{\partial \theta}(\psi' + \psi_e) = 0,$$

where $\bar{\omega} = \bar{v}/r$ is the basic angular velocity, and q_e and ψ_e are the environmental PV and stream function, respectively. Substituting (4) into the above equation gives

$$\left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \theta}\right)q' - \frac{1}{r} \frac{d\bar{q}}{dr} \frac{\partial \psi'}{\partial \theta} = U_0 \frac{d\bar{q}}{dr} \cos \theta \cos \frac{\pi z}{H}, \\ q' = \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{f^2}{N^2} \frac{\partial^2}{\partial z^2}\right) \psi'. \quad (5)$$

In deriving the first equation of (5), we neglect a term $-\bar{\omega} \partial q_e / \partial \theta$ associated with the environmental PV q_e of the environmental flow (4) relative to $U_0(d\bar{q}/dr)\cos\theta\cos(\pi z/H)$. The reason for the neglect is as follows. Because

$$-\bar{\omega} \frac{\partial q_e}{\partial \theta} = -\bar{\omega} \frac{f^2}{N^2} \frac{\partial}{\partial \theta} \frac{\partial^2 \psi_e}{\partial z^2} \\ = \bar{\omega} \frac{f^2}{N^2} \frac{\partial}{\partial \theta} \frac{\partial^2}{\partial z^2} \{U_e(z)r \sin \theta\} \\ = -\bar{v} \left(\frac{f\pi}{NH}\right)^2 U_0 \cos \theta \cos \frac{\pi z}{H},$$

where the first equality comes from the fact that the environmental flow is a horizontally uniform zonal flow, the ratio becomes

$$\left| -\bar{\omega} \frac{\partial q_e}{\partial \theta} \right| / \left| U_0 \frac{d\bar{q}}{dr} \cos \theta \cos \frac{\pi z}{H} \right| = \left(\frac{f\pi}{NH}\right)^2 \bar{v} / \left| \frac{d\bar{q}}{dr} \right|.$$

This is less than one for a TC-like vortex whose horizontal vortex scale $L \approx 10^5$ m (e.g., twice the scale of the radius of maximum wind), i.e., $(f\pi/NH)^2 \bar{v} / |d\bar{q}/dr| \approx 10^{-11}/L^{-2} < 1$. Here, we assume that $f \approx 10^{-4} \text{ s}^{-1}$, $N \approx 10^{-2} \text{ s}^{-1}$, and $H \approx 10^4$ m. The neglected term $-\bar{\omega} \partial q_e / \partial \theta$ represents vorticity generation by azimuthal advection of the environmental q_e . The azimuthal advection is caused by the vortex flow. Hence, this term generates so-called the β gyres. The influence of the β gyres is neglected in some papers (e.g., Reasor et al. 2004; Reasor and Montgomery 2015) by similar scale analyses. To analytically solve (5), the basic PV is assumed to be piecewise constant in the radial direction (see Fig. 1).

$$\bar{q} = \sum_{j=1}^N \bar{q}_j h(r_j - r) + f \quad \text{with} \quad \bar{q}_j > 0 \\ \text{and} \quad 0 < r_1 < r_2 < \dots < r_N < \infty, \quad (6)$$

where $h(x)$ is the step function, i.e., $h(x) = 0$ for $x < 0$ and $h(x) = 1$ for $x \geq 0$.

From (6), the radial derivative of $\bar{q}$ is given in terms of Dirac's delta function.

$$\frac{d\bar{q}}{dr} = -\sum_{j=1}^N \bar{q}_j \delta(r - r_j). \quad (7)$$

From (6) and the relation between $\bar{q}$ and $\bar{\omega}$, i.e., $\bar{q} = (1/r)(d/dr)(r^2\bar{\omega}) + f$, the basic angular velocity $\bar{\omega}$ is determined (see Fig. 2).

Fig. 1. The PVs $\bar{q} = \bar{q}(r)$ in units of 10^{-3} s^{-1} of the assumed basic vortices, BV1 and BV2. BV1 is assumed in Figs. 3, 4, 5, and 7. BV2 is assumed in Fig. 6.

$$\bar{\omega} = \frac{1}{2} \sum_{j=J+1}^N \bar{q}_j + \frac{1}{2} \sum_{j=1}^J \bar{q}_j \left(\frac{r_j}{r} \right)^2 \quad \text{for } r_j \leq r < r_{j+1}, \quad (8)$$

where $J = 0, 1, 2, \dots, N$, and $r_0 = 0$ and $r_{N+1} = \infty$.

Because of the horizontal rigid boundaries on $z = 0$ and $z = H$, the vertical velocity vanishes there. This implies that the disturbance potential temperature vanishes there. As a result, the disturbance stream function ψ' must satisfy

$$\frac{\partial \psi'}{\partial z} = 0 \quad \text{on } z = 0 \quad \text{and} \quad z = H. \quad (9)$$

This is consistent with the assumption of the barotropic basic vortex in (3), and with the assumption of the sheared environmental flow $U_e(z)$ in (4), whose vertical derivative vanishes on $z = 0$ and $z = H$. As for the radial boundary condition, the disturbance must be finite at the origin and is naturally assumed to vanish at infinity.

$$\psi' < \infty \quad \text{at } r = 0 \quad \text{and} \quad \psi' \rightarrow 0 \quad \text{as } r \rightarrow \infty. \quad (10)$$

2.3 Governing equation

Because the forcing term, i.e., the right hand side

Fig. 2. The angular velocities $\bar{\omega} = \bar{\omega}(r)$ in units of 10^{-3} s^{-1} of BV1 and BV2.

(RHS) of the first equation of (5), is proportional to $\cos \theta \cos(\pi z/H)$, the disturbance ψ' and q' have vertical structure proportional to $\cos(\pi z/H)$ and azimuthal structure with wave number one under the null initial condition, and can be written as follows.

$$\begin{aligned} \psi'(t, r, \theta, z) &= \sum_{\pm} \hat{\psi}^{\pm}(t, r) \frac{e^{\pm i\theta}}{\sqrt{2\pi}} \cos \frac{\pi z}{H}, \\ q'(t, r, \theta, z) &= \sum_{\pm} \hat{q}^{\pm}(t, r) \frac{e^{\pm i\theta}}{\sqrt{2\pi}} \cos \frac{\pi z}{H}. \end{aligned} \quad (11)$$

By the substitution of (11), (5) become

$$\begin{aligned} \left(\frac{\partial}{\partial t} \pm i\bar{\omega} \right) \hat{q}^{\pm} \mp \frac{i}{r} \frac{d\bar{q}}{dr} \hat{\psi}^{\pm} &= U_0 \sqrt{\frac{\pi}{2}} \frac{d\bar{q}}{dr}, \\ \hat{q}^{\pm} &= \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{1}{l_R^2} \right) \hat{\psi}^{\pm}, \end{aligned} \quad (12)$$

where $l_R = NH/f\pi$ is the internal Rossby deformation radius for a QG system. However, in Section 3, we assume that l_R is not a fixed constant (QG value), but takes the value of $NH/\sqrt{\bar{q}\xi}\pi$ for an AB system and varies so as to cover the wide range of the vertical interaction between VRWs. Under the radial boundary conditions (10), the second equation of (12) is inverted, and the disturbance stream function $\hat{\psi}^{\pm}$ is expressed in terms of the Green function $G(r, r')$ and

the disturbance PV $\hat{q}^\pm$ as follows.

$$\hat{\psi}^\pm(t, r) = -\int_0^\infty dr' r' G(r, r') \hat{q}^\pm(t, r'), \quad (13)$$

$$G(r, r') = I\left(\frac{r'}{l_R}\right) K\left(\frac{r}{l_R}\right) \quad \text{for } r' < r, \\ G(r, r') = I\left(\frac{r}{l_R}\right) K\left(\frac{r'}{l_R}\right) \quad \text{for } r < r', \quad (14)$$

where $I(x)$ is the Bessel function of the first kind of order 1, and $K(x)$ of the second kind of order 1 (e.g., Oliver 1964). The derivation of the Green function G is briefly described in Appendix A. Because of the presence of $d\bar{q}/dr$ in the first equation of (12), which is expressed in terms of Dirac's delta function in (7), and because of the null initial condition, the disturbance PV is also written in terms of Dirac's delta function.

$$\hat{q}^\pm(t, r) = \sum_{j=1}^N \hat{q}_j^\pm(t) \delta(r - r_j). \quad (15)$$

Substituting (7), (13), and (15) into the first equation of (12), gives the following governing equation.

$$\frac{d}{dt}(r_j \hat{q}_j^\pm) \pm i \bar{\omega}_j r_j \hat{q}_j^\pm \mp i \bar{q}_j \sum_{k=1}^N G_{jk} r_k \hat{q}_k^\pm \\ = -U_0 \sqrt{\frac{\pi}{2}} r_j \bar{q}_j, \quad j = 1, 2, \dots, N, \quad (16)$$

where $\bar{\omega}_j = \bar{\omega}(r_j)$ and $G_{jk} = G(r_j, r_k)$. Equation (16) is rewritten in a vector form.

$$\frac{d}{dt} |r \hat{q}^\pm(t)\rangle \pm i \Lambda |r \hat{q}^\pm(t)\rangle = -U_0 \sqrt{\frac{\pi}{2}} |r \bar{q}\rangle, \quad (17)$$

where $|r \hat{q}^\pm\rangle$ and $|r \bar{q}\rangle$ are column vectors whose j th components are $r_j \hat{q}_j^\pm$ and $r_j \bar{q}_j$, respectively, and Λ is an $N \times N$ matrix whose (j, k) th component is given by

$$\Lambda_{jk} = \bar{\omega}_j \delta_{jk} - \bar{q}_j G_{jk}. \quad (18)$$

2.4 Solution

Under the null initial condition, the solution to (17) is given by the following formula.

$$|r \hat{q}^\pm(t)\rangle = -U_0 \sqrt{\frac{\pi}{2}} \sum_{n=1}^N e^{\mp i \lambda_n t} \int_0^t dt' e^{\pm i \lambda_n t'} \frac{|r_n\rangle \langle l_n|}{\langle l_n | r_n \rangle} |r \bar{q}\rangle, \quad (19)$$

where λ_n ($n = 1, 2, \dots, N$) are the eigenvalues of the

matrix Λ in (17), and $|r_n\rangle$ and $\langle l_n|$ are the corresponding right and left eigenvectors, respectively (see Appendix C). $\langle l_n | r_n \rangle = \sum_{j=1}^N r_{nj} l_{nj}$ is the scalar product of $\langle l_n|$ and $|r_n\rangle$, and $|r_n\rangle \langle l_n|$ is the dyadic product of $|r_n\rangle$ and $\langle l_n|$, which is an $N \times N$ matrix whose (j, k) th component is $r_{nj} l_{nk}$. From the definition of the eigenvectors and eigenvalues

$$\Lambda |r_n\rangle = \lambda_n |r_n\rangle, \quad n = 1, 2, \dots, N,$$

and the spectral decomposition of the identity matrix

$$I = \sum_{n=1}^N \frac{|r_n\rangle \langle l_n|}{\langle l_n | r_n \rangle},$$

it is easily seen that (19) is indeed the solution to (17). Because the basic PV monotonically decreases with radius, there are no growing disturbances due to the barotropic instability (e.g., Gent and McWilliams 1986). As a result, the eigenvalues and the eigenvectors are all real. Since $|r \bar{q}\rangle$ is independent of time t , the integration in (19) with respect to t is easily performed to give

$$|r \hat{q}^\pm(t)\rangle = \pm i U_0 \sqrt{\frac{\pi}{2}} \sum_{n=1}^N \frac{1 - e^{\mp i \lambda_n t}}{\lambda_n} \frac{|r_n\rangle \langle l_n|}{\langle l_n | r_n \rangle} |r \bar{q}\rangle. \quad (20)$$

By substituting (20) into (15), and further into the second equation of (11), the solution in physical space is obtained.

$$|r q(t, \theta)\rangle = -U_0 \sum_{n=1}^N \frac{\sin \theta - \sin(\theta - \lambda_n t)}{\lambda_n} \frac{|r_n\rangle \langle l_n|}{\langle l_n | r_n \rangle} |r \bar{q}\rangle, \quad (21)$$

where $|r q(t, \theta)\rangle$ is a column vector whose j th component is $r_j q_j(t, \theta)$, which is so defined that the disturbance PV $q'(t, r, \theta, z)$ is given by the following expression,

$$q'(t, r, \theta, z) = \sum_{j=1}^N q_j(t, \theta) \cos\left(\frac{\pi z}{H}\right) \delta(r - r_j). \quad (22)$$

3. Dependence on the vertical interaction

In this section, we consider how the solution depends on the strength of the vertical interaction (VI) between VRWs.

3.1 Vertical interaction and l_R

The last term $-(1/l_R^2) \hat{\psi}^\pm = -(f\pi/NH)^2 \hat{\psi}^\pm$ on the RHS of the second equation of (12), which is derived

from the last term $(f^2/N^2)\partial^2\psi'/\partial z^2$ on the RHS of the second equation of (5), represents the VI between the VRWs. This is because the problem is reduced to a horizontally two dimensional one in the absence of that term. Specifically, the VI becomes stronger (weaker) for a smaller (larger) l_R . Here and hereafter, based on the similarity between the AB equations (1) and the quasigeostrophic equations (5), we assume that $l_R = NH/f\pi$ is not a fixed constant and that l_R takes the value of $NH/\sqrt{q\xi}\pi$ which varies so as to cover the wide range of the VI.

VI between VRWs $\rightarrow \infty$ as $l_R \rightarrow 0$,

VI between VRWs $\rightarrow 0$ as $l_R \rightarrow \infty$.

In the present case of the first baroclinic vertical structure in (11), the horizontal circulation induced by the upper disturbance PV $q'(z = H)$ is opposite-signed to that by the lower one $q'(z = 0)$. The two tend to cancel each other. As l_R decreases, i.e., as the VI becomes stronger, the two cancel each other more strongly, and the horizontal circulation induced by the disturbance PV q' is more and more suppressed. As a result, the retrograde propagation angular velocity of the VRWs is reduced by the decrease in l_R because the propagation is caused by the advection of the basic PV $\bar{q}$ by the horizontal circulation. In the limiting case of $l_R \rightarrow 0$, the retrograde propagation vanishes and the VRWs are simply cyclonically advected by the basic vortex angular velocity $\bar{\omega}$. On the other hand, in the opposite limiting case of $l_R \rightarrow \infty$, the retrograde propagation angular velocity becomes maximum.

Retrograde propagation of VRWs $\rightarrow 0$
as $l_R \rightarrow 0$.

Retrograde propagation of VRWs $\rightarrow$ maximum
as $l_R \rightarrow \infty$.

The simple cyclonic advection of VRWs by the basic vortex angular velocity $\bar{\omega}$ implies the absence of the horizontal interaction of the VRWs. This is confirmed by the asymptotic form of the Green function introduced in (14).

$$G(r, r') \rightarrow \frac{1}{2} \frac{l_R}{\sqrt{rr'}} \exp\left(-\frac{|r - r'|}{l_R}\right) \rightarrow 0 \quad \text{as } l_R \rightarrow 0,$$

$$G(r, r') \rightarrow \frac{1}{2} \min\left(\frac{r}{r'}, \frac{r'}{r}\right) \quad \text{as } l_R \rightarrow \infty.$$

The Green function $G(r, r')$ represents the horizontal interaction of the VRWs between r and r' , and decreases as l_R decreases.

3.2 Solution for $l_R \rightarrow \infty$

In the case of $l_R \rightarrow \infty$ (although such a limit is difficult to defend when applied to the real atmosphere or ocean), the VI between the VRWs vanishes. The horizontally two dimensional system on each level temporally evolves independently of the systems on the other levels. In the horizontally two dimensional system, the eigenvalue λ_n introduced in (19) becomes the basic vortex angular velocity $\bar{\omega}_{n+1}$ at r_{n+1} (e.g., Ito and Kanehisa 2013),

$$\lambda_n = \bar{\omega}_{n+1}, \quad n = 1, 2, \dots, N,$$

where the smallest eigenvalue $\lambda_N = \bar{\omega}_{N+1} = 0$ is the basic vortex angular velocity at $r_{N+1} = \infty$. The corresponding eigenvectors are presented in Appendix D. The mode with eigenvalue $\lambda_N = 0$ is a steady mode, which is called the pseudo mode. The existence of the pseudo mode is associated with the displacement of the origin of the coordinate system or, equivalently, with the displacement of the basic vortex by a small distance. In the present case, the pseudo mode is selectively forced by the steady environmental flow introduced in (4), i.e., the scalar product $\langle l_n | r\bar{q} \rangle$ in (21) vanishes except for $n = N$.

$$\langle l_n | r\bar{q} \rangle = 0 \quad \text{for } n = 1, 2, \dots, N-1.$$

Further, because of the steadiness, the pseudo mode resonantly interacts with the steady environmental flow, and the physical space solution (21) grows linearly in time,

$$\begin{aligned} & \lim_{l_R \rightarrow \infty} |rq(t, \theta)\rangle \\ &= -U_0 \lim_{l_R \rightarrow \infty} \sum_{n=1}^N \frac{\sin \theta - \sin(\theta - \lambda_n t)}{\lambda_n} \frac{|r_n\rangle \langle l_n|}{\langle l_n | r_n \rangle} |r\bar{q}\rangle \\ &= -U_0 \lim_{l_R \rightarrow \infty} \frac{\sin \theta - \sin(\theta - \lambda_N t)}{\lambda_N} \frac{|r_N\rangle \langle l_N|}{\langle l_N | r_N \rangle} |r\bar{q}\rangle \\ & \quad [\langle l_n | r\bar{q} \rangle \rightarrow 0 \text{ for } n \neq N \text{ as } l_R \rightarrow \infty] \\ &= -U_0 t \cos \theta \frac{|r_N\rangle \langle l_N|}{\langle l_N | r_N \rangle} |r\bar{q}\rangle \quad [\lambda_N \rightarrow 0 \text{ as } l_R \rightarrow \infty]. \end{aligned} \tag{23}$$

As a result of the linear growth of the disturbance, the vortex is destroyed. The linear growth in (23) is simply a result of the vertically differential horizontal advection by the vertically sheared environmental flow.

For $l_R < \infty$, the solution in (21) precesses (represented by $\sin(\theta - \lambda_n t)$) about the downshear-left tilt

equilibrium (represented by $\sin \theta$) instead of growing linearly in time. However, as l_R becomes larger, the smallest eigenvalue $\lambda_N > 0$ in the denominator of (21) becomes smaller, and the amplitude of the precession becomes larger. The precession continuously changes into linear growth. Therefore, for a finite but sufficiently large l_R , regardless of finite or infinite, the vortex is practically destroyed.

For the disturbance PV at $r = r_j$, i.e., $q_j(t, \theta) \cos(\pi z/H) \delta(r - r_j)$ in (22), the radial displacement $d_j(t, \theta, z)$ of the Iso-PV line at $r = r_j$ is estimated as

$$d_j(t, \theta, z) = \frac{q_j(t, \theta) \cos \frac{\pi z}{H}}{\bar{q}_j}, \quad j = 1, 2, \dots, N. \quad (24)$$

For the linearly growing solution (23) for $l_R \rightarrow \infty$, the displacement in (24) becomes independent of radius r ,

$$d_j(t, \theta, z) = -U_0 t \cos \theta \cos \frac{\pi z}{H}, \quad j = 1, 2, \dots, N. \quad (25)$$

An example of the linearly growing solution (23) is shown in Fig. 3 in terms of (25).

3.3 Solution for $l_R \rightarrow 0$

In the case of $l_R \rightarrow 0$, the VI between the VRWs becomes so strong that the retrograde propagation vanishes and the VRWs are simply cyclonically advected by the basic vortex angular velocity $\bar{\omega}$ at each radius r . The simple advection implies the absence of the horizontal interaction between the VRWs, which is represented by the reduction of the Green function, $G(r, r') \rightarrow 0$ as $l_R \rightarrow 0$ ((A6) in Appendix A). Because of the reduction of G , the matrix Λ_{jk} in (18) is reduced to a diagonal one as $l_R \rightarrow 0$,

$$\Lambda_{jk} = \bar{\omega}_j \delta_{jk}. \quad (26)$$

The eigenvalues λ_n of (26), and the j th components r_{nj} and l_{nj} of the corresponding right and left eigenvectors $|r_n\rangle$ and $\langle l_n|$ are respectively given by

$$\lambda_n = \bar{\omega}_n, \quad \text{and} \quad r_{nj} = \delta_{nj} \quad \text{and} \quad l_{nj} = \delta_{nj}, \quad n = 1, 2, \dots, N. \quad (27)$$

The eigenvalues and eigenvectors in (27) are consistent with the simple advection by $\bar{\omega}$ at each radius r . By the substitution of (27), the analytical solution (21) is reduced to the following simple form,

$$q_j(t, \theta) = -U_0 \frac{\bar{q}_j}{\bar{\omega}_j} \{ \sin \theta - \sin(\theta - \bar{\omega}_j t) \}, \quad j = 1, 2, \dots, N. \quad (28)$$

The second term on the RHS of (28) represents simple advection by the basic angular velocity $\bar{\omega}_j$ at each radius r_j . In a continuous model, the simple advection implies the spiral wind up and resulting axisymmetrization. In our discrete model, the spiral wind up or axisymmetrization does not exist because the solution in (28) is periodic and eventually recurs. However, some kind of pseudo-axisymmetrization occurs (see Appendix E) around the first term on the RHS of (28), which represents the downshear-left tilt equilibrium. Because of this pseudo-axisymmetrization around the downshear-left tilt equilibrium, the vortex maintains vertical coherence in spite of the presence of the vertically sheared environmental flow.

For $l_R > 0$, the analytical solution (21) precesses about the downshear-left tilt equilibrium. However, for small $l_R > 0$, i.e., for strong VI, the retrograde

Fig. 3. The analytical solution on $z = H$ for $l_R \rightarrow \infty$ and $U_0 = 5 \text{ m s}^{-1}$. BV1 is assumed. The temporal evolution of the VRWs is represented by the solid curves. The dashed circles are the Iso-PV lines of BV1. The VRWs grow linearly in time (pseudo mode) and the vortex cannot maintain vertical coherence.

propagation of the VRWs is reduced and the cyclonic advection by the basic vortex is dominant. As a result, the VRWs are somewhat axisymmetrized by the radially differential advection by the basic angular velocity. Therefore, for sufficiently small l_R , whether zero or nonzero, the vortex maintains vertical coherence by the pseudo-axisymmetrization around the downshear-left tilt equilibrium.

For the simple advection solution (28) for $l_R \rightarrow 0$, the displacement of the Iso-PV line in (24) becomes

$$d_j(t, \theta, z) = -\frac{U_0}{\bar{\omega}_j} \{\sin \theta - \sin(\theta - \bar{\omega}_j t)\} \cos \frac{\pi z}{H},$$

$$j = 1, 2, \dots, N. \quad (29)$$

An example of the simple advection solution (28) is shown in Fig. 4 in terms of (29).

3.4 Solution for a moderate l_R

The analytical solution (21) precesses, which is represented by $\sin(\theta - \lambda_n t)$, about the downshear-left tilt equilibrium, which is represented by $\sin \theta$. As stated in the above subsections, as l_R increases, the amplitude of the precession increases, and the precession is continuously changed into linear growth. While, as l_R decreases, the advection by the basic vortex angular velocity becomes dominant, and the precession is continuously changed into simple advection.

$$\begin{aligned} \text{precessing} &\rightarrow \text{linearly growing} & \text{as } l_R \rightarrow \infty, \\ \text{precessing} &\rightarrow \text{simple advection} & \text{as } l_R \rightarrow 0. \end{aligned}$$

For a moderate l_R , which is of the order of the horizontal length scale of the vortex, i.e., for $l_R \sim 100$ km, the analytical solution (21) shows genuine precession behavior. That is, the VRW at r_j cyclonically moves with almost the same angular phase velocity C for any $j = 1, 2, \dots, N$. This almost form-preserving mode

is called the quasi mode (QM). The QM consists mainly of the N th mode. For $l_R \rightarrow \infty$, the N th mode is steady $\lambda = 0$, and is selectively excited by the sheared environmental flow. It grows linearly to destroy the vortex. However, for $l_R \sim 100$ km, the N th mode is nonsteady $\lambda > 0$, and is dominantly excited by the sheared environmental flow. It constitutes the QM. Practically, the QM cannot be distinguished from the N th mode. Because of the precession of the QM about the downshear-left tilt equilibrium, the vortex maintains vertical coherence in spite of the presence of the vertically sheared environmental flow. An example of the genuine precession solution (21) for $l_R \sim 100$ km is shown in Fig. 5 in terms of the displacement $d_j(t, \theta, z)$ of the Iso-PV line in (24).

The QM propagates retrograde, and is cyclonically advected by the basic vortex angular velocity $\bar{\omega}$. Because the cyclonic advection is dominant over the retrograde propagation, the QM moves cyclonically with a slower angular phase velocity C than the basic vortex angular velocity $\bar{\omega}$ in the central region of the QM. Because of the slower cyclonic phase velocity C , there exists such a radius r_c in the outer region that the angular phase velocity C of the QM is equal to the basic vortex angular velocity $\bar{\omega}$ there. This radius r_c is called the critical radius for the QM.

$$C = \bar{\omega} \quad \text{at } r = r_c \text{ critical radius.}$$

If the jump of the basic PV $\bar{q}$ at r_c is negative, i.e., $\bar{q}_c = \lim_{\varepsilon \rightarrow 0} \{\bar{q}(r_c - \varepsilon) - \bar{q}(r_c + \varepsilon)\} > 0$, then the horizontal circulation induced by the QM advects the basic PV $\bar{q}$ there, and then a retrograde propagating VRW is generated by the advection there.

$$\bar{q}_c > 0 \quad \text{at } r = r_c \Rightarrow \text{VRW at } r = r_c.$$

Further, if the magnitude of the jump $|\bar{q}_c|$ is small, then the retrograde propagation angular velocity of

Fig. 4. As in Fig. 3, except that $l_R \rightarrow 0$. The VRW at each radius r_j is simply advected by $\bar{\omega}(r_j)$ and precesses about the downshear-left tilt equilibrium.

Fig. 5. As in Fig. 3, except that $l_r = 100$ km. The VRWs move with nearly the same angular phase velocity and precess about the downshear-left tilt equilibrium with preserving the form.

VRW($r = r_c$) is small, and then the VRW($r = r_c$) is almost simply advected by the basic vortex angular velocity $\bar{\omega}(r = r_c)$, which is equal to the angular phase velocity C of the QM. That is, the phase difference between the VRW($r = r_c$) and the QM is nearly constant. Because of this, the VRW($r = r_c$) is successively enhanced by the horizontal circulation induced by the QM, and the VRW($r = r_c$) grows. If two waves are propagating in the opposite direction to each other and the phase difference is nearly constant, then both of them grow by mutual amplification. In the present case, however, both the VRW($r = r_c$) and the QM are retrograde propagating (i.e., propagating in the same direction), and then the growth of the VRW($r = r_c$) implies damping of the QM (see Appendix F).

$$\begin{aligned} |\bar{q}_c| \text{ is small} &\Rightarrow \text{nearly constant phase difference} \\ &\quad \text{between VRW}(r = r_c) \text{ and QM} \\ &\Rightarrow \text{growth of VRW}(r = r_c) \\ &\Rightarrow \text{damping of QM.} \end{aligned}$$

Therefore, in the presence of $\bar{q}_c > 0$ which is small in magnitude, the precession is damped and the QM eventually approaches the downshear-left tilt equilibrium. This is called the critical radius damping of the QM. Also in our discrete model, the following conservation equation of wave activity is derived from the governing equation (16),

$$\frac{d}{dt} \int_0^{2\pi} d\theta \sum_{j=1}^N \frac{r_j^2 \tilde{q}_j^2}{2\bar{q}_j} = 0,$$

where $r_j \tilde{q}_j = r_j \tilde{q}_j(t, \theta)$ is the j th component of the time-dependent part of the solution in (21), i.e.,

$$|r\tilde{q}(t, \theta)\rangle = U_0 \sum_{n=1}^N \frac{\sin(\theta - \lambda_n t)}{\lambda_n} \frac{|r_n\rangle \langle l_n|}{\langle l_n | r_n \rangle} |r\bar{q}\rangle.$$

Because of the rigorous conservation of this wave activity, growth at the critical radius implies decay at the central radii. An example of the precession solution with the critical radius damping is shown in Fig. 6 in terms of the displacement $d_j(t, \theta, z)$ of the Iso-PV line in (24).

4. Diabatic effect

In this section, we consider the diabatic effect on the evolution of the solution.

4.1 Governing equation

To examine the diabatic effect, i.e., condensational heating and evaporative cooling, we consider the solution with l_r being small and large (i.e., the VI being strong and weak) in the central and outer regions, respectively. This is because diabatic processes take place mainly in the central region, and to a first approximation the diabatic effect is accounted for by the reduction of the buoyancy frequency N . The reduction is roughly explained as follows. In the presence of diabatic heating/cooling, the linearized PV and thermodynamic equations are

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \theta} \right) q' + \frac{d\bar{q}}{dr} u' &= f \frac{\partial \dot{b}}{\partial z}, \\ \left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \theta} \right) b' + N^2 w' &= \dot{b}, \end{aligned}$$

where $\dot{b}$ represents diabatic heating/cooling. Here, we assume that the basic PV is not substantially changed by the introduction of diabatic heating/cooling. Usually, diabatic heating/cooling occurs in updrafts/downdrafts. Hence, it is assumed that

$$\dot{b} = \alpha N^2 w',$$

Fig. 6. As in Fig. 3, except that $l_R = 100$ km and that BV2 is assumed instead of BV1. BV2 has a PV jump $\bar{q}_4 > 0$ at the critical radius r_c , which is the outermost radius, where the angular phase velocity of the quasi mode is equal to $\bar{\omega}(r_c)$. By the critical radius damping, the quasi mode damps to the downshear-left tilt equilibrium, accompanied with the growth of the disturbance at r_c . The damping quasi mode is represented by the inner three solid curves. The growing critical radius disturbance is represented by the outermost solid curve.

where $\alpha (> 0)$ is, first of all, dependent on r because diabatic processes mainly take place in the central region. Therefore, we roughly assume that $\alpha = \alpha(r)$. Eliminating $\hat{b}$ from the above three equations, and expressing q' and b' in terms of the stream function ψ' , gives

$$\left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \theta} \right) \left\{ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \psi' + \frac{1}{r^2} \frac{\partial^2 \psi'}{\partial \theta^2} + \frac{f^2}{(1-\alpha)N^2} \frac{\partial^2 \psi'}{\partial z^2} \right\} - \frac{1}{r} \frac{d\bar{q}}{dr} \frac{\partial \psi'}{\partial \theta} = 0.$$

To this approximation, the equations of the disturbance PV q' (5) are unchanged except that the buoyancy frequency N in the second equation is replaced with the moist one $\tilde{N}$, which is reduced where the diabatic processes take place.

$$q' = \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{f^2}{\tilde{N}^2} \frac{\partial^2}{\partial z^2} \right) \psi'.$$

The equations of the disturbance PV $\hat{q}^\pm$ (12), which are derived from (5), are also unchanged except that the internal Rossby deformation radius $l_R = NH/f\pi$ in

the second equation of (12) is replaced with the moist one $\tilde{l}_R = \tilde{N}H/f\pi$, which is reduced where the diabatic processes take place.

$$\hat{q}^\pm = \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{1}{\tilde{l}_R^2} \right) \hat{\psi}^\pm. \quad (30)$$

Besides the diabatic effects, i.e., besides the reduction of N , the internal Rossby deformation radius $l_R = NH/\sqrt{\bar{q}\bar{\xi}}\pi$ in the AB model is small and large in the central and outer regions, respectively. This is because both $\bar{q}$ and $\bar{\xi}$ are large in the central region and small in the outer region. For a usual tropical cyclone, $\bar{q}, \bar{\xi} \sim$ a few 10^{-3} s^{-1} in the central region, and $10^{-3} \text{ s}^{-1} > \bar{q}, \bar{\xi} > 10^{-4} \text{ s}^{-1}$ in the outer region. As a result, the governing equation (16), which is derived from the first equation of (12), is also unchanged except that the Green function $G_{jk} = G(r_j, r_k)$ is replaced with the moist one $\tilde{G}_{jk} = \tilde{G}(r_j, r_k)$.

$$\frac{d}{dt}(r_j \hat{q}_j^\pm) \pm i \bar{\omega}_j r_j \hat{q}_j^\pm \mp i \bar{q}_j \sum_{k=1}^N \tilde{G}_{jk} r_k \hat{q}_k^\pm = -U_0 \sqrt{\frac{\pi}{2}} r_j \bar{q}_j, \quad j = 1, 2, \dots, N.$$

The moist Green function $\tilde{G}$ is obtained by inverting (30) under the radial boundary conditions (10) in the same way as the Green function G , which is obtained by inverting the second equation of (12). The derivation of the moist Green function $\tilde{G}$ is briefly described in Appendix A. The equation in the vector form (17) is also unchanged except that the matrix Λ is replaced with the moist one $\tilde{\Lambda}$.

$$\frac{d}{dt}|r\hat{q}^\pm(t)\rangle \pm i\tilde{\Lambda}|r\hat{q}^\pm(t)\rangle = -U_0\sqrt{\frac{\pi}{2}}|r\bar{q}\rangle$$

with

$$\tilde{\Lambda}_{jk} = \bar{\omega}_j\delta_{jk} - \bar{q}_j\tilde{G}_{jk}. \quad (31)$$

4.2 Solution

The closed form solution (21) is also unchanged except that the eigenvalues λ_n and the right and left eigenvectors $|r_n\rangle$ and $\langle l_n|$ of the matrix Λ (18) are respectively replaced with the moist counterparts $\tilde{\lambda}_n$, $|\tilde{r}_n\rangle$, and $\langle \tilde{l}_n|$ of the moist matrix $\tilde{\Lambda}$ (31).

$$|rq(t, \theta)\rangle = -U_0 \sum_{n=1}^N \frac{\sin\theta - \sin(\theta - \tilde{\lambda}_n t)}{\tilde{\lambda}_n} \frac{|\tilde{r}_n\rangle\langle \tilde{l}_n|}{\langle \tilde{l}_n|\tilde{r}_n\rangle} |r\bar{q}\rangle. \quad (32)$$

As an example, we consider a limiting case that the stratification is moist neutral $\tilde{N} \rightarrow 0$ implying $\tilde{l}_R \rightarrow 0$ in the central region $0 \leq r \leq r_v = 50$ km, and that the stratification is very strong $\tilde{N} \rightarrow \infty$ implying $\tilde{l}_R \rightarrow \infty$ in the outer region $r > r_v = 50$ km. Of course, this is a crude approximation because the diabatic processes take place where the moist air is rising and are very complicated nonlinear processes. The evolution of

the solution (32) is shown in Fig. 7 in terms of the displacement $d_j(t, \theta, z)$ of the Iso-PV line in (24).

As expected, the VRW is simply advected by the basic vortex angular velocity $\bar{\omega}$ in the central region. While, contrary to the expectation for linear growth in time, the VRWs behave just like a quasi mode, and precess about the downshear-left tilt equilibrium in the outer region. In spite of $\tilde{N} \rightarrow \infty$ (i.e., no VI) in the outer region, the vortex maintains vertical coherence due to $\tilde{N} \rightarrow 0$ (i.e., very strong VI) in the central region.

5. Concluding remarks

In this paper, we analytically investigated the vortex resiliency in the quasigeostrophic system. We considered vortex Rossby waves (VRWs) on a barotropic basic vortex, which were persistently forced by a vertically sheared environmental flow. The forced VRWs have the first baroclinic vertical structure and the wave number one azimuthal structure. To obtain a closed form analytical solution of the VRWs, we assumed that the potential vorticity (PV) of the basic vortex is piecewise constant in the radial direction. Specifically, we examined the dependency of the evolution of the analytical solution on the strength of the vertical interaction (VI) between the VRWs. Because the VI becomes stronger (weaker) as the internal Rossby deformation radius l_R becomes smaller (larger), dependency on the strength of VI implies dependency on l_R . Although l_R in the quasigeostrophic system is a constant parameter, we assumed that l_R takes the typical value of l_R in the AB system, which varies as the vortex flow varies, based on the similarity between the quasigeostrophic and AB equations. The results are summarized as follows.

When $l_R \sim l$, where $l \sim 100$ km is the horizontal

Fig. 7. As in Fig. 3, except that $l_R \rightarrow 0$ for $0 \leq r \leq r_v = 50$ km and that $l_R \rightarrow \infty$ for $r > r_v$. The innermost VRW is simply advected by $\bar{\omega}(r_1)$. The outer VRWs at outer two radii behave just like a precessing quasi mode. By the precession, the vortex maintains vertical coherence.

scale of the vortex, the VI is moderate. In this case, the VRWs move just like a true form-preserving mode called the quasi mode. The quasi mode precesses about the downshear-left tilt equilibrium, which is a stable equilibrium because of the balance between the downshear advection by the environmental wind shear and the upshear advection by the horizontal circulation induced by the upper and lower PVs. The precession does not grow, and the vortex maintains vertical coherence in spite of the presence of vertical wind shear. In the presence of a small negative radial gradient of the basic PV at the critical radius of the quasi mode, the quasi mode damps and the vortex approaches the downshear-left tilt equilibrium state.

As l_r increases, the VI becomes weak, and the amplitude and angular phase velocity of the precession become large and small, respectively.

When $l_r \rightarrow \infty$, the VI between the VRWs vanishes, and the VRWs on each level evolve independently of the VRWs on the other levels. This implies that the vortex is simply differentially advected by the vertically sheared environmental flow. As a result, the vortex is tilted, the tilt grows linearly in time, and the vortex is eventually destroyed. Another way to explain this behavior is as follows. When $l_r \rightarrow \infty$, a steady mode called the pseudo mode appears. The pseudo mode resonantly interacts with the steady environmental wind shear, grows linearly in time, and eventually destroys the vortex. As l_r increases, the precessing quasi mode with large amplitude and small angular phase velocity continuously changes into the linearly growing pseudo mode.

As l_r decreases, the VI becomes strong, and the cyclonic advection by the basic vortex flow becomes more and more dominant over the retrograde propagation of the VRWs. When $l_r \rightarrow 0$, the VI is so strong that the horizontal circulations induced by the lower and upper disturbance PVs, which are opposite-signed to each other, cancel each other. This implies that the retrograde propagation of the VRWs disappears, and that the VRWs are simply advected by the basic vortex flow at each radius. As a result, the VRWs are axisymmetrized to some extent by the simple advection by the basic vortex flow, and the vortex maintains vertical coherence in spite of the presence of vertical wind shear. As l_r decreases, the precessing quasi mode under the radially differential cyclonic advection continuously changes into the simple advection of the VRWs.

To examine the diabatic effects, we consider the solution with $l_r = NH/f\pi$ being small and large (i.e., the VI being strong and weak) in the central and outer

regions, respectively. This is because the diabatic processes take place mainly in the central region, and the diabatic effects are accounted for by the reduction of N to a first approximation. In particular, the case of $N \rightarrow 0$ (maximum VI) and $N \rightarrow \infty$ (null VI) in the central and outer regions, respectively, is examined. As expected, the VRWs in the central region are simply advected by the basic vortex flow, which means pseudo-axisymmetrization. While, contrary to the expectation for linear growth destroying the vortex, the VRWs in the outer region behave just like a quasi mode precessing about the downshear-left tilt equilibrium. By the precession, the vortex maintains vertical coherence. In spite of $N \rightarrow \infty$ (null VI) in the outer region, the outer VRWs do not linearly grow but precess due to $N \rightarrow 0$ (strong VI) in the central region, and the vortex is not destroyed.

Besides the diabatic effects, i.e., besides the reduction of N , the internal Rossby deformation radius $l_r = NH/\sqrt{\bar{q}\bar{\xi}}\pi$ in the AB model is small and large in the central and outer regions, respectively. This is because both $\bar{q}$ and $\bar{\xi}$ are large in the central region and small in the outer region. For a usual tropical cyclone, the central $l_r = NH/\sqrt{\bar{q}\bar{\xi}}\pi < l \sim 100$ km, and the VI is strong in the central region. From this, we suppose the following. Because of the large $\sqrt{\bar{q}\bar{\xi}}$ (strong VI) in the central region, the central VRWs are almost simply cyclonically advected by the basic vortex flow at each radius. This implies that the central region maintains vertical alignment of the downshear-left tilt equilibrium in spite of the presence of vertical wind shear. Further, even if $l_r = NH/\sqrt{\bar{q}\bar{\xi}}\pi > l \sim 100$ km (the VI is weak) in the outer region, the outer VRWs do not grow linearly in time, but precess about the downshear-left tilt equilibrium because of the central large $\sqrt{\bar{q}\bar{\xi}}$ (strong VI). As a result, the vortex can maintain vertical coherence.

Both the diabatic effects (central small N) and the radial distribution of vortex flow (large $\sqrt{\bar{q}\bar{\xi}}$ in the central region) are supposed to contribute to the vortex resiliency. However weak the VI between the VRWs in the outer region is (however large the outer N is, and although the outer $\sqrt{\bar{q}\bar{\xi}}$ is relatively small), the vortex can maintain vertical coherence because of the strong VI in the central region due to the small $l_r = NH/\sqrt{\bar{q}\bar{\xi}}\pi$ there.

Acknowledgments

We thank Dr. Michael Montgomery and an anonymous reviewer for their many valuable comments and suggestions that substantially improved the manuscript.

Appendix A: Green function

The Green function $G(r, r')$ in (13) is so defined that the invertibility equation in (12) and the boundary conditions (10) are satisfied. From the second equation of (12), the Green function $G(r, r')$ must satisfy

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{1}{l_R^2} \right) G(r, r') = -\frac{1}{r} \delta(r - r'). \quad (\text{A1})$$

For $r \neq r'$, the solution to (A1) is expressed as a linear combination of $I(r/l_R)$ and $K(r/l_R)$, where $I(x)$ and $K(x)$ are the modified Bessel functions of the first and second kind of order 1, respectively.

$$G(r, r') = aI\left(\frac{r}{l_R}\right) + bK\left(\frac{r}{l_R}\right) \quad \text{for } r \neq r', \quad (\text{A2})$$

with some coefficients a and b , whose values for $r < r'$ are different from those for $r > r'$. Further, (A1) implies the continuity of $G(r, r')$ and the discontinuity of $\partial G(r, r')/\partial r$ at $r = r'$.

$$\lim_{\varepsilon \rightarrow 0} \{G(r' - \varepsilon, r') - G(r' + \varepsilon, r')\} = 0, \\ \lim_{\varepsilon \rightarrow 0} \left\{ \left[\frac{\partial G(r, r')}{\partial r} \right]_{r=r'-\varepsilon} - \left[\frac{\partial G(r, r')}{\partial r} \right]_{r=r'+\varepsilon} \right\} = \frac{1}{r'}. \quad (\text{A3})$$

From (10), the Green function $G(r, r')$ must satisfy

$$\lim_{r \rightarrow 0} G(r, r') < \infty \quad \text{and} \quad \lim_{r \rightarrow \infty} G(r, r') \rightarrow 0. \quad (\text{A4})$$

Under conditions (A3) and (A4), the coefficients a and b in (A2) are determined to give the Green function,

$$G(r, r') = I\left(\frac{r'}{l_R}\right) K\left(\frac{r}{l_R}\right) \quad \text{for } r' < r, \\ G(r, r') = I\left(\frac{r}{l_R}\right) K\left(\frac{r'}{l_R}\right) \quad \text{for } r < r'. \quad (\text{A5})$$

The Green function $G(r, r')$ (A5) asymptotically becomes

$$G(r, r') \rightarrow \frac{1}{2} \frac{l_R}{\sqrt{rr'}} \exp\left(-\frac{|r - r'|}{l_R}\right) \rightarrow 0 \quad \text{as } l_R \rightarrow 0, \quad (\text{A6})$$

$$G(r, r') \rightarrow \frac{1}{2} \min\left(\frac{r}{r'}, \frac{r'}{r}\right) \quad \text{as } l_R \rightarrow \infty. \quad (\text{A7})$$

In Subsection 4.2, the moist internal Rossby deformation radius $\tilde{l}_R$ is assumed to be piecewise constant

in the radial direction,

$$\tilde{l}_R = \tilde{l}_{RA} \rightarrow 0 \quad \text{for } r < r_v, \quad \text{and} \\ \tilde{l}_R = \tilde{l}_{RB} \rightarrow \infty \quad \text{for } r > r_v, \quad (\text{A8})$$

In this case, the equation for the moist Green function $\tilde{G}(r, r')$ is also given by (A1) except that l_R is replaced with $\tilde{l}_R$,

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{1}{\tilde{l}_R^2} \right) \tilde{G}(r, r') = -\frac{1}{r} \delta(r - r'). \quad (\text{A9})$$

For $r \neq r'$, the solution to (A9) is expressed in the same form as (A2),

$$\tilde{G}(r, r') = \tilde{a}I\left(\frac{r}{\tilde{l}_R}\right) + \tilde{b}K\left(\frac{r}{\tilde{l}_R}\right) \quad \text{for } r \neq r'. \quad (\text{A10})$$

The coefficients $\tilde{a}$ and $\tilde{b}$ take different values in each case, $r < r' < r_v$, $r < r_v < r'$, $r_v < r < r'$, $r' < r < r_v$, $r' < r_v < r$, $r_v < r' < r$. In addition to the continuity and discontinuity conditions (A3) at $r = r'$, the continuity conditions up to the first derivative at $r = r_v$ are also required by (A9).

$$\lim_{\varepsilon \rightarrow 0} \{\tilde{G}(r_v - \varepsilon, r') - \tilde{G}(r_v + \varepsilon, r')\} = 0, \\ \lim_{\varepsilon \rightarrow 0} \left\{ \left[\frac{\partial \tilde{G}(r, r')}{\partial r} \right]_{r=r_v-\varepsilon} - \left[\frac{\partial \tilde{G}(r, r')}{\partial r} \right]_{r=r_v+\varepsilon} \right\} = 0. \quad (\text{A11})$$

Under conditions (A3), (A4) and (A11), the coefficients $\tilde{a}$ and $\tilde{b}$ in (A10) are determined to give the moist Green function $\tilde{G}(r, r')$ in the limit of $\tilde{l}_{RA} \rightarrow 0$ and $\tilde{l}_{RB} \rightarrow \infty$.

$$\tilde{G}(r, r') = \frac{1}{2} \min\left(\frac{r}{r'}, \frac{r'}{r}\right) - \frac{1}{2} \frac{r_v^2}{rr'} \\ \quad \text{for } r > r_v \quad \text{and } r' > r_v, \quad (\text{A12})$$

$$\tilde{G}(r, r') = 0 \quad \text{otherwise.} \quad (\text{A13})$$

In the central region $r < r_v$ where $\tilde{l}_{RA} \rightarrow 0$, the VI becomes maximum. As a result, the horizontal interaction in the central region, and that between the central and the outer regions are absent, as represented by (A13). In the outer region $r > r_v$ where $\tilde{l}_{RB} \rightarrow \infty$, the VI vanishes. However, the horizontal interaction there is not maximum, but is reduced by the strong VI in the central region, as represented by the second term on the RHS of (A12).

Appendix B:

QG Green function and AB Green function

The balance equation of the AB system for disturbances of the first baroclinic and azimuthal wave number one mode $\propto e^{i\theta} \cos(\pi z/H)$ on a barotropic axisymmetric basic vortex is given by

$$\hat{q}_{AB} = \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \left(1 - r \frac{d}{dr} \log |\bar{q} \bar{\xi}| \right) \frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{1}{l_R^2} \right\} \hat{\psi}$$

$$\text{with } \frac{1}{l_R^2} = \frac{\pi^2 \bar{q} \bar{\xi}}{N^2 H^2}, \quad (\text{B1})$$

where $\bar{q} = f + \bar{\xi} = f + (1/r)(d/dr)(r\bar{v})$ and $\bar{\xi} = f + 2\bar{\omega} = f + 2(\bar{v}/r)$ are the basic absolute vorticity and the inertia parameter, respectively. Here, we consider a basic vortex of Gauss type, i.e., $\bar{\xi} = \zeta_0 e^{-ar^2}$. Then,

$$\bar{q} = f + \zeta_0 e^{-ar^2}, \quad \bar{\xi} = f + \frac{\zeta_0}{ar^2} (1 - e^{-ar^2}). \quad (\text{B2})$$

The Green function $G_{AB}(r, r')$ for the balance equation in (B1) is numerically calculated. This is shown as a function of r' for a fixed r in Figs. B1–3 together with the QG Green function $G_{QG}(r, r')$ that is calculated from the balance equation of the QG system,

$$\hat{q} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{1}{l_R^2} \right) \hat{\psi}, \quad (\text{B3})$$

where the last term $f^2 \pi^2 / N^2 H^2$ is replaced with $1/l_R^2 = \pi^2 \bar{q} \bar{\xi} / N^2 H^2$.

The QG Green function $G_{QG}(r, r')$ is qualitatively similar to the AB Green function $G_{AB}(r, r')$, although $G_{QG}(r, r')$ overestimates the horizontal interaction in the central region. The overestimation and underestimation in the prognostic equation mitigate each other to some extent, as shown by the graphs of $\{\bar{\xi}(r)/\bar{q}(r)\} G_{AB}(r, r')$.

As shown in Fig. B1, $G_{QG}(r, r')$ and $\{\bar{\xi}(r)/\bar{q}(r)\} G_{AB}(r, r')$ are almost indistinguishable from each other for $r = 50$ km. This is because $-r(d/dr) \log |\bar{q}(r) \bar{\xi}(r)|$ is small and $\bar{\xi}(r)/\bar{q}(r)$ is nearly equal to one for $r \lesssim 50$ km, as shown in Fig. B4 for the same parameter values as in Fig. B1. Also, for $r = 100$ km, $G_{QG}(r, r')$ and $\{\bar{\xi}(r)/\bar{q}(r)\} G_{AB}(r, r')$ are very similar to each other (Fig. B2).

As shown in Fig. B3, the underestimation in the prognostic equation becomes somewhat dominant for $r = 150$ km. This is primarily because $\bar{\xi}(r)/\bar{q}(r)$ becomes evidently larger than one for $r \gtrsim 150$ km.

Fig. B1. Example of $G_{QG}(r, r')$, $G_{AB}(r, r')$, and $\{\bar{\xi}(r)/\bar{q}(r)\} G_{AB}(r, r')$, for $a = 5 \times 10^{-11} \text{ m}^{-2}$, $\zeta_0 = 1.2 \times 10^{-3} \text{ s}^{-1}$, $f = 10^{-4} \text{ s}^{-1}$, $N = 10^{-2} \text{ s}^{-1}$, $H = 10^4 \text{ m}$, and $r = 50 \text{ km}$.

Fig. B2. As in Fig. B1 except for $r = 100 \text{ km}$.

Although the underestimation becomes much more dominant for $r = 200$ km (not shown), $G_{QG}(r, r')$ and $\{\bar{\xi}(r)/\bar{q}(r)\} G_{AB}(r, r')$ are not so seriously different from

Fig. B3. As in Fig. B1 except for $r = 150$ km.

Fig. B4. The graphs of $-r(d/dr) \log |\bar{q} \bar{\xi}|$, $\bar{\xi}/\bar{q}$, and $l_R/(100 \text{ km})$ for the same parameter values as in Fig. B1.

each other for $r \lesssim 150$ km. In particular, in the central region of $r \lesssim 100$ km, $G_{QG}(r, r')$ and $\{\bar{\xi}(r)/\bar{q}(r)\} G_{AB}(r, r')$ are practically the same.

Appendix C: Eigenvalues and eigenvectors

For the $N \times N$ matrix Λ in (18), the eigenvalues λ_n , right eigenvectors $|r_n\rangle$, and left eigenvectors $\langle l_n|$ ($n = 1, 2, 3, \dots, N$) are so defined that

$$\Lambda |r_n\rangle = \lambda_n |r_n\rangle, \quad \langle l_n| \Lambda = \lambda_n \langle l_n|, \quad n = 1, 2, 3, \dots, N. \quad (C1)$$

Here, $|r_n\rangle = \begin{bmatrix} r_{n1} \\ r_{n2} \\ \vdots \\ r_{nN} \end{bmatrix}$ is an N -component column vector,

and $\langle l_n| = [l_{n1} \ l_{n2} \ \dots \ l_{nN}]$ is an N -component row vector. The scalar product of $\langle l_n|$ and $|r_m\rangle$, which is a scalar, is denoted as $\langle l_n | r_m \rangle$,

$$\langle l_n | r_m \rangle = [l_{n1} \ l_{n2} \ \dots \ l_{nN}] \begin{bmatrix} r_{m1} \\ r_{m2} \\ \vdots \\ r_{mN} \end{bmatrix} = \sum_{j=1}^N l_{nj} r_{mj}.$$

The dyadic product of $|r_n\rangle$ and $\langle l_m|$, which is an $N \times N$ matrix, is denoted as $|r_n\rangle \langle l_m|$,

$$|r_n\rangle \langle l_m| = \begin{bmatrix} r_{n1} \\ r_{n2} \\ \vdots \\ r_{nN} \end{bmatrix} [l_{m1} \ l_{m2} \ \dots \ l_{mN}] = \begin{bmatrix} r_{n1} l_{m1} & r_{n1} l_{m2} & \dots & r_{n1} l_{mN} \\ r_{n2} l_{m1} & r_{n2} l_{m2} & \dots & r_{n2} l_{mN} \\ \vdots & \vdots & \ddots & \vdots \\ r_{nN} l_{m1} & r_{nN} l_{m2} & \dots & r_{nN} l_{mN} \end{bmatrix}.$$

They satisfy the associative law, i.e., $(|r_n\rangle \langle l_m|) |r_l\rangle = |r_n\rangle (\langle l_m | r_l \rangle) = |r_n\rangle \langle l_m | r_l \rangle$. On the assumption of no degeneration of eigenvalues (indeed, it can be shown that the QG model used here has no degenerate eigenvalues), from the definition (C1), the eigenvectors satisfy the following orthogonality relation.

$$\langle l_n | r_m \rangle = 0 \quad \text{if} \quad n \neq m. \quad (C2)$$

Proof

$$\begin{aligned} \Lambda |r_m\rangle &= \lambda_m |r_m\rangle \quad [\text{the first equation of (C1)}] \\ \Rightarrow \langle l_n | \Lambda |r_m\rangle &= \lambda_m \langle l_n | r_m \rangle. \\ \langle l_n | \Lambda &= \lambda_n \langle l_n| \quad [\text{the second equation of (C1)}] \\ \Rightarrow \langle l_n | \Lambda |r_m\rangle &= \lambda_n \langle l_n | r_m \rangle. \\ \Rightarrow \lambda_m \langle l_n | r_m \rangle &= \lambda_n \langle l_n | r_m \rangle \\ \Rightarrow (\lambda_n - \lambda_m) \langle l_n | r_m \rangle &= 0 \\ \Rightarrow \langle l_n | r_m \rangle &= 0 \quad \text{if} \quad \lambda_n \neq \lambda_m \\ \Rightarrow \langle l_n | r_m \rangle &= 0 \quad \text{if} \quad n \neq m \quad [\text{no degeneration}]. \end{aligned}$$

Q.E.D.

The identity matrix I can be decomposed in terms of the eigenvectors as follows.

$$I = \sum_{n=1}^N \frac{|r_n\rangle\langle l_n|}{\langle l_n | r_n \rangle}. \quad (\text{C3})$$

Proof

Any column vector $|a\rangle$ is decomposed in terms of the right eigenvectors $|r_n\rangle$ as $|a\rangle = \sum_{n=1}^N C_n |r_n\rangle$ with some coefficients C_n .

$$\begin{aligned} \Rightarrow \langle l_m | a \rangle &= \sum_{n=1}^N C_n \langle l_m | r_n \rangle = C_m \langle l_m | r_m \rangle \quad [(\text{C2})] \\ \Rightarrow C_n &= \frac{\langle l_n | a \rangle}{\langle l_n | r_n \rangle} \\ \Rightarrow |a\rangle &= \sum_{n=1}^N \frac{\langle l_n | a \rangle}{\langle l_n | r_n \rangle} |r_n\rangle = \sum_{n=1}^N |r_n\rangle \frac{\langle l_n | a \rangle}{\langle l_n | r_n \rangle} \\ &= \sum_{n=1}^N \frac{|r_n\rangle\langle l_n|}{\langle l_n | r_n \rangle} |a\rangle \quad [\text{associative law}] \\ \Rightarrow \sum_{n=1}^N \frac{|r_n\rangle\langle l_n|}{\langle l_n | r_n \rangle} &= I \quad [|a\rangle \text{ is arbitrary}]. \end{aligned}$$

Q.E.D.

The $N \times N$ matrix Λ is decomposed in terms of the eigenvalues and eigenvectors as follows.

$$\Lambda = \sum_{n=1}^N \lambda_n \frac{|r_n\rangle\langle l_n|}{\langle l_n | r_n \rangle}. \quad (\text{C4})$$

Proof

$$\begin{aligned} \Lambda &= \Lambda I = \Lambda \sum_{n=1}^N \frac{|r_n\rangle\langle l_n|}{\langle l_n | r_n \rangle} \quad [(\text{C3})] \\ &= \sum_{n=1}^N \frac{\Lambda |r_n\rangle\langle l_n|}{\langle l_n | r_n \rangle} = \sum_{n=1}^N \lambda_n \frac{|r_n\rangle\langle l_n|}{\langle l_n | r_n \rangle} \quad [(\text{C1})]. \end{aligned}$$

Q.E.D.

Appendix D:

Eigenvalues and eigenvectors for $l_R \rightarrow \infty$

In the limiting case of $l_R \rightarrow \infty$, the Green function $G_{jk} = G(r_j, r_k)$ in the matrix Λ given in (18),

$$\Lambda_{jk} = \bar{\omega}_j \delta_{jk} - \bar{q}_j G_{jk}, \quad (\text{D1})$$

becomes

$$G_{jk} = \frac{1}{2} \min \left(\frac{r_j}{r_k}, \frac{r_k}{r_j} \right), \quad (\text{D2})$$

which is derived from (A7). The basic vortex angular velocity $\bar{\omega}_j = \bar{\omega}(r_j)$ is given by

$$\bar{\omega}_j = \frac{1}{2} \sum_{k=1}^{j-1} \bar{q}_k \left(\frac{r_k}{r_j} \right)^2 + \frac{1}{2} \sum_{k=j}^N \bar{q}_k, \quad (\text{D3})$$

which is derived from (8). The eigenvalues λ_n , $n = 1, 2, \dots, N$, of the matrix Λ are given by

$$\lambda_n = \bar{\omega}_{n+1}, \quad (\text{D4})$$

where $\bar{\omega}_{N+1} = \bar{\omega}(r_{N+1}) = \bar{\omega}(\infty) = 0$. The corresponding right and left eigenvectors, $|r_n\rangle$ and $\langle l_n|$, are respectively given by

$$\begin{aligned} r_{nj} &= r_j \bar{q}_j \quad \text{for } 1 \leq j \leq n, \quad n = 1, 2, \dots, N-1, \\ r_{m+1} &= - \sum_{k=1}^n \frac{\bar{q}_k r_k^2}{r_{n+1}}, \quad n = 1, 2, \dots, N-1, \\ r_{nj} &= 0 \quad \text{for } n+2 \leq j \leq N, \quad n = 1, 2, \dots, N-1, \\ r_{Nj} &= r_j \bar{q}_j \quad \text{for } 1 \leq j \leq N, \end{aligned} \quad (\text{D5})$$

and

$$\begin{aligned} l_{nj} &= r_j \quad \text{for } 1 \leq j \leq n, \quad n = 1, 2, \dots, N-1, \\ l_{m+1} &= - \sum_{k=1}^n \frac{\bar{q}_k r_k^2}{\bar{q}_{n+1} r_{n+1}}, \quad n = 1, 2, \dots, N-1, \\ l_{nj} &= 0 \quad \text{for } n+2 \leq j \leq N, \quad n = 1, 2, \dots, N-1, \\ l_{Nj} &= r_j \quad \text{for } 1 \leq j \leq N. \end{aligned} \quad (\text{D6})$$

By the substitution of (D1), (D2) and (D3), we can see that the eigenvalues given in (D4) and the eigenvectors given in (D5) and (D6) satisfy the following eigenequations,

$$\sum_{k=1}^N \Lambda_{jk} r_{nk} = \lambda_n r_{nj} \quad \text{and} \quad \sum_{j=1}^N l_{nj} \Lambda_{jk} = \lambda_n l_{nk}, \quad n = 1, 2, \dots, N. \quad (\text{D7})$$

By the substitution of (D6), we can see that the scalar product $\langle l_n | r \bar{q} \rangle$ in the analytical solution (21) vanishes except for $n = N$.

$$\langle l_n | r \bar{q} \rangle = \sum_{j=1}^N l_{nj} r_j \bar{q}_j = 0 \quad \text{for } n = 1, 2, \dots, N-1. \quad (\text{D8})$$

That is, the pseudo mode with $\lambda_N = 0$ is selectively forced by the environmental flow.

Appendix E:

Pseudo-axisymmetrization in the discrete model

After Reasor and Montgomery (2001), we defined the following position of vorticity center.

$$\begin{aligned}\bar{x}(t) &= \frac{\int_0^{2\pi} d\theta \int_0^\infty dr r x \zeta}{\int_0^{2\pi} d\theta \int_0^\infty dr r \zeta}, \\ \bar{y}(t) &= \frac{\int_0^{2\pi} d\theta \int_0^\infty dr r y \zeta}{\int_0^{2\pi} d\theta \int_0^\infty dr r \zeta},\end{aligned}\quad (E1)$$

where $\zeta = \bar{\zeta} + q'$, and $\bar{\zeta} = \bar{\zeta}(r) = \bar{q}(r) - f$ is the basic vorticity given in (6), and $q' = q'(t, r, \theta, z)$ is the disturbance vorticity given in (21) and (22). Substituting (6), (21), and (22) into the denominator of (E1) gives

$$\int_0^{2\pi} d\theta \int_0^\infty dr r \zeta = \pi \sum_{j=1}^N r_j^2 \bar{q}_j = \pi \langle I_N^\infty | r_N^\infty \rangle. \quad (E2)$$

Here $\langle I_N^\infty | = \lim_{l_R \rightarrow \infty} \langle I_N |$ and $| r_N^\infty \rangle = \lim_{l_R \rightarrow \infty} | r_N \rangle$, and we used

(D5) and (D6) in Appendix D. In the same way, the numerators in (E1) are given by

$$\begin{aligned}\int_0^{2\pi} d\theta \int_0^\infty dr r x \zeta &= -\pi U_0 \sum_{n=1}^N \frac{\langle I_N^\infty | r_n \rangle \langle I_n | r_N^\infty \rangle}{\langle I_n | r_n \rangle} \frac{\sin \lambda_n t}{\lambda_n} \cos\left(\frac{\pi z}{H}\right), \\ \int_0^{2\pi} d\theta \int_0^\infty dr r y \zeta &= -\pi U_0 \sum_{n=1}^N \frac{\langle I_N^\infty | r_n \rangle \langle I_n | r_N^\infty \rangle}{\langle I_n | r_n \rangle} \frac{1 - \cos \lambda_n t}{\lambda_n} \cos\left(\frac{\pi z}{H}\right).\end{aligned}\quad (E3)$$

Dividing (E3) by (E2) gives the position of the vorticity center,

$$\begin{aligned}\bar{x}(t) &= -U_0 \sum_{n=1}^N \frac{\langle I_N^\infty | r_n \rangle \langle I_n | r_N^\infty \rangle}{\langle I_N^\infty | r_N^\infty \rangle \langle I_n | r_n \rangle} \frac{\sin \lambda_n t}{\lambda_n} \cos\left(\frac{\pi z}{H}\right), \\ \bar{y}(t) &= -U_0 \sum_{n=1}^N \frac{\langle I_N^\infty | r_n \rangle \langle I_n | r_N^\infty \rangle}{\langle I_N^\infty | r_N^\infty \rangle \langle I_n | r_n \rangle} \frac{1 - \cos \lambda_n t}{\lambda_n} \cos\left(\frac{\pi z}{H}\right).\end{aligned}\quad (E4)$$

In the case of $l_R \rightarrow \infty$, $\langle I_n | r_N^\infty \rangle = 0$ if $n \neq N$ and $\lambda_n = 0$. Hence, as expected, the position in (E4) becomes $\bar{x}(t) = -U_0 t \cos(\pi z/H)$ and $\bar{y}(t) = 0$, which represents the simple differential advection by the sheared envi-

Fig. E1. Temporal evolution of $D(t)$ for BV1 of $N = 3$ and for BV3 of $N = 12$.

ronmental flow in (4).

In the case of $l_R \rightarrow 0$, $r_{nj} = l_{nj} = \delta_{nj}$ and $\lambda_n = \bar{\omega}_n$. Therefore, the position in (E4) becomes

$$\begin{aligned}\bar{x}(t) &= U_0 \sum_{n=1}^N \frac{r_n^2 \bar{q}_n}{\langle I_N^\infty | r_N^\infty \rangle} \frac{\sin \lambda_n t}{\lambda_n}, \\ \bar{y}(t) &= U_0 \sum_{n=1}^N \frac{r_n^2 \bar{q}_n}{\langle I_N^\infty | r_N^\infty \rangle} \frac{1 - \cos \lambda_n t}{\lambda_n},\end{aligned}\quad (E5)$$

apart from the vertical dependence $-\cos(\pi z/H)$. From (E5), we define the following distance from the oscillation center (downshear-left tilt equilibrium) $(x_0, y_0) = (0, U_0 \sum_{n=1}^N \lambda_n^{-1} r_n^2 \bar{q}_n / \langle I_N^\infty | r_N^\infty \rangle)$

$$D(t) = \sqrt{\{\bar{x}(t) - x_0\}^2 + \{\bar{y}(t) - y_0\}^2}. \quad (E6)$$

Decreasing $D(t)$ in (E6) implies symmetrization about the oscillation center. The temporal evolution of $D(t)$ for the basic vortex BV1 of $N = 3$ in Fig. 1 is shown in Fig. E1 together with that for another basic vortex BV3 of $N = 12$ (region $0 < r < 175$ km is divided into 12 subregions).

For the basic vortex BV1, the distance $D(t)$ decreases during the initial period $0 \leq t \lesssim 12000$ s. As N increases, the initial decreasing period becomes longer.

Appendix F: Explanation for decay of the QM

In the case without critical radius, the QM in our model is nearly equal to the N th mode. For $l_R \rightarrow \infty$, the N th mode becomes steady $\lambda_N \rightarrow 0$. By the sheared environmental flow, only the steady N th mode is excited and grows linearly in time, resulting in destruction of the vortex. For a finite large l_R , the N th mode, which is nonsteady $\lambda_N > 0$, is dominantly excited by the sheared environmental flow. The N th mode is the main component of the QM. Practically, the QM cannot be distinguished from the N th mode. The QM, which is nearly equal to the N th mode, does not grow (or decay) in spite of the presence of the sheared environmental flow.

In the case with critical radius, and for a finite large l_R , the N th and $(N - 1)$ th modes are dominantly excit-

ed by the sheared environmental flow. The two modes are the main components of the QM, as schematically shown in Fig. F1. The N th and $(N - 1)$ th modes are of the same sign at the central radii and of opposite sign at the critical radius (left figure). Because of this, and because of their different angular phase velocities $\lambda_{N-1} > \lambda_N > 0$, decay at the central radii and growth at the critical radius ensue (right figure), subsequently followed by regrowth at the central radii and decay at the critical radius, and so on.

As for growth at the critical radius and decay at the central radii, we rely on the following simple reasoning, schematically shown in Fig. F2. Let the basic PV jumps $\bar{q}_a$ at r_a and $\bar{q}_b$ at r_b ($> r_a$) be of the same sign, say positive (toward the center). The same positive sign implies that both $W_a = \text{VRW}(r = r_a)$ and $W_b = \text{VRW}(r = r_b)$ (if they exist) propagate retrograde.

Fig. F1. Left: N th ($\circ$) and $(N - 1)$ th ($\bullet$) modes are in phase at central radii. Right: N th ($\circ$) and $(N - 1)$ th ($\bullet$) modes are in anti-phase at central radii. From Left to Right: Perturbations ($\bullet$) at central and critical radii, respectively, decay and grow.

Fig. F2. The sinusoidal curves represent VRWs at r_a and r_b ($r_b > r_a > 0$). The + and - signs represent the positive and negative vorticity perturbations, respectively. The vertical arrows represent the horizontal circulations induced by the vorticity perturbations.

us assume an initial W_a (left figure). The horizontal circulation induced by W_a advects the basic PV at r_b , and then generates W_b (middle figure). The phase difference of the original W_a and the generated W_b is $\pi/2$. If the phase difference is kept nearly constant, then the horizontal circulation induced by W_a continues to enhance W_b , resulting in the growth of W_b . As the amplitude of W_b becomes large, the horizontal circulation induced by W_b becomes strong. The strong horizontal circulation induced by W_b advects the basic PV at r_a , resulting in the damping of W_a (right figure).

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