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論文

On Numerical Prediction (I)

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§1 Introduction.

The purpose of the present paper is to put forward a method for numerical prediction of geophysical phenomena, especially of the change in pressure patterns.

First we explain the basic idea.

Now we consider a state-function S , which is to be obtained by observations and analyses. S contains variables which characterize not only the state at any instant but also their timely variations or the past state, if necessary. If we assume that S_0 at $t=t_0$ changes to S_1 at $t=t_1$ ($t_1 > t_0$) following some laws, we can express the change symbolically by

$$S_1 = L_0 S_0 + F_0,$$

where L_0 and F_0 are laws, which govern the phenomenon during t_0 and t_1 . L_0 is the sum of laws which relate directly to S_0 , and F_0 is the sum of laws which may be considered to be independent of S_0 . In many cases, L_0 may be forecast or, at least theoretically, will be obtained exactly in a satisfactory degree. On the other hand, generally, F_0 cannot be forecast from S_0 . We shall give an example. In summer clear days, we can forecast the generation of cumulus-clouds in the afternoon from the morning observations. In this example, we can easily see what are meant by S_0 , S_1 and L_0 . But if a conflagration breaks out after the morning observations, the cumulus-clouds or even shower generate. But we cannot forecast such cumulus-clouds or shower from the morning observations. In this case, the conflagration and its influence are F_0 .

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If F_0 gives finite effects in all cases, we must abandon the trial of forecasting. So we shall put F_0 out of account. The problem of forecasting is to obtain S_1 from L_0 and S_0 . Theoretically, if we take sufficient number of elements, S_1 is determined by L_0 and S_0 . In this case, if we know S_1 and S_0 , we can obtain L_0 . This relation is written symbolically as follows:

$$L_0 = S_1 S_0^{-1} \quad \text{where} \quad S_0 S_0^{-1} = 1.$$

In practical problems, we cannot obtain L_0 and S_0 exactly; some errors must exist in S_0 and L_0 . If we denote these errors by a symbol δ , we can write

$$S_1 + \delta S_1 = L_0 S_0 + \delta L_0 S_0 + L_0 \delta S_0 + \delta L_0 \delta S_0,$$

where S_0 , S_1 and L_0 are actual states and actual law respectively. If δS_0 and δL_0 are unavoidable and δS_1 is considerably large, we cannot forecast the phenomenon satisfactorily. So we shall put this case out of account too.

Now we return to the forecasting problem. If S_0 changes to S_1 during t_0 and t_1 , and S_1 changes to S_2 during t_1 and t_2 , we can write this case as follows:

$$S_1 = L_0 S_0, \quad S_2 = L_1 S_1.$$

Generally, L_0 is not equal to L_1 . But if we assume that L_1 is equal to L_0 , we can predict S_2 by using S_0 and S_1 . This relation is written symbolically as follows:

$$S_2 = S_1 S_0^{-1} S_1.$$

If we know the timely variation of L , we can also obtain S_2 . This relation is written as follows:

$$S_0 = L_{-1} S_{-1} = (L_0 - \Delta L) S_{-1}, \quad S_1 = L_0 S_0, \quad S_2 = L_1 S_1 = (L_0 + \Delta L) S_1.$$

Solving the equations symbolically, we obtain;

$$S_0 S_0^{-1} = L_0 - \Delta L = S_1 S_0^{-1} - \Delta L, \quad \Delta L = S_1 S_0^{-1} - S_0 S_0^{-1}$$

$$\therefore S_2 = (S_1 S_0^{-1} + S_1 S_0^{-1} - S_0 S_0^{-1}) S_1 = 2 S_1 S_0^{-1} S_1 - S_0 S_0^{-1} S_1.$$

The above explanation is the symbolic representation of the basic idea of the forecasting problem. In many geophysical phenomena, especially in the weather phenomena, the laws are very complicated, but the final goal of meteorological scientists and geophysicists is to obtain the exact representation of the state function S and the law L . But even at present we know that many geophysical and meteorological problems are represented by simple differential equations. The above symbolic equations are considered as integral equations. In the following paragraphs, we shall discuss the forecasting problem more concretely.

§2 Preliminary Analysis:

We consider a function Z of time t and spatial coordinates (x, y) . The function Z

satisfies the following differential equation

$$L(Z) = 0, \quad (1)$$

where L is a linear operator with constant coefficients:

$$L = \sum_{l,m,n} A_{l,m,n} \frac{\partial^{l+m+n}}{\partial t^l \partial x^m \partial y^n}. \quad (2)$$

If we put

$$Z = e^{i(p_x x + p_y y - \alpha x - \beta y)} \quad (3)$$

and introduce it into (1), we get

$$\sum_{l,m,n} (-1)^{m+n} i^{l+m+n} A_{l,m,n} (p_{\alpha,\beta})^l \alpha^m \beta^n = 0. \quad (4)$$

This equation gives $p_{\alpha,\beta}$ as functions of α and β . For a given couple of α and β , l roots of $p_{\alpha,\beta}$ are obtained. Changing α or β into $-\alpha$ or $-\beta$, we get four groups of roots:

$$p^{\kappa}_{\alpha,\beta}, p^{\kappa}_{-\alpha,\beta}, p^{\kappa}_{\alpha,-\beta}, p^{\kappa}_{-\alpha,-\beta} \quad (\kappa = 1 \cdots l)$$

Then the solution of (2) is given by

$$\begin{aligned} Z = \sum_{\kappa=1}^l \{ & A^{(\kappa)}_{\alpha,\beta} e^{i(p^{\kappa}_{\alpha,\beta} t - \alpha x - \beta y)} + A^{(\kappa)}_{-\alpha,\beta} e^{i(p^{\kappa}_{-\alpha,\beta} t + \alpha x - \beta y)} \\ & + A^{(\kappa)}_{\alpha,-\beta} e^{i(p^{\kappa}_{\alpha,-\beta} t - \alpha x + \beta y)} + A^{(\kappa)}_{-\alpha,-\beta} e^{i(p^{\kappa}_{-\alpha,-\beta} t + \alpha x + \beta y)} \}. \end{aligned} \quad (5)$$

We consider a domain $(0-a, 0-b)$, in which Z will be expanded into double Fourier series.

Putting $\alpha = \frac{\pi}{a} u$, $\beta = \frac{\pi}{b} v$, the general solution of (1) is given by

$$\begin{aligned} Z = \sum_{\kappa=1}^l \sum_{u,v} \{ & A^{(\kappa)}_{u,v} e^{i(p^{\kappa}_{u,v} t - \alpha x - \beta y)} + A^{(\kappa)}_{-u,v} e^{i(p^{\kappa}_{-u,v} t + \alpha x - \beta y)} \\ & + A^{(\kappa)}_{u,-v} e^{i(p^{\kappa}_{u,-v} t - \alpha x + \beta y)} + A^{(\kappa)}_{-u,-v} e^{i(p^{\kappa}_{-u,-v} t + \alpha x + \beta y)} \} \end{aligned} \quad (6)$$

where u and v are positive integers. $A^{(\kappa)}_{uv}$'s are determined by the initial conditions. In order to determine these conditions, l conditions are necessary. As l initial conditions, we can take groups of distributions of $Z(0)$, $Z'(0)$, $\dots$ and $Z^{(l-1)}(0)$ or $Z(0)$, $Z(-T)$, $\dots$ and $Z(-(l-1)T)$ or other combinations of l quantities. This is an ordinary initial value problem. Our problem of prediction is to obtain $Z(t)$ for $t > 0$ from given combinations of above quantities, not knowing the differential equation. As differential equation is not given, p^{κ}_{uv} 's are not known. Then we must determine not only $A^{(\kappa)}_{uv}$'s but also p^{κ}_{uv} 's, so that the $2l$ initial conditions must be given. The phenomena which are described by the equation of diffusion $\frac{\partial Z}{\partial t} = k^2 \left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} \right)$ can be predicted by using the states at two instants, but those which are described by the equation of wave motion $\frac{\partial^2 Z}{\partial t^2} = c^2 \left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} \right)$ cannot be predicted by using the states at two instants but are done by using past four states.

§ 3 Problem of prediction.

If the law of change of Z , *i. e.* the linear operator L , is given, the distribution of Z at an arbitrary instant is obtained as above. On the other hand, if L is not given, p 's are unknown. Our problem is to determine A 's and p 's from the given distribution of Z and predict Z at later instants.

In order to determine A 's and p 's, $2l$ conditions are necessary. If $Z(0)$, $Z(-T)$, $\dots$, $Z(-(2l-1)T)$ are given, expanding them into double Fourier series, we get

$$\begin{aligned} Z(-\kappa T) = & \sum_{u,v} G_{u,v}(-\kappa T) e^{i(-\alpha u - \beta v)} + \sum_{u,v} G_{-u,v}(\kappa T) e^{i(\alpha u - \beta v)} \\ & + \sum_{u,v} G_{u,-v}(-\kappa T) e^{i(-\alpha u + \beta v)} + \sum_{u,v} G_{-u,-v}(\kappa T) e^{i(\alpha u + \beta v)} : (\kappa = 0, \dots, 2l-1). \quad (7) \end{aligned}$$

For the sake of simplicity, we shall drop the subscripts u and v , so far as any confusion does not occur.

Comparing (6) and (7), we get

$$\left. \begin{aligned} A^{(1)} + A^{(2)} + \dots + A^{(l)} &= G(0), \\ A^{(1)} e^{-i p^1 T} + A^{(2)} e^{-i p^2 T} + \dots + A^{(l)} e^{-i p^l T} &= G(-T), \\ A^{(1)} e^{-i 2 p^1 T} + A^{(2)} e^{-i 2 p^2 T} + \dots + A^{(l)} e^{-i 2 p^l T} &= G(-2T), \\ &\dots \dots \dots \\ A^{(1)} e^{-i l p^1 T} + A^{(2)} e^{-i l p^2 T} + \dots + A^{(l)} e^{-i l p^l T} &= G(-lT), \\ &\dots \dots \dots \\ A^{(1)} e^{-i (2l-1) p^1 T} + A^{(2)} e^{-i (2l-1) p^2 T} + \dots + A^{(l)} e^{-i (2l-1) p^l T} &= G(-(2l-1)T). \end{aligned} \right\} \quad (8)$$

From the above, $2l$ equations we can determine lA 's and lp 's. Putting $e^{-i p^k T} = x_k$, (8) becomes as follows:

$$\left. \begin{aligned} A^{(1)} + A^{(2)} + \dots + A^{(l)} &= G(0), \\ A^{(1)} x_1 + A^{(2)} x_2 + \dots + A^{(l)} x_l &= G(-T), \\ A^{(1)} x_1^2 + A^{(2)} x_2^2 + \dots + A^{(l)} x_l^2 &= G(-2T), \\ &\dots \dots \dots \\ A^{(1)} x_1^l + A^{(2)} x_2^l + \dots + A^{(l)} x_l^l &= G(-lT), \\ &\dots \dots \dots \\ A^{(1)} x_1^{2l-1} + A^{(2)} x_2^{2l-1} + \dots + A^{(l)} x_l^{2l-1} &= G(-(2l-1)T). \end{aligned} \right\} \quad (9)$$

Putting

$$\sum_{u,v} \text{ denotes the double summation } \sum_{u=0}^{\infty} \sum_{v=0}^{\infty}$$

$$D = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_l \\ \cdots & \cdots & \cdots & \cdots \\ x_1^l & x_2^l & \cdots & x_l^l \end{vmatrix}, \quad \Delta_i^{(1)} = (-1)^i \begin{vmatrix} G(0) & 1 & \cdots & 1 \\ G(-T) & x_1 & \cdots & x_{i-1} & x_{i+1} \cdots x_l \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ G(-(l-1)T) & x_1^{l-1} \cdots x_{i-1}^{l-1} & x_{i+1}^{l-1} \cdots x_l^{l-1} \end{vmatrix}$$

$$\Delta_i^{(2)} = (-1)^i \begin{vmatrix} G(-lT) & 1 & \cdots & 1 & \cdots & 1 \\ G(-(l+1)T) & 1 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_l \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ G(-(2l-1)T) & 1 & \cdots & x_{i-1}^{l-1} & x_{i+1}^{l-1} & \cdots & x_l^{l-1} \end{vmatrix},$$

we get

$$A^{(i)} = \frac{\Delta_i^{(1)}}{D} = \frac{\Delta_i^{(2)}}{D} \quad \text{or} \quad \Delta_i^{(1)} = \Delta_i^{(2)}, \quad i=1, \dots, l, \quad (9)$$

$$\begin{vmatrix} G(0) & 1 & \cdots & 1 \\ G(-T) & x_1 & \cdots & x_l \\ \vdots & \vdots & \ddots & \vdots \\ G(-(l-1)T) & x_1^{l-1} & \cdots & x_l^{l-1} \end{vmatrix} = \begin{vmatrix} G(-lT) & 1 & \cdots & 1 \\ G(-(l+1)T) & x_1 & \cdots & x_l \\ \vdots & \vdots & \ddots & \vdots \\ G(-(2l+1)T) & x_1^{l-1} & \cdots & x_l^{l-1} \end{vmatrix}$$

$$\begin{vmatrix} G(-T) - G(-lT) & 1 & \cdots & 1 \\ G(-T) - G(-(l+1)T) & x_1 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_l \\ \cdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ G(-(l-1)T) - G(-(2l-1)T) & x_1^{l-1} & \cdots & x_{i-1}^{l-1} & x_{i+1}^{l-1} & \cdots & x_l^{l-1} \end{vmatrix} = 0 \quad (i=1, \dots, l) \quad (10)$$

From the l equations above, we get $x_1 \cdots x_l$, then we get A^i from (9). In meteorological problems, cases $l=1$ and $l=2$ are important, so that we shall discuss these cases in detail.

§ 4 The case where $l=1$.

We have two cases of practical importance. The first case is when we know the distributions of Z and $\frac{\partial Z}{\partial t}$ at $t=0$. The second case is when we know the distributions of Z at $t=0$ and $t=-T$.

i) The case when Z and $\frac{\partial Z}{\partial t}$ at $t=0$ are given.

Expanding Z and $\frac{\partial Z}{\partial t}$ into double Fourier series, we get

$$Z(0) = \sum_{u,v} \{ G_{u,v}(0) e^{i(-\alpha x - \beta y)} + G_{-u,v}(0) e^{i(\alpha x - \beta y)} + G_{u,-v}(0) e^{i(-\alpha x + \beta y)} + G_{-u,-v}(0) e^{i(\alpha x + \beta y)} \}, \quad (11)$$

$$\left(\frac{\partial Z}{\partial t} \right)_0 = \sum_{u,v} \{ G_{u,v}^I(0) e^{i(-\alpha x - \beta y)} + G_{-u,v}^I(0) e^{i(\alpha x - \beta y)} + G_{u,-v}^I(0) e^{i(-\alpha x + \beta y)} + G_{-u,-v}^I(0) e^{i(\alpha x + \beta y)} \}. \quad (12)$$

On the other hand, from (6) we get

$$-Z(0) = \sum_{u,v} \{ A_{u,v} e^{i(-\alpha x - \beta y)} + A_{-u,v} e^{i(\alpha x - \beta y)} \\ + A_{u,-v} e^{i(-\alpha x + \beta y)} + A_{-u,-v} e^{i(\alpha x + \beta y)} \}, \quad (13)$$

$$\left(\frac{\partial Z}{\partial t} \right)_0 = \sum_{u,v} \{ i A_{u,v} p_{u,v} e^{i(-\alpha x - \beta y)} + i A_{-u,v} p_{-u,v} e^{i(\alpha x - \beta y)} \\ + i A_{u,-v} p_{u,-v} e^{i(-\alpha x + \beta y)} + i A_{-u,-v} p_{-u,-v} e^{i(\alpha x + \beta y)} \}. \quad (14)$$

Comparing (11) and (12) with (13) and (14) respectively, we get

$$\left. \begin{aligned} G_{u,v}(0) &= A_{u,v}, \\ G_{-u,v}(0) &= A_{-u,v}, \\ G_{u,-v}(0) &= A_{u,-v}, \\ G_{-u,-v}(0) &= A_{-u,-v}, \end{aligned} \right\} \quad (15) \quad \left. \begin{aligned} G^I_{u,v}(0) &= i A_{u,v} p_{u,v}, \\ G^I_{-u,v}(0) &= i A_{-u,v} p_{-u,v}, \\ G^I_{u,-v}(0) &= i A_{u,-v} p_{u,-v}, \\ G^I_{-u,-v}(0) &= i A_{-u,-v} p_{-u,-v}. \end{aligned} \right\} \quad (16)$$

From (15) and (16), we get

$$i p_{u,v} = \frac{G^I_{u,v}(0)}{G_{u,v}(0)}, \quad i p_{-u,v} = \frac{G^I_{-u,v}(0)}{G_{-u,v}(0)}, \quad i p_{u,-v} = \frac{G^I_{u,-v}(0)}{G_{u,-v}(0)}, \quad i p_{-u,-v} = \frac{G^I_{-u,-v}(0)}{G_{-u,-v}(0)} \quad (17)$$

Then the state Z at t is given by

$$Z = \sum_{u,v} \left\{ G_{u,v}(0) e^{\frac{G^I_{u,v}(0)}{G_{u,v}(0)} t + i(-\alpha x - \beta y)} + G_{-u,v}(0) e^{\frac{G^I_{-u,v}(0)}{G_{-u,v}(0)} t + i(\alpha x - \beta y)} \right. \\ \left. + G_{u,-v}(0) e^{\frac{G^I_{u,-v}(0)}{G_{u,-v}(0)} t + i(-\alpha x + \beta y)} + G_{-u,-v}(0) e^{\frac{G^I_{-u,-v}(0)}{G_{-u,-v}(0)} t + i(\alpha x + \beta y)} \right\} \quad (18)$$

ii) When $Z(0)$ and $Z(-T)$ are given.

In this case we can use (11) and (13), so that we get (15). Expanding $Z(-T)$ into double Fourier series, we get

$$Z(-T) = \sum_{u,v} \{ G_{u,v}(-T) e^{i(-\alpha x - \beta y)} + G_{-u,v}(-T) e^{i(\alpha x - \beta y)} \\ + G_{u,-v}(-T) e^{i(-\alpha x + \beta y)} + G_{-u,-v}(-T) e^{i(\alpha x + \beta y)} \} \quad (19)$$

From (6), we get

$$Z(-T) = \sum_{u,v} \{ A_{u,v} e^{-i p_{u,v} T + i(-\alpha x - \beta y)} + A_{-u,v} e^{-i p_{-u,v} T + i(\alpha x - \beta y)} \\ + A_{u,-v} e^{-i p_{u,-v} T + i(-\alpha x + \beta y)} + A_{-u,-v} e^{-i p_{-u,-v} T + i(\alpha x + \beta y)} \} \quad (20)$$

Comparing (19) and (20), we get

$$\left. \begin{aligned} G_{u,v}(-T) &= G_{u,v}(0) e^{-i p_{u,v} T}, & G_{-u,v}(-T) &= G_{-u,v}(0) e^{-i p_{-u,v} T}, \\ G_{u,-v}(-T) &= G_{u,-v}(0) e^{-i p_{u,-v} T}, & G_{-u,-v}(-T) &= G_{-u,-v}(0) e^{-i p_{-u,-v} T} \\ e^{i p_{u,v} T} &= \frac{G_{u,v}(0)}{G_{u,v}(-T)}, & e^{i p_{-u,v} T} &= \frac{G_{-u,v}(0)}{G_{-u,v}(-T)} \end{aligned} \right\} \quad (21)$$

$$e^{ip_{u,v}T} = \frac{G_{u,v}(0)}{G_{u,v}(-T)}, \quad e^{ip_{-u,-v}T} = \frac{G_{-u,-v}(0)}{G_{-u,-v}(-T)} \quad \Bigg\}$$

Then we get

$$\begin{aligned} Z(T) = \sum_{u,v} \left\{ G_{u,v}(0) \left(\frac{G_{u,v}(0)}{G_{u,v}(-T)} \right) e^{i(-\alpha x - \beta y)} + G_{-u,v}(0) \left(\frac{G_{-u,v}(0)}{G_{-u,v}(-T)} \right) e^{i(\alpha x - \beta y)} \right. \\ \left. + G_{u,-v}(0) \left(\frac{G_{u,-v}(0)}{G_{u,-v}(-T)} \right) e^{i(-\alpha x + \beta y)} + G_{-u,-v}(0) \left(\frac{G_{-u,-v}(0)}{G_{-u,-v}(-T)} \right) e^{i(\alpha x + \beta y)} \right\}. \quad (22) \end{aligned}$$

This is the final solution. From the above solution, we can predict the distribution at $t=T$. But we can not obtain the distribution of an arbitrary instant. But this method is convenient, when we predict the distribution at every definite integral interval. From (21), we get $e^{iNpT} = \left(\frac{G(0)}{G(-T)} \right)^N$, so that the distribution at $t=NT$ is given by

$$\begin{aligned} Z(NT) = \sum_{u,v} \left\{ G_{u,v}(0) \left(\frac{G_{u,v}(0)}{G_{u,v}(-T)} \right)^N e^{i(-\alpha x - \beta y)} + G_{-u,v}(0) \left(\frac{G_{-u,v}(0)}{G_{-u,v}(-T)} \right)^N e^{i(\alpha x - \beta y)} \right. \\ \left. + G_{u,-v}(0) \left(\frac{G_{u,-v}(0)}{G_{u,-v}(-T)} \right)^N e^{i(-\alpha x + \beta y)} + G_{-u,-v}(0) \left(\frac{G_{-u,-v}(0)}{G_{-u,-v}(-T)} \right)^N e^{i(\alpha x + \beta y)} \right\}. \quad (23) \end{aligned}$$

On the other hand, if we put NT instead of T in (21), we get

$$e^{ipNT} = \frac{G(0)}{G(-NT)} = \left\{ \frac{G(0)}{G(-T)} \right\}^N \quad (24)$$

If a phenomenon is described by the differential equation of the first order with respect to time, the above equation must hold. This gives a criterion, whether the phenomenon obeys the differential equation of the first order with respect to time or not.

iii) Remark on the case where the coefficients vary slowly with time.

When the coefficients of the differential equation are slowly varying functions of t , p 's vary slowly with time. If we put the solution as

$$Z = A e^{ip_0 t},$$

we get the following relations:

$$\begin{aligned} Z &= Z_0 = A & : t=0 \\ Z &= Z_1 = A e^{ip_0 T} = B & : t=T \\ Z &= Z_2 = B e^{ip_1 T} = A e^{2ip_0 T + i\Delta p_0 T} = C & : t=2T \\ Z &= Z_3 = C e^{ip_2 T} = A e^{3ip_0 T + i2\Delta p_0 T} & : t=3T \end{aligned}$$

Eliminating A , $e^{ip_0 T}$ and $e^{i\Delta p_0 T}$ from the above four equations, we get

$$Z_1 Z_3 = Z_2^2. \quad (25)$$

This is the condition that the coefficients are slowly varying functions of time.

§ 5 The case where $l=2$.

In this case, we must determine four groups of unknown quantities, i. e., $A^{(1)}$, $A^{(2)}$, $p^{(1)}$, $p^{(2)}$. We can determine these quantities, if we know $\{Z(0), Z(-T), Z(-2T), Z(-3T)\}$, $\left\{Z(0), \left(\frac{\partial Z}{\partial t}\right)_0, Z(-T), \left(\frac{\partial Z}{\partial t}\right)_{-T}\right\}$, $\left\{Z(0), \left(\frac{\partial Z}{\partial t}\right)_0, \left(\frac{\partial^2 Z}{\partial t^2}\right)_0, \left(\frac{\partial^3 Z}{\partial t^3}\right)_0\right\}$. We can take other combinations of the four quantities included in these, if we want. Now we confine ourselves to the above three cases.

i) When $Z(0)$, $\left(\frac{\partial Z}{\partial t}\right)_0$, $\left(\frac{\partial^2 Z}{\partial t^2}\right)_0$, and $\left(\frac{\partial^3 Z}{\partial t^3}\right)_0$ are given.

Expanding these quantities, we get

$$\begin{aligned} Z(0) = \sum_{u,v} \{ & G_{uv}(0) e^{i(-\alpha x - \beta y)} + G_{-u,v}(0) e^{i(\alpha x - \beta y)} \\ & + G_{u,-v}(0) e^{i(-\alpha x + \beta y)} + G_{-u,-v}(0) e^{i(\alpha x + \beta y)} \}, \end{aligned} \quad (26)$$

$$\begin{aligned} \left(\frac{\partial Z}{\partial t}\right)_0 = \sum_{u,v} \{ & G_{uv}^I(0) e^{i(-\alpha x - \beta y)} + G_{-u,v}^I(0) e^{i(\alpha x - \beta y)} \\ & + G_{u,-v}^I(0) e^{i(-\alpha x + \beta y)} + G_{-u,-v}^I(0) e^{i(\alpha x + \beta y)} \}, \end{aligned} \quad (27)$$

$$\begin{aligned} \left(\frac{\partial^2 Z}{\partial t^2}\right)_0 = \sum_{u,v} \{ & G_{uv}^{II}(0) e^{i(-\alpha x - \beta y)} + G_{-u,v}^{II}(0) e^{i(\alpha x - \beta y)} \\ & + G_{u,-v}^{II}(0) e^{i(-\alpha x + \beta y)} + G_{-u,-v}^{II}(0) e^{i(\alpha x + \beta y)} \}, \end{aligned} \quad (28)$$

$$\begin{aligned} \left(\frac{\partial^3 Z}{\partial t^3}\right)_0 = \sum_{u,v} \{ & G_{uv}^{III}(0) e^{i(-\alpha x - \beta y)} + G_{-u,v}^{III}(0) e^{i(\alpha x - \beta y)} \\ & + G_{u,-v}^{III}(0) e^{i(-\alpha x + \beta y)} + G_{-u,-v}^{III}(0) e^{i(\alpha x + \beta y)} \}. \end{aligned} \quad (29)$$

From (6), we get

$$\begin{aligned} Z(0) = \sum_{u,v} \{ & (A^{(1)}_{uv} + A^{(2)}_{uv}) e^{i(-\alpha x - \beta y)} + (A^{(1)}_{-u,v} + A^{(2)}_{-u,v}) e^{i(\alpha x - \beta y)} \\ & + (A^{(1)}_{u,-v} + A^{(2)}_{u,-v}) e^{i(-\alpha x + \beta y)} + (A^{(1)}_{-u,-v} + A^{(2)}_{-u,-v}) e^{i(\alpha x + \beta y)} \}, \end{aligned} \quad (30)$$

$$\begin{aligned} \left(\frac{\partial Z}{\partial t}\right)_0 = \sum_{u,v} \{ & i(A^{(1)}_{uv} p^{(1)}_{uv} + A^{(2)}_{uv} p^{(2)}_{uv}) e^{i(-\alpha x - \beta y)} \\ & + i(A^{(1)}_{-u,v} p^{(1)}_{-u,v} + A^{(2)}_{-u,v} p^{(2)}_{-u,v}) e^{i(\alpha x - \beta y)} \\ & + i(A^{(1)}_{u,-v} p^{(1)}_{u,-v} + A^{(2)}_{u,-v} p^{(2)}_{u,-v}) e^{i(-\alpha x + \beta y)} \\ & + i(A^{(1)}_{-u,-v} p^{(1)}_{-u,-v} + A^{(2)}_{-u,-v} p^{(2)}_{-u,-v}) e^{i(\alpha x + \beta y)} \}, \end{aligned} \quad (31)$$

$$\begin{aligned} \left(\frac{\partial^2 Z}{\partial t^2}\right)_0 = \sum_{u,v} \{ & (i)^2 (A^{(1)}_{uv} p^{(1)2}_{uv} + A^{(2)}_{uv} p^{(2)2}_{uv}) e^{i(-\alpha x - \beta y)} \\ & + (i)^2 (A^{(1)}_{-u,v} p^{(1)2}_{-u,v} + A^{(2)}_{-u,v} p^{(2)2}_{-u,v}) e^{i(\alpha x - \beta y)} \\ & + (i)^2 (A^{(1)}_{u,-v} p^{(1)2}_{u,-v} + A^{(2)}_{u,-v} p^{(2)2}_{u,-v}) e^{i(-\alpha x + \beta y)} \\ & + (i)^2 (A^{(1)}_{-u,-v} p^{(1)2}_{-u,-v} + A^{(2)}_{-u,-v} p^{(2)2}_{-u,-v}) e^{i(\alpha x + \beta y)} \}. \end{aligned}$$

$$\begin{aligned}
& + (i)^2 (A^{(1)} - u, -v p^{(1)2} - u, -v + A^{(2)} - u, -v p^{(2)2} - u, -v) e^{i(\alpha x + \beta y)}, \\
\left(\frac{\partial^3 Z}{\partial t^3} \right)_0 = & \sum_{u,v} \{ (i)^3 (A^{(1)} u, v p^{(1)3} u, v + A^{(2)} u, v p^{(2)3} u, v) e^{i(-\alpha x - \beta y)} \\
& + (i)^3 (A^{(1)} - u, v p^{(1)3} - u, v + A^{(2)} - u, v p^{(2)3} - u, v) e^{i(\alpha x - \beta y)} \\
& + (i)^3 (A^{(1)} u, -v p^{(1)3} u, -v + A^{(2)} u, -v p^{(2)3} u, -v) e^{i(-\alpha x + \beta y)} \\
& + (i)^3 (A^{(1)} - u, -v p^{(1)3} - u, -v + A^{(2)} - u, -v p^{(2)3} - u, -v) e^{i(\alpha x + \beta y)} \}.
\end{aligned} \tag{32}$$

Comparing (26)~(29) and (30)~(33), we get

$$G(0) = A^{(1)} + A^{(2)}, \tag{34}$$

$$G^I(0) = i(A^{(1)} p^{(1)} + A^{(2)} p^{(2)}), \tag{35}$$

$$G^{II}(0) = (i)^2 (A^{(1)} p^{(1)2} + A^{(2)} p^{(2)2}), \tag{36}$$

$$G^{III}(0) = (i)^3 (A^{(1)} p^{(1)3} + A^{(2)} p^{(2)3}), \tag{37}$$

From (34) and (35), we get

$$A^{(1)} = \frac{G(0)ip^{(2)} - G^I(0)}{i(p^{(2)} - p^{(1)})}, \quad A^{(2)} = \frac{G^I(0) - ip^{(1)}G(0)}{i(p^{(2)} - p^{(1)})}. \tag{38}$$

From (35) and (36), we get

$$A^{(1)} ip^{(1)} = \frac{G^I(0)ip^{(2)} - G^{II}(0)}{i(p^{(2)} - p^{(1)})}, \quad A^{(2)} ip^{(2)} = \frac{G^{II}(0) - G^I(0)ip^{(1)}}{i(p^{(2)} - p^{(1)})}. \tag{39}$$

From (36) and (37), we get

$$A^{(1)} (ip^{(1)})^2 = \frac{G^{II}(0)ip^{(2)} - G^{III}(0)}{i(p^{(2)} - p^{(1)})}, \quad A^{(2)} (ip^{(2)})^2 = \frac{G^{III}(0) - G^{II}(0)ip^{(1)}}{i(p^{(2)} - p^{(1)})}. \tag{40}$$

Eliminating $A^{(1)}$ and $A^{(2)}$ from (38) and (39), we get

$$\frac{G^I(0)ip^{(2)} - G^{II}(0)}{G(0)ip^{(2)} - G^I(0)} = ip^{(1)}, \quad \frac{G^{II}(0) - G^I(0)ip^{(1)}}{G^I(0) - G(0)ip^{(1)}} = ip^{(2)}. \tag{41}$$

Eliminating $A^{(1)}ip^{(1)}$ and $A^{(2)}ip^{(2)}$ from (39) and (40), we get

$$\frac{G^{II}(0)ip^{(2)} - G^{III}(0)}{G^I(0)ip^{(2)} - G^{II}(0)} = ip^{(1)}, \quad \frac{G^{III}(0) - G^{II}(0)ip^{(1)}}{G^{II}(0) - G^I(0)ip^{(1)}} = ip^{(2)}. \tag{42}$$

From (41) and (42), we get

$$\begin{aligned}
& \{G^I(0)x - G^{II}(0)\} \{G^I(0)x - G^{II}(0)\} = \{G^{II}(0)x - G^{III}(0)\} \{G^I(0)x - G^I(0)\} \\
\text{or } & \{(G^I(0)^2 - G(0)G^{II}(0))x^2 - \{G^I(0)G^{II}(0) - G(0)G^{III}(0)\}x + \{G^{II}(0)^2 - G^I(0)G^{III}(0)\}\} = 0
\end{aligned} \tag{43}$$

where $x = ip^{(1)}$ or $ip^{(2)}$.

Then we get

$$\frac{ip^{(1)}}{ip^{(2)}} = \frac{G^I(0)G^{II}(0) - G(0)G^{III}(0) \pm \sqrt{\{G^I(0)G^{II}(0) - G(0)G^{III}(0)\}^2 - 4\{G^I(0)^2 - G(0)G^{II}(0)\} \times \{G^{II}(0)^2 - G^I(0)G^{III}(0)\}}}{2\{G^I(0)^2 - G(0)G^{II}(0)\}}$$

$$\overline{\{G^{II}(0)\}^2 - G^{II}G^{III}(0)} \quad (44)$$

Introducing these values into (38), we get $A^{(1)}$ and $A^{(2)}$.

The function at any instant is given by

$$\begin{aligned} Z(t) = \sum_{u,v} \{ & (A^{(1)}_{u,v} e^{ip^{(1)}_{u,v}t} + A^{(2)}_{u,v} e^{ip^{(2)}_{u,v}t}) e^{i(-\alpha x - \beta y)} \\ & + (A^{(1)}_{-u,v} e^{ip^{(1)}_{-u,v}t} + A^{(2)}_{-u,v} e^{ip^{(2)}_{-u,v}t}) e^{i(\alpha x - \beta y)} \\ & + (A^{(1)}_{u,-v} e^{ip^{(1)}_{u,-v}t} + A^{(2)}_{u,-v} e^{ip^{(2)}_{u,-v}t}) e^{i(-\alpha x + \beta y)} \\ & + (A^{(1)}_{-u,-v} e^{ip^{(1)}_{-u,-v}t} + A^{(2)}_{-u,-v} e^{ip^{(2)}_{-u,-v}t}) e^{i(\alpha x + \beta y)} \}. \end{aligned} \quad (45)$$

where $ip^{(1)}$'s, $ip^{(2)}$'s, $A^{(1)}$'s and $A^{(2)}$'s are given by (38) and (44).

ii) When $Z(0)$, $\left(\frac{\partial Z}{\partial t}\right)_0$, $Z(-T)$, and $\left(\frac{\partial Z}{\partial t}\right)_{-T}$ are given.

Expanding these quantities into double Fourier series, we get

$$\begin{aligned} Z(0) = \sum_{u,v} G_{u,v}(0) e^{i(-\alpha x - \beta y)} + \sum_{u,v} G_{-u,v}(0) e^{i(\alpha x - \beta y)} \\ + \sum_{u,v} G_{u,-v}(0) e^{i(-\alpha x + \beta y)} + \sum_{u,v} G_{-u,-v}(0) e^{i(\alpha x + \beta y)}, \end{aligned} \quad (46)$$

$$\begin{aligned} \left(\frac{\partial Z}{\partial t}\right)_0 = \sum_{u,v} G^I_{u,v}(0) e^{i(-\alpha x - \beta y)} + \sum_{u,v} G^I_{-u,v}(0) e^{i(\alpha x - \beta y)} \\ + \sum_{u,v} G^I_{u,-v}(0) e^{i(-\alpha x + \beta y)} + \sum_{u,v} G^I_{-u,-v}(0) e^{i(\alpha x + \beta y)}, \end{aligned} \quad (47)$$

$$\begin{aligned} Z(-T) = \sum_{u,v} G_{u,v}(-T) e^{i(-\alpha x - \beta y)} + \sum_{u,v} G_{-u,v}(-T) e^{i(\alpha x - \beta y)} \\ + \sum_{u,v} G_{u,-v}(-T) e^{i(-\alpha x + \beta y)} + \sum_{u,v} G_{-u,-v}(-T) e^{i(\alpha x + \beta y)} \end{aligned} \quad (48)$$

$$\begin{aligned} \left(\frac{\partial Z}{\partial t}\right)_{-T} = \sum_{u,v} G^I_{u,v}(-T) e^{i(-\alpha x - \beta y)} + \sum_{u,v} G^I_{-u,v}(-T) e^{i(\alpha x - \beta y)} \\ + \sum_{u,v} G^I_{u,-v}(-T) e^{i(-\alpha x + \beta y)} + \sum_{u,v} G^I_{-u,-v}(-T) e^{i(\alpha x + \beta y)} \end{aligned} \quad (49)$$

From (6), we get

$$\begin{aligned} Z(0) = \sum_{u,v} (A^1_{u,v} + A^2_{u,v}) e^{i(-\alpha x - \beta y)} + \sum_{u,v} (A^1_{-u,v} + A^2_{-u,v}) e^{i(\alpha x - \beta y)} \\ + \sum_{u,v} (A^1_{u,-v} + A^2_{u,-v}) e^{i(-\alpha x + \beta y)} + \sum_{u,v} (A^1_{-u,-v} + A^2_{-u,-v}) e^{i(\alpha x + \beta y)}. \end{aligned} \quad (50)$$

$$\begin{aligned} \left(\frac{\partial Z}{\partial t}\right)_0 = \sum_{u,v} (iA^1_{u,v} p^1_{u,v} + iA^2_{u,v} p^2_{u,v}) e^{i(-\alpha x - \beta y)} \\ + \sum_{u,v} (iA^1_{-u,v} p^1_{-u,v} + iA^2_{-u,v} p^2_{-u,v}) e^{i(\alpha x - \beta y)} \\ + \sum_{u,v} (iA^1_{u,-v} p^1_{u,-v} + iA^2_{u,-v} p^2_{u,-v}) e^{i(-\alpha x + \beta y)} \\ + \sum_{u,v} (iA^1_{-u,-v} p^1_{-u,-v} + iA^2_{-u,-v} p^2_{-u,-v}) e^{i(\alpha x + \beta y)}. \end{aligned} \quad (51)$$

$$\begin{aligned}
Z(-T) = & \sum_{u,v} (A^1_{uv} e^{-ip^{(1)}_{uv}T} + A^2_{uv} e^{-ip^{(2)}_{uv}T}) e^{i(-\alpha x - \beta y)} \\
& + \sum_{u,v} (A^1_{-u,v} e^{-ip^{(1)}_{-u,v}T} + A^2_{-u,v} e^{-ip^{(2)}_{-u,v}T}) e^{i(\alpha x - \beta y)} \\
& + \sum_{u,v} (A^1_{u,-v} e^{-ip^{(1)}_{u,-v}T} + A^2_{u,-v} e^{-ip^{(2)}_{u,-v}T}) e^{i(-\alpha x + \beta y)} \\
& + \sum_{u,v} (A^1_{-u,-v} e^{-ip^{(1)}_{-u,-v}T} + A^2_{-u,-v} e^{-ip^{(2)}_{-u,-v}T}) e^{i(\alpha x + \beta y)} , \quad (52)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial Z}{\partial t} \right)_{-T} = & \sum_{u,v} (iA^1_{uv} p^{(1)}_{uv} e^{-ip^{(1)}_{uv}T} + iA^2_{uv} p^{(2)}_{uv} e^{-ip^{(2)}_{uv}T}) e^{i(-\alpha x - \beta y)} \\
& + \sum_{u,v} (iA^1_{-u,v} p^{(1)}_{-u,v} e^{-ip^{(1)}_{-u,v}T} + iA^2_{-u,v} p^{(2)}_{-u,v} e^{-ip^{(2)}_{-u,v}T}) e^{i(\alpha x - \beta y)} \\
& + \sum_{u,v} (iA^1_{u,-v} p^{(1)}_{u,-v} e^{-ip^{(1)}_{u,-v}T} + iA^2_{u,-v} p^{(2)}_{u,-v} e^{-ip^{(2)}_{u,-v}T}) e^{i(-\alpha x + \beta y)} \\
& + \sum_{u,v} (iA^1_{-u,-v} p^{(1)}_{-u,-v} e^{-ip^{(1)}_{-u,-v}T} + iA^2_{-u,-v} p^{(2)}_{-u,-v} e^{-ip^{(2)}_{-u,-v}T}) e^{i(\alpha x + \beta y)} \quad (53)
\end{aligned}$$

Comparing (46)~(49) with (50)~(53), we get

$$G(0) = A^{(1)} + A^{(2)} . \quad (54)$$

$$G^I(0) = i(A^{(1)} p^{(1)} + A^{(2)} p^{(2)}) , \quad (55)$$

$$G(-T) = A^{(1)} e^{-ip^{(1)}T} + A^{(2)} e^{-ip^{(2)}T} , \quad (56)$$

$$G^I(-T) = i(A^{(1)} p^{(1)} e^{-ip^{(1)}T} + A^{(2)} p^{(2)} e^{-ip^{(2)}T}) \quad (57)$$

From (54) and (55), we get

$$A^{(1)} = \frac{G(0)ip^{(2)} - G^I(0)}{i(p^{(2)} - p^{(1)})} , \quad A^{(2)} = \frac{G^I(0) - ip^{(1)}G(0)}{i(p^{(2)} - p^{(1)})} \quad (58)$$

From (54) and (56), we get

$$A^{(1)} = \frac{G(0)x^{(2)} - G(-T)}{x^{(2)} - x^{(1)}} , \quad A^{(2)} = \frac{G(-T) - G(0)x^{(1)}}{x^{(2)} - x^{(1)}} , \quad (59)$$

where $x^{(1)} = e^{-ip^{(1)}T}$ and $x^{(2)} = e^{-ip^{(2)}T}$

From (55) and (57), we get

$$iA^{(1)}p^{(2)} = \frac{G^I(0)x^{(2)} - G^I(-T)}{x^{(2)} - x^{(1)}} , \quad iA^{(2)}p^{(2)} = \frac{G^I(-T) - G^I(0)x^{(1)}}{x^{(2)} - x^{(1)}} \quad (60)$$

From (56) and (57), we get

$$A^{(1)}x^{(1)} = \frac{G(-T)ip^{(2)} - G^I(-T)}{i(p^{(2)} - p^{(1)})} , \quad A^{(2)}x^{(2)} = \frac{G^I(-T) - G(-T)ip^{(1)}}{i(p^{(2)} - p^{(1)})} \quad (61)$$

Eliminating $A^{(1)}$ and $A^{(2)}$ from (59) and (60), we get

$$ip^{(1)} = \frac{G^I(0)x^{(2)} - G^I(-T)}{G(0)x^{(2)} - G(-T)} , \quad ip^{(2)} = \frac{G^I(0)x^{(1)} - G^I(-T)}{G(0)x^{(1)} - G(-T)} \quad (62)$$

Eliminating $A^{(1)}$ and $A^{(2)}$ from (58) and (61), we get

$$x^{(1)} = \frac{G(-T)ip^{(2)} - G^{(1)}(-T)}{G(0)ip^{(2)} - G^{(1)}(0)}, \quad x^{(2)} = \frac{G(-T)ip^{(1)} - G^{(1)}(-T)}{G(0)ip^{(1)} - G^{(1)}(0)}. \quad (63)$$

From (62) or (63), we can obtain $x^{(1)}$ and $x^{(2)}$, or $p^{(1)}$ and $p^{(2)}$. But the equation which is obtained by eliminating one of $p^{(1)}$ and $p^{(2)}$, is a very complicated transcendental equation.

$$e^{-i p T} = \frac{G(-T) \log \frac{G(-T)ip - G(-T)}{G(0)ip - G(0)} + T G^{(1)}(-T)}{G(0) \log \frac{G(-T)ip - G(-T)}{G(0)ip - G(0)} + T G^{(1)}(0)}. \quad (64)$$

Then this group is inconvenient for the practical purpose.

iii) When $Z(0)$, $Z(-T)$, $Z(-2T)$, and $Z(-3T)$ are given.

Expanding these quantities into double Fourier series, we get

$$\begin{aligned} Z(-\nu T) = \sum_{u,v} \{ & G_{u,v}(-\nu T) e^{i(-\alpha x - \beta y)} \\ & + G_{-u,v}(-\nu T) e^{i(\alpha x - \beta y)} \\ & + G_{u,-v}(-\nu T) e^{i(-\alpha x + \beta y)} \\ & + G_{-u,-v}(-\nu T) e^{i(\alpha x + \beta y)} \}, \quad \nu = 0, 1, 2, 3. \end{aligned} \quad (65)$$

From (6), we get

$$\begin{aligned} Z(-\nu T) = \sum_{u,v} \{ & (A^{(1)}_{u,v} e^{-i\nu p^{(1)}_{u,v} T} + A^{(2)}_{u,v} e^{-i\nu p^{(2)}_{u,v} T}) e^{i(-\alpha x - \beta y)} \\ & + (A^{(1)}_{-u,v} e^{-i\nu p^{(1)}_{-u,v} T} + A^{(2)}_{-u,v} e^{-i\nu p^{(2)}_{-u,v} T}) e^{i(\alpha x - \beta y)} \\ & + (A^{(1)}_{u,-v} e^{-i\nu p^{(1)}_{u,-v} T} + A^{(2)}_{u,-v} e^{-i\nu p^{(2)}_{u,-v} T}) e^{i(-\alpha x + \beta y)} \\ & + (A^{(1)}_{-u,-v} e^{-i\nu p^{(1)}_{-u,-v} T} + A^{(2)}_{-u,-v} e^{-i\nu p^{(2)}_{-u,-v} T}) e^{i(\alpha x + \beta y)} \} \end{aligned} \quad (66)$$

Comparing (65) and (66), we get

$$G(0) = A^{(1)} + A^{(2)}, \quad (67)$$

$$G(-T) = A^{(1)} e^{-i p^{(1)} T} + A^{(2)} e^{-i p^{(2)} T}, \quad (68)$$

$$G(-2T) = A^{(1)} e^{-i 2 p^{(1)} T} + A^{(2)} e^{-i 2 p^{(2)} T}, \quad (69)$$

$$G(-3T) = A^{(1)} e^{-i 3 p^{(1)} T} + A^{(2)} e^{-i 3 p^{(2)} T} \quad (70)$$

If we substitute $ip^{(1)}$ and $ip^{(2)}$ instead of $e^{-i p^{(1)} T}$ and $e^{-i p^{(2)} T}$ in (67)~(70), we get (34)~(37). Then, from (38) and (44), we get unknown quantities :

$$A^{(1)} = \frac{G(0)e^{-i p^{(2)} T} - G(-T)}{e^{-i p^{(1)} T} - e^{-i p^{(2)} T}}, \quad A^{(2)} = \frac{G(0)e^{-i p^{(1)} T} - G(-T)}{e^{-i p^{(1)} T} - e^{-i p^{(2)} T}}, \quad (71)$$

$$\frac{e^{-ip^{(1)T}}}{e^{-ip^{(2)T}}} = \frac{G(-T)G(-2T) - G(0)G(-3T) \pm \sqrt{\{G(-T)G(-2T) - G(0)G(-3T)\}^2 - 4\{(G(-T))^2 - G(0)G(-2T)\}}}{2\{(G(-T))^2 - G(0)G(-2T)\}} \quad (72)$$

Then we get

$$\frac{e^{ip^{(1)T}}}{e^{ip^{(2)T}}} = \frac{2\{(G(-T))^2 - G(0)G(-2T)\}}{G(-T)G(-2T) - G(0)G(-3T) \pm \sqrt{\{G(-T)G(-2T) - G(0)G(-3T)\}^2 - 4\{(G(-T))^2 - G(0)G(-2T)\}}} \quad (73)$$

Putting the right hand side of (73) as $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$, we get the solution at $t=NT$:

$$\begin{aligned} Z(NT) = \sum_{u,v} \left[\left\{ \frac{G_{u,v}(0) - \mathcal{E}^{(2)}_{u,v} G_{u,v}(-T)}{\mathcal{E}^{(1)}_{u,v} - \mathcal{E}^{(2)}_{u,v}} (\mathcal{E}^{(1)}_{u,v})^{N+1} \right. \right. \\ + \frac{\mathcal{E}^{(1)}_{u,v} G_{u,v}(-T) - G_{u,v}(0)}{\mathcal{E}^{(1)}_{u,v} - \mathcal{E}^{(2)}_{u,v}} (\mathcal{E}^{(2)}_{u,v})^{N+1} \Big\} e^{i(-\alpha x - \beta y)} \\ + \left\{ \frac{G_{-u,v}(0) - \mathcal{E}^{(2)}_{-u,v} G_{-u,v}(-T)}{\mathcal{E}^{(1)}_{-u,v} - \mathcal{E}^{(2)}_{-u,v}} (\mathcal{E}^{(1)}_{-u,v})^{N+1} \right. \\ + \frac{\mathcal{E}^{(1)}_{-u,v} G_{-u,v}(-T) - G_{-u,v}(0)}{\mathcal{E}^{(1)}_{-u,v} - \mathcal{E}^{(2)}_{-u,v}} (\mathcal{E}^{(2)}_{-u,v})^{N+1} \Big\} e^{i(\alpha x - \beta y)} \\ + \left\{ \frac{G_{u,-v}(0) - \mathcal{E}^{(2)}_{u,-v} G_{u,-v}(-T)}{\mathcal{E}^{(1)}_{u,-v} - \mathcal{E}^{(2)}_{u,-v}} (\mathcal{E}^{(1)}_{u,-v})^{N+1} \right. \\ + \frac{\mathcal{E}^{(1)}_{u,-v} G_{u,-v}(-T) - G_{u,-v}(0)}{\mathcal{E}^{(1)}_{u,-v} - \mathcal{E}^{(2)}_{u,-v}} (\mathcal{E}^{(2)}_{u,-v})^{N+1} \Big\} e^{i(-\alpha x + \beta y)} \\ + \left\{ \frac{G_{-u,-v}(0) - \mathcal{E}^{(2)}_{-u,-v} G_{-u,-v}(-T)}{\mathcal{E}^{(1)}_{-u,-v} - \mathcal{E}^{(2)}_{-u,-v}} (\mathcal{E}^{(1)}_{-u,-v})^{N+1} \right. \\ + \frac{\mathcal{E}^{(1)}_{-u,-v} G_{-u,-v}(-T) - G_{-u,-v}(0)}{\mathcal{E}^{(1)}_{-u,-v} - \mathcal{E}^{(2)}_{-u,-v}} (\mathcal{E}^{(2)}_{-u,-v})^{N+1} \Big\} e^{i(\alpha x + \beta y)} \Big]. \quad (74) \end{aligned}$$

§ 6 Remarks on the differential equation.

For the purpose of numerical prediction, it is convenient to know some characters of the differential equation. But it is not so easy to discuss it completely, so we confine ourselves to some of them.

i) Now we consider the equation (4).

$$\sum_{l,m,n} A_{l,m,n} (ip_{\alpha,\beta})^l (-i\alpha)^m (-i\beta)^n = 0. \quad (4)$$

If m 's are even, then $p_{\alpha,\beta} = p_{-\alpha,\beta}$ and $p_{\alpha,-\beta} = p_{-\alpha,-\beta}$.

These relations are verified as follows:

For the case $l=1$.

From (17), we get

$$\frac{G_{u,v}^I(0)}{G_{u,v}(0)} = \frac{G_{-u,v}^I(0)}{G_{-u,v}(0)} \quad \frac{G_{u,-v}^I(0)}{G_{u,-v}(0)} = \frac{G_{-u,-v}^I(0)}{G_{-u,-v}(0)} \quad (75)$$

From (21), we get

$$\frac{G_{u,v}(0)}{G_{u,v}(-T)} = \frac{G_{-u,v}(0)}{G_{-u,v}(-T)} \quad \frac{G_{u,-v}(0)}{G_{u,-v}(-T)} = \frac{G_{-u,-v}(0)}{G_{-u,-v}(-T)} \quad (76)$$

For the case $l=2$, equivalent relations are obtained from (44) and (72).

If n 's are even, $p_{\alpha,\beta} = p_{\alpha,-\beta}$ and $p_{-\alpha,\beta} = p_{-\alpha,-\beta}$.

These relations are verified in the same manner as above. Further, as m 's and n 's are even, the relation $p_{\alpha,-} = p_{-\alpha,\beta} = p_{\alpha,-} = p_{-\alpha,-\beta}$ holds.

ii) If the operator (2) is of the form :

$$\begin{aligned} L &= \sum_{l,m,n} A_{l,m,n} \frac{\partial^{l+m+n}}{\partial t^l \partial x^m \partial y^n} \\ &= \sum_{l,m,n} \bar{A}_{l,m,n} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y} \right)^l \frac{\partial^{m+n}}{\partial x^m \partial y^n} + \sum_{m,n} A_{0,m,n} \frac{\partial^{m+n}}{\partial x^m \partial y^n} \end{aligned} \quad (77)$$

We put $X = x - Ut$, $Y = y - Vt$, then the function $\bar{Z}(X, Y, t)$ satisfies the following differential equation

$$\sum_{l,m,n} \bar{A}_{l,m,n} \frac{\partial^{l+m+n} \bar{Z}}{\partial t^l \partial X^m \partial Y^n} = 0. \quad (78)$$

The convectonal velocities U and V are determined from the translation of the whole pattern of Z . This process referring to the moving coordinates is convenient for practical purpose.

iii) In our method, we expand Z into double Fourier series, but from the practical point of view, we can not take sufficiently large number of terms.

Then it is useful to know the feature of $p_{\alpha,\beta}$ for large values of α and β . For the sake of simplicity, we consider one dimensional case. Then from (4) we get

$$\sum_{l,m} A_{l,m} (ip_{\alpha})^l (-i\alpha)^m = 0. \quad (4')$$

This equation may be written

$$\begin{aligned} (ip_{\alpha})^l \sum_{m=0}^{m_l} A_{l,m} (-i\alpha)^m + (ip_{\alpha})^{l-1} \sum_{m=0}^{m_{l-1}} A_{l-1,m} (-i\alpha)^m \\ + \cdots + ip_{\alpha} \sum_{m=0}^{m_1} A_{1,m} (-i\alpha)^m + \sum_{m=0}^{m_0} A_{0,m} (-i\alpha)^m = 0, \end{aligned} \quad (79)$$

where $m_l, m_{l-1}, \dots, m_1$ and m_0 are of orders of differential coefficients with respect to x for each order with respect to t . The smaller the scale of disturbance, the larger becomes α .

For a large value of α , (79) becomes

$$(ip_\alpha)^l A_{l,m_l} (-i\alpha)^m + (ip_\alpha)^{l-1} A_{l-1,m_{l-1}} (-i\alpha)^{m_{l-1}} + \cdots + ip_\alpha A_{1,m_1} (-i\alpha)^{m_1} + A_{0,m_0} (-i\alpha)^{m_0} = 0. \quad (80)$$

If m_k is the largest among m_l , m_{l-1} , $\dots$, m_1 , we get

$$(ip)^m \alpha^k A_{k,m_k} (-i\alpha)^{m_k} + A_{0,m_0} (-i\alpha)^{m_0} = 0 \quad (81)$$

for $\alpha \rightarrow \infty$. Then we get

$$(ip_\alpha)^{m_k} = -\frac{A_{0,m_0}}{A_{k,m_k}} (-i\alpha)^{m_0-m_k} \quad \text{for } \alpha \rightarrow \infty \quad (82)$$

Generally, roots of (4') p_α 's are not purely imaginary or real, but complex quantities. If the roots are real, the amplitudes of all harmonics are constant with respect to time. Then the harmonics whose initial amplitudes are large, remain predominant. Our present problem is whether higher harmonics of initially small amplitudes develop or not. Then we confine ourselves to the case when p_α 's are not real. Now we return to (82).

If $m_0 > m_k$, p_α 's increase with the increase of α , and *vice versa*. If $m_0 = m_k$, p_α 's tend to $-i\left(-\frac{A_{0,m_0}}{A_{k,m_k}}\right)^{1/m_k}$ asymptotically.

If p_α 's increase with the increase of α and $R\left(-\frac{A_{0,m_0}}{A_{k,m_k}}\right)^{1/m_k}$ is negative, the higher harmonics damp out with time, so that considerable error will not be caused by adopting finite sum of harmonics instead of the Fourier series. On the other hand, if p_α 's increase with the increase of α and $R\left(-\frac{A_{0,m_0}}{A_{k,m_k}}\right)^{1/m_k}$ is positive, considerable error may possibly be caused by the above process. If p_α 's tend to the above asymptotic value, all higher harmonics change similarly with time.

Now we consider examples, given by J. G. Charney (1949).

If the planetary wave is given by ((10))*

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial^2 Z}{\partial x^2} + \beta\frac{\partial Z}{\partial x} - \lambda^2\frac{\partial Z}{\partial t} = 0, \quad ((10))$$

we get

$$l_k = 1, m_k = 2, m_0 = 1, A_{k,m_k} = 1, A_{0,m_0} = \beta, \\ ip_\alpha = -\beta(-i\alpha)^{-1} = -i\beta/\alpha \quad \text{for } \alpha \rightarrow \infty.$$

Then amplitudes of higher harmonics do not change but their periods increase with the increase of α .

If the planetary wave is given by ((39))

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \kappa F\right)\left(\frac{\partial^2 Z}{\partial x^2} - m^2 Z\right) + \beta\frac{\partial Z}{\partial x} - \lambda^2\frac{\partial Z}{\partial t} = 0, \quad ((39))$$

The symbol R denotes the real part of the following quantities.

* The number in ((10)) denotes that in the Charney's paper.

where the right hand side is neglected. Then we get

$$l_k=1, m_k=2, m_0=2, A_{k,m_k}=1, A_{0,m_0}=\kappa F, ip_\alpha=-\kappa F \quad \text{for } \alpha \rightarrow \infty.$$

In this case the amplitudes of higher harmonics damp out with the time, so that if their initial amplitudes are not large, the neglect of higher harmonics does not cause considerable error.

The above features of p_α for large values of α give an useful technique for the determination problem of the differential equations concerned.

iv) Now we consider the solution (23) as an example.

$$\begin{aligned} Z(NT) = & \sum_{u,v} \left\{ G_{u,v}(0) \left(\frac{G_{u,v}(0)}{G_{u,v}(-T)} \right)^N e^{i(-\alpha x - \beta y)} \right. \\ & + G_{-u,v}(0) \left(\frac{G_{-u,v}(0)}{G_{-u,v}(-T)} \right)^N e^{i(\alpha x - \beta y)} \\ & + G_{u,-v}(0) \left(\frac{G_{u,-v}(0)}{G_{u,-v}(-T)} \right)^N e^{i(-\alpha x + \beta y)} \\ & \left. + G_{-u,-v}(0) \left(\frac{G_{-u,-v}(0)}{G_{-u,-v}(-T)} \right)^N e^{i(\alpha x + \beta y)} \right\} \end{aligned} \quad (23)$$

For $t=0$, we get

$$\begin{aligned} Z(0) = & \sum_{u,v} \{ G_{u,v}(0) e^{i(-\alpha x - \beta y)} + G_{-u,v}(0) e^{i(\alpha x - \beta y)} \\ & + G_{u,-v}(0) e^{i(-\alpha x + \beta y)} + G_{-u,-v}(0) e^{i(\alpha x + \beta y)} \} \end{aligned} \quad (83)$$

From (23) and (83), we get

$$\begin{aligned} Z(NT) = & Z(0) + \sum_{u,v} \left[G_{u,v}(0) \left\{ \left(\frac{G_{u,v}(0)}{G_{u,v}(-T)} \right)^N - 1 \right\} e^{i(-\alpha x - \beta y)} \right. \\ & + G_{-u,v}(0) \left\{ \left(\frac{G_{-u,v}(0)}{G_{-u,v}(-T)} \right)^N - 1 \right\} e^{i(\alpha x - \beta y)} \\ & + G_{u,-v}(0) \left\{ \left(\frac{G_{u,-v}(0)}{G_{u,-v}(-T)} \right)^N - 1 \right\} e^{i(-\alpha x + \beta y)} \\ & \left. + G_{-u,-v}(0) \left\{ \left(\frac{G_{-u,-v}(0)}{G_{-u,-v}(-T)} \right)^N - 1 \right\} e^{i(\alpha x + \beta y)} \right] \end{aligned} \quad (84)$$

The above form of the solution is convenient for the case $m_0 \leq m_k$. When $m_0 > m_k$, the pattern changes rapidly, so the above form of the solution is not convenient.

The same transformation of the solution is, of course, possible for other cases.

§ 7 Practical Method.

In order to apply the above theory to practical problems, we shall confine ourselves to the one dimensional case.

If we expand $Z(-T)$ and $Z(0)$ into Fourier series, we get

$$\begin{aligned}
Z(-T) &= \sum \varepsilon_u A_u(-T) \cos \alpha x + \sum \varepsilon'_u B_u(-T) \sin \alpha x, \\
Z(0) &= \sum \varepsilon_u A_u(0) \cos \alpha x + \sum \varepsilon'_u B_u(0) \sin \alpha x
\end{aligned} \tag{85}$$

where

$$\begin{aligned}
\varepsilon_u &= \frac{1}{2}, & \varepsilon'_u &= 0 & \text{for } u=0 \\
\varepsilon_u &= 1, & \varepsilon'_u &= 1 & \text{for } u \neq 0.
\end{aligned}$$

On the other hand, we have

$$Z(-T) = \sum G_u(-T) e^{-i\alpha x} + \sum G_{-u}(-T) e^{i\alpha x} \tag{19}$$

$$Z(0) = \sum G_u(0) e^{-i\alpha x} + \sum G_{-u}(0) e^{i\alpha x} \tag{11}$$

Comparing these equations, we get

$$\left. \begin{aligned}
G_u(-T) &= \frac{1}{2} \{ \varepsilon_u A_u(-T) + i \varepsilon'_u B_u(-T) \} = R_u(-T) e^{i\theta_u(-T)}, \\
G_{-u}(-T) &= \frac{1}{2} \{ \varepsilon_u A_u(-T) - i \varepsilon'_u B_u(-T) \} = R_u(-T) e^{-i\theta_u(-T)}, \\
G_u(0) &= \frac{1}{2} \{ \varepsilon_u A_u(0) + i \varepsilon'_u B_u(0) \} = R_u(0) e^{i\theta_u(0)}, \\
G_{-u}(0) &= \frac{1}{2} \{ \varepsilon_u A_u(0) - i \varepsilon'_u B_u(0) \} = R_u(0) e^{-i\theta_u(0)}
\end{aligned} \right\} \tag{86}$$

Then we get from (84)

$$\begin{aligned}
Z(NT) &= Z(0) + \sum R_u(0) e^{i\theta_u(0)} \left\{ \left(\frac{R_u(0)}{R_u(-T)} \right)^N e^{iN(\theta_u(0) - \theta_u(-T))} - 1 \right\} e^{-i\alpha x} \\
&\quad + \sum R_u(0) e^{-i\theta_u(0)} \left\{ \left(\frac{R_u(0)}{R_u(-T)} \right)^N e^{-iN(\theta_u(0) - \theta_u(-T))} - 1 \right\} e^{i\alpha x} \\
&= Z(0) + \sum R_u(0) S_u^N \cos(\Theta_u^N + \theta_u(0) - \alpha x) - \sum 2R_u(0) \cos(\theta_u(0) - \alpha x),
\end{aligned} \tag{87}$$

where

$$\begin{aligned}
S_u^N &= \left\{ 4 \left(\frac{R_u(0)}{R_u(-T)} \right)^{2N} - \left(\frac{R_u(0)}{R_u(-T)} \right)^N 4 \cos N(\theta_u(0) - \theta_u(-T)) + 1 \right\}^{\frac{1}{2}}, \\
\Theta_u^N &= \tan^{-1} \left[2 \left(\frac{R_u(0)}{R_u(-T)} \right)^N \sin N(\theta_u(0) - \theta_u(-T)) \middle/ \left\{ 2 \left(\frac{R_u(0)}{R_u(-T)} \right)^N \times \right. \right. \\
&\quad \left. \left. \cos N(\theta_u(0) - \theta_u(-T)) - 1 \right\} \right].
\end{aligned} \tag{88}$$

The amplitudes and phase angles of complex quantities are easily obtained diagrammatically on the complex plane in the polar form.

Next we shall explain an alternative method of analysis.

Subtracting (19) from (11), we get

$$\begin{aligned}
Z(0) - Z(-T) &= \sum_u [\{G_u(0) - G_u(-T)\} e^{-i\alpha x} + \{G_{-u}(0) - G_{-u}(-T)\} e^{i\alpha x}] \\
&= \sum_u [\varepsilon_u \{A_u(0) - A_u(-T)\} \cos \alpha x + \varepsilon'_u \{B_u(0) - B_u(-T)\} \sin \alpha x] \\
&= \sum_u [\varepsilon_u \bar{A}_u(0) \cos \alpha x + \varepsilon'_u \bar{B}_u(0) \sin \alpha x], \tag{89}
\end{aligned}$$

where

$$\begin{aligned}
G_u(0) - G_u(-T) &= \frac{1}{2} \{\varepsilon_u \bar{A}_u(0) + i\varepsilon'_u \bar{B}_u(0)\} = \bar{R}_u(0) e^{i\bar{\theta}_u(0)} \\
G_{-u}(0) - G_{-u}(-T) &= \frac{1}{2} \{\varepsilon_u \bar{A}_u(0) - i\varepsilon'_u \bar{B}_u(0)\} = \bar{R}_u(0) e^{-i\bar{\theta}_u(0)} \tag{90}
\end{aligned}$$

and from (86)

$$\begin{aligned}
G_u(0) &= \frac{1}{2} \{\varepsilon_u A_u(0) + i\varepsilon'_u B_u(0)\} \\
G_{-u}(0) &= \frac{1}{2} \{\varepsilon_u \bar{A}_u(0) - i\varepsilon'_u \bar{B}_u(0)\}. \tag{91}
\end{aligned}$$

From (90) and (91), we get

$$\begin{aligned}
G_u(-T) &= \frac{1}{2} [\varepsilon_u \{A_u(0) - \bar{A}_u(0)\} + i\varepsilon'_u \{B_u(0) - \bar{B}_u(0)\}], \\
G_{-u}(-T) &= \frac{1}{2} [\varepsilon_u \{A_u(0) - \bar{A}_u(0)\} + i\varepsilon'_u \{B_u(0) - \bar{B}_u(0)\}] \tag{92}
\end{aligned}$$

and

$$\begin{aligned}
\frac{G_u(0)}{G_u(-T)} - 1 &= \frac{G_u(0) - G_u(-T)}{G_u(-T)} = \frac{\bar{R}_u(0)}{R_u(-T)} e^{i(\bar{\theta}_u(0) - \theta_u(-T))} \\
\frac{G_{-u}(0)}{G_{-u}(-T)} - 1 &= \frac{G_{-u}(0) - G_{-u}(-T)}{G_{-u}(-T)} = \frac{\bar{R}_u(0)}{R_u(-T)} e^{-i(\bar{\theta}_u(0) - \theta_u(-T))} \tag{93}
\end{aligned}$$

Introducing (93) into (84), we get

$$Z(X, T) - Z(X, 0) = \sum_u \frac{R_u(0) \bar{R}_u(0)}{R_u(-T)} \cos[\{\theta_u(0) + \bar{\theta}_u(0) - \theta_u(-T) - \alpha x\}] \tag{94}$$

If the pattern moves as a whole, we may obtain better results by referring to moving coordinates, whose velocities are obtained empirically by taking mean values of displacements of some major troughs and wedges.

§ 8 Application to upper westerlies.

Recently J. G. Charney published an article on the numerical prediction of upper westerlies (1949). He obtained differential equations of upper westerlies, which were of the first order with respect to time. He treated the prediction-problem as an ordinary initial value problem and obtained a successful result. If upper westerlies satisfy the dif-

ferential equation of the first order with respect to time, our theory for the case $l=1$ can be applied. Using data of Charney obtained from Figs. 8~13, we discussed our method. In our investigation, we took 24 terms of Fourier series.

Charney gave the following solution:

$$Z(x+ut, t) = Z(x, 0) + \int_0^{2\pi} Z(\alpha, 0) I_{a^2}(x-\alpha, t) d\alpha, \quad ((40))$$

where

$$I_{a^2}(x, t) = \frac{1}{2\pi} \sum_{u=-\infty}^{+\infty} (e^{\frac{iut}{u^2+a^2}} - 1) e^{iux}. \quad ((41))$$

Here the term which represents the topographical effect is neglected. Commuting the order of the integration and summation, we get

$$Z(x+ut, t) = Z(x, 0) + \sum_{-\infty}^{\infty} (e^{\frac{iut}{u^2+a^2}} - 1) \left(\frac{1}{2\pi} \int_0^{2\pi} Z(\alpha, 0) e^{-iua} d\alpha \right) e^{iux}. \quad (95)$$

Comparing (84) for $N=1$ and (95), we get

$$G_u(0) = \frac{1}{2\pi} \int_0^{2\pi} Z(\beta, 0) e^{iub} d\beta, \quad (96)$$

$$G_{-u}(0) = \frac{1}{2\pi} \int_0^{2\pi} Z(\beta, 0) e^{-iub} d\beta, \quad (97)$$

$$\frac{G_u(0)}{G_u(-T)} = e^{-i \frac{ubT}{u^2+a^2}}, \quad \frac{G_{-u}(0)}{G_{-u}(-T)} = e^{i \frac{ubT}{u^2+a^2}} \quad (98)$$

Introducing appropriate values for b and a^2 , we get $e^{i \frac{ubT}{u^2+a^2}} = e^{i \frac{7u}{u^2+19}}$ for $t=1$ day and $e^{i \frac{14u}{u^2+19}}$ for $t=2$ days. Taking 24 terms, we calculated $Z(x+ut, t)$. We call this method "modified Charney's method".

The results obtained by modified Charney's method are shown in Fig. 1 and Fig. 2 for $t=1$ and in Fig. 3 for $t=2$.

In the figures, the chained curves and the continuous curves show the observed and the predicted height profiles of the 500 mb surface at 45°N for the dates indicated in the figures respectively. And the points in the figures show the predicted values by Charney.

Fig. 4 and Fig. 5 show the results, which are obtained by our method (using (93) and (94)) from the data of the preceding two days.

Fig. 6 shows the predicted and observed curves of 13 Jan., which is obtained from the data of 9 and 11 Jan..

Fig. 7, 8, 9 and 10 show the components of $\left(\frac{G_u(0)}{G_u(-T)} - 1 \right)$ for various u 's. In the figures, the chain line and the full line denote the real and imaginary parts respectively.

Figure 1. One-day forecasting. Predicted results of 10 Jan., by Modified Charney's Method, using the data of 9 Jan..

Figure 2. One-day forecasting. Predicted results of 12 Jan., by Modified Charney's Method, using the data of 11 Jan..

Figure 3. Two-day forecasting. Predicted results of 12 Jan., by Modified Charney's Method, using the data of 10 Jan..

Figure 4. One-day forecasting. Predicted results of 13 Jan., by Syōno's Method, using the data of 11 and 12 Jan..

Figure 5. One-day forecasting. Predicted results of 12 Jan., by Syōno's Method, using the data of 10 and 11 Jan..

Figure 6. Two-day forecasting. Predicted results of 13 Jan., by Syōno's Method, using the data of 9 and 11 Jan..

Figure 7. Components of $\left(\frac{G_u(0)}{G_u(-T)} - 1\right)$ for various u 's. They are obtained, by using the data of 8 and 9 Jan..

Figure 8. Components of $\left(\frac{G_u(0)}{G_u(-T)} - 1\right)$ for various u 's. They are obtained, by using the data of 9 and 10 Jan..

Figure 9. Components of $\left(\frac{G_u(0)}{G_u(-T)} - 1\right)$ for various u 's. They are obtained, by using the data of 11 and 12 Jan..

Figure 10. Components of $\left(\frac{G_u(0)}{G_u(-T)} - 1\right)$ for various u 's. They are obtained, by using the data of 12 and 13 Jan..

Figure 11. Mean values components of $\left(\frac{G_u(0)}{G_u(-T)} - 1\right)$ for various u 's from 8 Jan. to 13 Jan. The dotted lines represent the components of $\left(e^{i\frac{7u}{u^2+10}} - 1\right)$ for various u 's.

Further investigation is now continuing.

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REFERENCE

Charney, J. G. and Eliassen, A., 1949: A numerical method for predicting the perturbations of the middle latitude westerlies. *Tellus*, Vol. 1, No. 2, pp. 38~54.

The results shown in Fig. 7 are calculated from the data of 8 and 9 Jan. and in Fig. 8 they are calculated from that of 9 and 10 Jan. and so on.

The mean values of these components are shown in Fig. 11. They show irregular changes for various u 's. The components of $\left(e^{i\frac{7u}{u^2+10}} - 1\right)$ are also shown in Fig. 11, which are represented by the dotted lines.

If the differential equation of Z is of the first order with respect to time, we must obtain a smooth curve such as the dotted line shown in Fig. 11, so that it is doubtful that the change of the upper westerlies obeys the differential equation of the first order.

This may be one of the reasons that two-day forecasting is not satisfactory. Another reason may be that the number of terms of Fourier series is too small.

数 値 豫 報 (I)

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今或時刻における状態を S_0 とし, その状態から或法則 L_0 に従つて状態が S_1 に變つた場合を記號的に $S_1 = L_0 S_0$ と表わす. もし同じ法則が働いて状態が S_1 から S_2 に變つたものとすれば, $S_2 = L_0 S_1$ と表わせる. 記號的に $L_0 = S_2 S_0^{-1}$ ($S_0 S_0^{-1} = 1$) と書ければ, $S_2 = S_1 S_0^{-1} S_1$ となる. 上の記號は具體的には積分方程式を表わすものと解釋出来る. 現象が常數係數の微分方程式 $\sum_{l,m,n} A_{l,m,n} \frac{\partial^{l+m+n} Z}{\partial t^l \partial x^m \partial y^n} = 0$ を満足する場合について, 上記の問題を具體化したものである. この微分方程式の解は $Z = \sum_{k,\alpha,\beta} A^k_{\alpha,\beta} e^{i(p^k_{\alpha,\beta} t - \alpha x - \beta y)}$ の形に展開出来るが, $A^k_{\alpha,\beta}$, $p^k_{\alpha,\beta}$ 等を實測より定めて, その定められた係数を使つて後の時刻における状態函數 Z を求めるのである.

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